

- Definitions: Here we list the definitions we use in the Mathematica file.

- $g = g_{\mu\nu}$; $inv g = g^{\mu\nu}$;
 $Yemag = g_{\mu\nu} Y e^{\mu} Y e^{\nu}$;

$Ymmag = g_{\mu\nu} Y m^{\mu} Y m^{\nu}$; $Rgrad Y e = r \nabla_{\mu} Y e_{\nu}$;

$econtra = e^{\mu}$, $mcontra = m^{\mu}$, $Rcontra = r^{\mu}$,
 $Tcontra = t^{\mu}$;

$Tol = T^{0l}_{\mu\nu}$, $Tot = T^{0t}_{\mu\nu}$,
 $T1m = T^{1m}_{\mu\nu}$, $T1e = T^{1e}_{\mu\nu}$

$h_{01} = h^{0r}$, $h_{02} = h^{0m}$,
 $P[X]_{\mu\nu\rho\sigma} = \nabla_{\mu} \nabla_{\nu} X_{\rho\sigma}$

$R = R_{\mu\nu\rho\sigma}$

$g = \{ \{-1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, r^2, 0\}, \{0, 0, 0, r^2 \sin[\theta]^2\} \}$
 $\{ \{-1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, r^2, 0\}, \{0, 0, 0, r^2 \sin[\theta]^2\} \}$

$inv g = \text{Inverse}[g]$

$\{ \{-1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, \frac{1}{r^2}, 0\}, \{0, 0, 0, \frac{\text{Csc}[\theta]^2}{r^2}\} \}$

$cord := \{t, r, \theta, \phi\}$;

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dim = Dimensions[cord][[1]]
```

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4
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```
Γ = Table[Sum[(1/2) * invg[[a, b]] *
  (D[g[[b, c]], cord[[d]] + D[g[[b, d]], cord[[c]] - D[g[[c, d]], cord[[b]]]),
  {b, 1, dim}], {a, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, -r, 0}, {0, 0, 0, -r Sin[θ]^2}},
{{0, 0, 0, 0}, {0, 0, 1/r, 0}, {0, 1/r, 0, 0}, {0, 0, 0, -Cos[θ] Sin[θ]}},
{{0, 0, 0, 0}, {0, 0, 0, 1/r}, {0, 0, 0, Cot[θ]}, {0, 1/r, Cot[θ], 0}}}
```

```
Yr := {0, Y[θ, ϕ], 0, 0}
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```
Ye := (1/(r (r + 1)))^(1/2) r {0, 0, D[Y[θ, ϕ], θ], D[Y[θ, ϕ], ϕ]}
```

```
Ymag = Assuming[r > 0 && Y'[θ] ∈ Reals && Y'[θ] > 0,
  FullSimplify[(Sum[invg[[i, j]] Ye[[i]] Ye[[j]], {i, 1, dim}, {j, 1, dim})^(1/2)]]
```

$$\sqrt{\frac{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}{r + r^2}}$$

```
econtra =
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```
Assuming[r > 0, FullSimplify[Table[Sum[invg[[i, j]] Ye[[j]]/Ymag, {j, 1, dim}], {i, 1, dim}]]]
```

$$\{0, 0, \frac{Y^{(1,0)}[\theta, \phi]}{r \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}}, \frac{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]}{r \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}}\}$$

```
Ym := (1/(r (r + 1)))^(1/2) {0, 0, -r D[Y[θ, ϕ], ϕ]/Sin[θ], r Sin[θ] D[Y[θ, ϕ], θ]}
```

```
Ymmag = Assuming[r > 0 && Y'[θ] ∈ Reals && Y'[θ] > 0,
  FullSimplify[(Sum[invg[[i, j]] Ym[[i]] Ym[[j]], {i, 1, dim}, {j, 1, dim})^(1/2)]]
```

$$\sqrt{\frac{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}{r + r^2}}$$

```
mcontra =
```

```
Assuming[r > 0, FullSimplify[Table[Sum[invg[[i, j]] Ym[[j]]/Ymmag, {j, 1, dim}], {i, 1, dim}]]]
```

$$\{0, 0, -\frac{\text{Csc}[\theta] Y^{(0,1)}[\theta, \phi]}{r \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}}, \frac{\text{Csc}[\theta] Y^{(1,0)}[\theta, \phi]}{r \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}}\}$$

```
Tol = Table[Y[θ, ϕ] KroneckerDelta[2, i] KroneckerDelta[2, j], {i, 1, dim}, {j, 1, dim}]
```

```
{{0, 0, 0, 0}, {0, Y[θ, ϕ], 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

Tol[[1, 1]]

0

$$T1e = \left(\frac{2}{\ell(\ell+1)} \right)^{\frac{1}{2}} \frac{r}{2} \{ \{ \theta, \theta, \theta, \theta \}, \{ \theta, \theta, D[Y[\theta, \phi], \theta], D[Y[\theta, \phi], \phi] \}, \\ \{ \theta, D[Y[\theta, \phi], \theta], \theta, \theta \}, \{ \theta, D[Y[\theta, \phi], \phi], \theta, \theta \} \} \\ \{ \{ \theta, \theta, \theta, \theta \}, \{ \theta, \theta, \frac{r \sqrt{\frac{1}{\ell(\ell+1)}} Y^{(1,\theta)}[\theta, \phi]}{\sqrt{2}}, \frac{r \sqrt{\frac{1}{\ell(\ell+1)}} Y^{(\theta,1)}[\theta, \phi]}{\sqrt{2}} \}, \\ \{ \theta, \frac{r \sqrt{\frac{1}{\ell(\ell+1)}} Y^{(1,\theta)}[\theta, \phi]}{\sqrt{2}}, \theta, \theta \}, \{ \theta, \frac{r \sqrt{\frac{1}{\ell(\ell+1)}} Y^{(\theta,1)}[\theta, \phi]}{\sqrt{2}}, \theta, \theta \} \}$$

divT1e[j_] = Table[Sum[(inv[g[[i, k]]] * (D[T1e[[i, j]], cord[[k]]] -
(Sum[r[[m, k, i]] * T1e[[m, j]] + r[[m, k, j]] * T1e[[i, m]] , {m, 1, dim}))),
{i, 1, dim}, {k, 1, dim}], {j, 1, dim}] // FullSimplify

$$\{ \theta, \frac{1}{r} \sqrt{\frac{1}{2\ell+2\ell^2}} (\csc[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \cot[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi]) , \\ 3 \sqrt{\frac{1}{2\ell+2\ell^2}} Y^{(1,\theta)}[\theta, \phi], 3 \sqrt{\frac{1}{2\ell+2\ell^2}} Y^{(\theta,1)}[\theta, \phi] \}$$

$$T1m = \left(\frac{2}{\ell(\ell+1)} \right)^{\frac{1}{2}} \frac{r}{2} \{ \{ \theta, \theta, \theta, \theta \}, \{ \theta, \theta, \frac{-D[Y[\theta, \phi], \phi]}{\sin[\theta]}, \sin[\theta] D[Y[\theta, \phi], \theta] \}, \\ \{ \theta, \frac{-D[Y[\theta, \phi], \phi]}{\sin[\theta]}, \theta, \theta \}, \{ \theta, \sin[\theta] D[Y[\theta, \phi], \theta], \theta, \theta \} \};$$

divT1m[j_] = Table[Sum[(inv[g[[i, k]]] * (D[T1m[[i, j]], cord[[k]]] -
(Sum[r[[m, k, i]] * T1m[[m, j]] + r[[m, k, j]] * T1e[[i, m]] , {m, 1, dim}))),
{i, 1, dim}, {k, 1, dim}], {j, 1, dim}] // FullSimplify

$$\{ \theta, \theta, -3 \sqrt{\frac{1}{2\ell+2\ell^2}} \csc[\theta] Y^{(\theta,1)}[\theta, \phi], 3 \sqrt{\frac{1}{2\ell+2\ell^2}} \sin[\theta] Y^{(1,\theta)}[\theta, \phi] \}$$

$$Tot = \frac{1}{\sqrt{2}} r^2 \{ \{ \theta, \theta, \theta, \theta \}, \{ \theta, \theta, \theta, \theta \}, \{ \theta, \theta, Y[\theta, \phi], \theta \}, \{ \theta, \theta, \theta, Y[\theta, \phi] (\sin[\theta])^2 \} \};$$

divTot[j_] = Table[Sum[(inv[g[[i, k]]] * (D[Tot[[i, j]], cord[[k]]] -
(Sum[r[[m, k, i]] * Tot[[m, j]] + r[[m, k, j]] * Tot[[i, m]] , {m, 1, dim}))),
{i, 1, dim}, {k, 1, dim}], {j, 1, dim}] // FullSimplify

$$\{ \theta, -\frac{\sqrt{2} Y[\theta, \phi]}{r}, \frac{Y^{(1,\theta)}[\theta, \phi]}{\sqrt{2}}, \frac{Y^{(\theta,1)}[\theta, \phi]}{\sqrt{2}} \}$$

RgradYe1 := r Table[(D[Ye[[i]], cord[[j]]] - (Sum[r[[k, j, i]] Ye[[k]], {k, 1, dim}))),
{i, 1, dim}, {j, 1, dim}] // FullSimplify

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RgradYe :=
Table[
$$\frac{\text{RgradYe1}[[i, j]] + \text{RgradYe1}[[j, i]]}{2}$$
, {i, 1, dim}, {j, 1, dim}] // FullSimplify

T2e := RgradYe +  $\frac{1}{\sqrt{2}}$  T1e +  $\left(\ell \frac{(\ell + 1)}{2}\right)^{\frac{1}{2}}$  Tot // FullSimplify

hv = h01[t, r] Yr + h02[t, r] Ym // FullSimplify

{0, Y[0, ϕ] h1[t, r], -r  $\sqrt{\frac{1}{\ell + \ell^2}}$  Csc[0] h2[t, r] Y(0,1)[0, ϕ],
r  $\sqrt{\frac{1}{\ell + \ell^2}}$  Sin[0] h2[t, r] Y(1,0)[0, ϕ]}

hs = h0[t, r] Y[0, ϕ];

P[X_] :=
Table[D[(D[X[[c, d]], cord[[b]]] - (Sum[r[[f, b, c]] X[[f, d]] + r[[f, b, d]] X[[c, f]],
{f, 1, dim}))), cord[[a]]] - Sum[r[[e, a, b]] (D[X[[c, d]], cord[[e]]] -
(Sum[r[[f, e, c]] X[[f, d]] + r[[f, e, d]] X[[c, f]], {f, 1, dim}))), {e, 1, dim}] -
Sum[r[[e, a, c]] (D[X[[e, d]], cord[[b]]] -
(Sum[r[[f, b, e]] X[[f, d]] + r[[f, b, d]] X[[e, f]], {f, 1, dim}))), {e, 1, dim}] -
Sum[r[[e, a, d]] (D[X[[c, e]], cord[[b]]] -
(Sum[r[[f, b, c]] X[[f, e]] + r[[f, b, e]] X[[c, f]], {f, 1, dim}))), {e, 1, dim}],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}];

P[Tol][[1, 1, 1, 1]]
0

R[X_] :=
Table[(P[X][[b, c, a, d]] - P[X][[a, c, b, d]] + P[X][[a, d, b, c]] - P[X][[b, d, a, c]]),
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}];

R[h01[t, r] Tol][[1, 1, 1, 1]]
0

Rcontra = {0, 1, 0, 0};
Tcontra = {-1, 0, 0, 0};

sp := TensorProduct[Tcontra, hv] // FullSimplify

ss := TensorProduct[hs Tcontra, Tcontra] // FullSimplify

ss = TensorProduct[hs Tcontra, Tcontra] // MatrixForm

$$\begin{pmatrix} Y[0, \phi] h_0[t, r] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$


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■ R_{0eme}

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Sum[Tcontra[[a]] econtra[[b]] mcontra[[c]] econtra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] econtra[[b]] mcontra[[c]] econtra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
-
  1
  r^2 Sqrt[Csc[θ]^2 Y^(0,1)[θ, ϕ]^2 + Y^(1,0)[θ, ϕ]^2]
  Sqrt[
    1
    ρ + ρ^2
    (r h2^(0,1)[t, r] (Csc[θ]^2 Y^(0,1)[θ, ϕ]^2 + Y^(1,0)[θ, ϕ]^2) +
    h2[t, r] (Csc[θ]^2 Y^(0,1)[θ, ϕ]^2 + Csc[θ]^2 Y^(0,1)[θ, ϕ]
    (Csc[θ]^2 Y^(0,3)[θ, ϕ] + Cot[θ] Y^(1,1)[θ, ϕ] + Y^(2,1)[θ, ϕ]) + Y^(1,0)[θ, ϕ]
    (Y^(1,0)[θ, ϕ] + Csc[θ]^2 (-2 Cot[θ] Y^(0,2)[θ, ϕ] - Y^(1,0)[θ, ϕ] + Y^(1,2)[θ, ϕ]) +
    Cot[θ] Y^(2,0)[θ, ϕ] + Y^(3,0)[θ, ϕ]))
  )
Sum[Tcontra[[a]] econtra[[b]] mcontra[[c]] econtra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] econtra[[b]] mcontra[[c]] econtra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] econtra[[b]] mcontra[[c]] econtra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
1
r Sqrt[
  1
  2 ρ + 2 ρ^2
  h1m^(1,0)[t, r] Sqrt[Csc[θ]^2 Y^(0,1)[θ, ϕ]^2 + Y^(1,0)[θ, ϕ]^2]
]

Sum[Tcontra[[a]] econtra[[b]] mcontra[[c]] econtra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] econtra[[b]] mcontra[[c]] econtra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

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■ R_{erem}

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Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

```

```
Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

```
Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$-\frac{1}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}} \sqrt{\frac{1}{2\ell + 2\ell^2}} h1m[t, r] \left(2 \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + \right. \\ \left. \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(0,3)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,1)}[\theta, \phi] + Y^{(2,1)}[\theta, \phi] \right) + \right. \\ \left. Y^{(1,0)}[\theta, \phi] \left(\text{Csc}[\theta]^2 \left(-2 \text{Cot}[\theta] Y^{(0,2)}[\theta, \phi] - \text{Cos}[2\theta] Y^{(1,0)}[\theta, \phi] + Y^{(1,2)}[\theta, \phi] \right) + \right. \right. \\ \left. \left. \text{Cot}[\theta] Y^{(2,0)}[\theta, \phi] + Y^{(3,0)}[\theta, \phi] \right) \right)$$

```
Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[econtra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

■ R_{0rem}

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Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2} \sqrt{\frac{1}{\ell + \ell^2}} \left(-h_2[t, r] + r h_2^{(0,1)}[t, r] \right) \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] + Y^{(2,0)}[\theta, \phi] \right)$$

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r} \sqrt{\frac{1}{2\ell + 2\ell^2}} h1m^{(1,0)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] + Y^{(2,0)}[\theta, \phi] \right)$$

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] mcontra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

$$-\frac{\sqrt{\frac{1}{\ell + \ell^2}} (-h_2[t, r] + r h_2^{(0,1)}[t, r]) \left(\frac{1}{\ell + \ell^2}\right)^{-1}}{r^2} + \frac{\sqrt{\frac{1}{2\ell + 2\ell^2}} h1m^{(1,0)}[t, r] \left(\frac{1}{\ell + \ell^2}\right)^{-1}}{r} // FullSimplify$$

$$\frac{2 h_2[t, r] - 2 r h_2^{(0,1)}[t, r] + \sqrt{2} r h1m^{(1,0)}[t, r]}{2 r^2 \sqrt{\frac{1}{\ell + \ell^2}}}$$

■ R_{0erm}

```
Sum[Tcontra[[a]] econtra[[b]] Rcontra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

.

```
Sum[Tcontra[[a]] econtra[[b]] Rcontra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] Rcontra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$-\frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{2\ell + 2\ell^2}} h1m^{(1,0)}[t, r]$$

$$\left(Y^{(1,0)}[\theta, \phi]^2 \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] \right) - 2 \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \right.$$

$$\left. Y^{(1,0)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] + Y^{(2,0)}[\theta, \phi] \right) \right)$$

```
Sum[Tcontra[[a]] econtra[[b]] Rcontra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{2\ell + 2\ell^2}} \text{Csc}[\theta] \\ h1e^{(1,0)}[t, r] \left(\text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\ \left. Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] - Y^{(2,0)}[\theta, \phi] \right) \right)$$

```
Sum[Tcontra[[a]] econtra[[b]] Rcontra[[c]] mcontra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$- \frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta]^3 \left(h2e^{(1,0)}[t, r] + r h2e^{(1,1)}[t, r] \right) \\ \left(\text{Cot}[\theta] Y^{(0,1)}[\theta, \phi]^3 - Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \text{Sin}[\theta]^2 Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\ \left. Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(Y^{(0,2)}[\theta, \phi] - \text{Sin}[\theta]^2 Y^{(2,0)}[\theta, \phi] \right) \right)$$

■ R_{0mre}

```
Sum[Tcontra[[a]] mcontra[[b]] Rcontra[[c]] econtra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] mcontra[[b]] Rcontra[[c]] econtra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] mcontra[[b]] Rcontra[[c]] econtra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\left(\sqrt{\frac{1}{2\ell + 2\ell^2}} h1m^{(1,0)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \right. \right. \\ \left. \left(Y^{(0,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] \right) + 2 Y^{(1,0)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] \right) + \right. \\ \left. \left. Y^{(1,0)}[\theta, \phi]^2 Y^{(2,0)}[\theta, \phi] \right) \right) \Bigg/ \left(r \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right) \right)$$

$$\begin{aligned}
& - \frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{2\ell + 2\ell^2}} h1m^{(1,0)}[t, r] \\
& \quad \left(Y^{(1,0)}[\theta, \phi]^2 \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] \right) - 2 \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \right. \\
& \quad \left. Y^{(1,0)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] + Y^{(2,0)}[\theta, \phi] \right) \right) - \\
& \quad \left(\sqrt{\frac{1}{2\ell + 2\ell^2}} h1m^{(1,0)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \right. \right. \\
& \quad \left. \left(Y^{(0,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] \right) + 2 Y^{(1,0)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] \right) + \right. \\
& \quad \left. \left. Y^{(1,0)}[\theta, \phi]^2 Y^{(2,0)}[\theta, \phi] \right) \right) \Bigg/ \left(r \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right) \right) // \text{FullSimplify} \\
& - \frac{1}{r} \sqrt{\frac{1}{2\ell + 2\ell^2}} h1m^{(1,0)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] + Y^{(2,0)}[\theta, \phi] \right)
\end{aligned}$$

■ R_{0r0e}

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] econtra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2} \left(h_0[t, r] - r h_0^{(0,1)}[t, r] \right) \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}$$

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] econtra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] econtra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] econtra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] econtra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] econtra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$- \sqrt{\frac{1}{2\ell + 2\ell^2}} \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2} h1e^{(2,0)}[t, r]$$

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] econtra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

■ R_{0r0m}

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\sqrt{\frac{1}{\ell + \ell^2}} \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} h_2^{(1,1)}[t, r]$$

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$-\sqrt{\frac{1}{2\ell + 2\ell^2}} \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} h_{1m}^{(2,\theta)}[t, r]$$

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] mcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

■ R_{0r0r}

```
Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] Rcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$-Y[\theta, \phi] h_0^{(\theta,2)}[t, r]$$

```

Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] Rcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
Y[θ, ϕ] h1(1,1)[t, r]

Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] Rcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
-Y[θ, ϕ] hol(2,0)[t, r]

Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] Rcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] Rcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] Rcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] Rcontra[[b]] Tcontra[[c]] Rcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

```

- One can remove h_0 using equation of motion . We will evaluate $R_{\{0\}r0r}$ again using equation of motion $R_{\{rr\}} = 0$.
-

■ $R_{\{0\}rer}$

```

Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] Rcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] Rcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
 $\frac{1}{r^2} (-h_1[t, r] + r h_1^{(0,1)}[t, r]) \sqrt{Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}$ 

Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] Rcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
 $-\frac{1}{r} hol^{(1,0)}[t, r] \sqrt{Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}$ 

```

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] Rcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] Rcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] Rcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r} \sqrt{\frac{1}{2\ell + 2\ell^2}} \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} (2 h1e^{(1,\theta)}[t, r] + r h1e^{(1,1)}[t, r])$$

```
Sum[Tcontra[[a]] Rcontra[[b]] econtra[[c]] Rcontra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

■ R_{0rmr}

```
Sum[Tcontra[[a]] Rcontra[[b]] mcontra[[c]] Rcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] mcontra[[c]] Rcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2} \sqrt{\frac{1}{\ell + \ell^2}} (h_2[t, r] - r (h_2^{(\theta,1)}[t, r] + r h_2^{(\theta,2)}[t, r])) \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

```
Sum[Tcontra[[a]] Rcontra[[b]] mcontra[[c]] Rcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] mcontra[[c]] Rcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] Rcontra[[b]] mcontra[[c]] Rcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r} \sqrt{\frac{1}{2\ell + 2\ell^2}} \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} (2 h1m^{(1,\theta)}[t, r] + r h1m^{(1,1)}[t, r])$$

$$\text{Assuming}[\ell \in \text{Integers} \&\& \ell > 0, \text{FullSimplify}[-\sqrt{\frac{1}{2\ell + 2\ell^2}} \frac{(2 h_{1m}^{(1,0)}[t, r] + r h_{1m}^{(1,1)}[t, r])}{r} - \sqrt{\frac{1}{\ell + \ell^2}} \frac{(h_2[t, r] - r (h_2^{(0,1)}[t, r] + r h_2^{(0,2)}[t, r]))}{r^2}]] - \left(\left(2 h_2[t, r] + r (-2 h_2^{(0,1)}[t, r] - 2 r h_2^{(0,2)}[t, r] + \sqrt{2} (2 h_{1m}^{(1,0)}[t, r] + r h_{1m}^{(1,1)}[t, r])) \right) \right) / (2 r^2 \sqrt{\ell (1 + \ell)}) \right)$$

```
Sum[Tcontra[[a]] Rcontra[[b]] mcontra[[c]] Rcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0

Sum[Tcontra[[a]] Rcontra[[b]] mcontra[[c]] Rcontra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

■ R_{0e0e}

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] econtra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
- \frac{1}{r^2 (Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2)} \left( r Y[\theta, \phi] h_0^{(0,1)}[t, r] (Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2) + h_0[t, r] (Csc[\theta]^2 Y^{(0,1)}[\theta, \phi] (Y^{(0,1)}[\theta, \phi] (Csc[\theta]^2 Y^{(0,2)}[\theta, \phi] - Cot[\theta] Y^{(1,0)}[\theta, \phi]) + 2 Y^{(1,0)}[\theta, \phi] Y^{(1,1)}[\theta, \phi]) + Y^{(1,0)}[\theta, \phi]^2 Y^{(2,0)}[\theta, \phi]) \right)

Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] econtra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
\frac{1}{r (Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2)} \left( Y[\theta, \phi] (Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2) h_1^{(1,0)}[t, r] - \sqrt{\frac{1}{\ell + \ell^2}} Csc[\theta] h_2^{(1,0)}[t, r] (Cot[\theta] Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^3 - Csc[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] (Csc[\theta]^2 Y^{(0,2)}[\theta, \phi] - Y^{(2,0)}[\theta, \phi])) \right)

Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] econtra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] econtra[[d]] R[hot[t, r] Tot] [[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
-  $\frac{Y[\theta, \phi] \text{hot}^{(2, \theta)}[t, r]}{\sqrt{2}}$ 
```

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] econtra[[d]] R[h1m[t, r] T1m] [[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] econtra[[d]] R[h1e[t, r] T1e] [[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] econtra[[d]] R[h2e[t, r] T2e] [[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
-  $\frac{1}{2 \left( \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2 \right)}$ 
  hte(2, θ)[t, r]  $\left( \sqrt{\ell(1+\ell)} Y[\theta, \phi] \left( \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2 \right) + \right.$ 
   $2 \sqrt{\frac{1}{\ell + \ell^2}} \left( \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi] \left( Y^{(\theta, 1)}[\theta, \phi] \left( \text{Csc}[\theta]^2 Y^{(\theta, 2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1, \theta)}[\theta, \phi] \right) + \right.$ 
   $\left. \left. 2 Y^{(1, \theta)}[\theta, \phi] Y^{(1, 1)}[\theta, \phi] \right) + Y^{(1, \theta)}[\theta, \phi]^2 Y^{(2, \theta)}[\theta, \phi] \right) \left. \right)$ 
```

■ R_{0e0m}

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] mcontra[[d]] R[ss] [[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
-  $\frac{1}{r^2 \left( \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2 \right)}$ 
  Csc[θ] h0[t, r]  $\left( \text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 Y^{(1, 1)}[\theta, \phi] + \right.$ 
   $\left. Y^{(1, \theta)}[\theta, \phi]^2 Y^{(1, 1)}[\theta, \phi] + Y^{(\theta, 1)}[\theta, \phi] Y^{(1, \theta)}[\theta, \phi] \left( \text{Csc}[\theta]^2 Y^{(\theta, 2)}[\theta, \phi] - Y^{(2, \theta)}[\theta, \phi] \right) \right)$ 
```

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] mcontra[[d]] R[sp] [[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
 $\left( \sqrt{\frac{1}{\ell + \ell^2}} h_2^{(1, \theta)}[t, r] \left( \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi] \right. \right.$ 
 $\left. \left( Y^{(\theta, 1)}[\theta, \phi] \left( \text{Csc}[\theta]^2 Y^{(\theta, 2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1, \theta)}[\theta, \phi] \right) + 2 Y^{(1, \theta)}[\theta, \phi] Y^{(1, 1)}[\theta, \phi] \right) + \right.$ 
 $\left. \left. Y^{(1, \theta)}[\theta, \phi]^2 Y^{(2, \theta)}[\theta, \phi] \right) \right) / \left( r \left( \text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2 \right) \right)$ 
```

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] mcontra[[d]] R[hol[t, r] Tol] [[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] Tcontra[[c]] mcontra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$-\frac{1}{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2} \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta]^3 \text{hte}^{(2,0)}[t, r]$$

$$\left(\text{Cot}[\theta] Y^{(0,1)}[\theta, \phi]^3 - Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \text{Sin}[\theta]^2 Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right.$$

$$\left. Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(Y^{(0,2)}[\theta, \phi] - \text{Sin}[\theta]^2 Y^{(2,0)}[\theta, \phi] \right) \right)$$

■ R_{0eem}

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}}$$

$$\sqrt{\frac{1}{\ell + \ell^2}} \left(r h_2^{(0,1)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right) + h_2[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + \right. \right.$$

$$\left. \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(0,3)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,1)}[\theta, \phi] + Y^{(2,1)}[\theta, \phi] \right) + \right.$$

$$\left. Y^{(1,0)}[\theta, \phi] \left(Y^{(1,0)}[\theta, \phi] + \text{Csc}[\theta]^2 \left(-2 \text{Cot}[\theta] Y^{(0,2)}[\theta, \phi] - Y^{(1,0)}[\theta, \phi] + Y^{(1,2)}[\theta, \phi] \right) + \right. \right.$$

$$\left. \left. \text{Cot}[\theta] Y^{(2,0)}[\theta, \phi] + Y^{(3,0)}[\theta, \phi] \right) \right)$$

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$-\frac{1}{r} \sqrt{\frac{1}{2\ell + 2\ell^2}} h_{1m}^{(1,0)}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}$$

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

■ R_{0eer}

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] Rcontra[[d]] R[ss][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] Rcontra[[d]] R[sp][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} \left(r \sqrt{\frac{1}{\ell + \ell^2}} \cot[\theta] \text{Csc}[\theta]^3 Y^{(0,1)}[\theta, \phi]^3 h_2^{(0,1)}[t, r] + \text{Csc}[\theta]^3 Y^{(0,1)}[\theta, \phi]^2 \right. \\ \left. \left(h_1[t, r] \left(\text{Csc}[\theta] Y^{(0,2)}[\theta, \phi] - \cos[\theta] Y^{(1,0)}[\theta, \phi] \right) - r \sqrt{\frac{1}{\ell + \ell^2}} h_2^{(0,1)}[t, r] Y^{(1,1)}[\theta, \phi] \right) + \right. \\ \left. \text{Csc}[\theta] Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(2 \text{Csc}[\theta] h_1[t, r] Y^{(1,1)}[\theta, \phi] + \right. \right. \\ \left. \left. r \sqrt{\frac{1}{\ell + \ell^2}} h_2^{(0,1)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] - Y^{(2,0)}[\theta, \phi] \right) \right) + \right. \\ \left. Y^{(1,0)}[\theta, \phi]^2 \left(r \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] h_2^{(0,1)}[t, r] Y^{(1,1)}[\theta, \phi] + h_1[t, r] Y^{(2,0)}[\theta, \phi] \right) + \right. \\ \left. \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta]^3 h_2[t, r] \left(\cot[\theta] Y^{(0,1)}[\theta, \phi]^3 - Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \sin[\theta]^2 \right. \right. \\ \left. \left. Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(Y^{(0,2)}[\theta, \phi] - \sin[\theta]^2 Y^{(2,0)}[\theta, \phi] \right) \right) \right)$$

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] Rcontra[[d]] R[h0l[t, r] T0l][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$-\frac{Y[\theta, \phi] h_{0l}^{(1,0)}[t, r]}{r}$$

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] Rcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{Y[\theta, \phi] \left(\text{hot}^{(1, \theta)}[t, r] + r \text{hot}^{(1, 1)}[t, r] \right)}{\sqrt{2} \, r}$$

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] Rcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$- \left(\left(\sqrt{\frac{1}{2\ell + 2\ell^2}} \, \text{h1e}^{(1, \theta)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi] \right. \right. \right. \\ \left. \left. \left(Y^{(\theta, 1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta, 2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1, \theta)}[\theta, \phi] \right) + 2 Y^{(1, \theta)}[\theta, \phi] Y^{(1, 1)}[\theta, \phi] \right) + \right. \right. \\ \left. \left. Y^{(1, \theta)}[\theta, \phi]^2 Y^{(2, \theta)}[\theta, \phi] \right) \right) \Bigg/ \left(r \left(\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2 \right) \right)$$

```
Sum[Tcontra[[a]] econtra[[b]] econtra[[c]] Rcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{2 r \left(\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2 \right)} \\ \left(\text{hte}^{(1, \theta)}[t, r] + r \text{hte}^{(1, 1)}[t, r] \right) \left(\sqrt{\ell (1 + \ell)} \, Y[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2 \right) + \right. \\ \left. 2 \sqrt{\frac{1}{\ell + \ell^2}} \left(\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi] \left(Y^{(\theta, 1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta, 2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1, \theta)}[\theta, \phi] \right) + \right. \right. \right. \\ \left. \left. 2 Y^{(1, \theta)}[\theta, \phi] Y^{(1, 1)}[\theta, \phi] \right) + Y^{(1, \theta)}[\theta, \phi]^2 Y^{(2, \theta)}[\theta, \phi] \right) \right)$$

■ R_{0mer}

```
Sum[Tcontra[[a]] mcontra[[b]] econtra[[c]] Rcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

0

Sum[Tcontra[[a]] mcontra[[b]] econtra[[c]] Rcontra[[d]] R[sp][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)}$$

$$\left(\text{Csc}[\theta] h_1[t, r] \left(\text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(1,\theta)}[\theta, \phi]^2 \right. \right.$$

$$Y^{(1,1)}[\theta, \phi] + Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - Y^{(2,\theta)}[\theta, \phi] \right) \left. \right) + \sqrt{\frac{1}{\ell + \ell^2}}$$

$$\left(-h_2[t, r] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] \left(Y^{(\theta,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) + \right. \right. \right.$$

$$2 Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] \left. \right) + Y^{(1,\theta)}[\theta, \phi]^2 Y^{(2,\theta)}[\theta, \phi] \left. \right) +$$

$$r h_2^{(\theta,1)}[t, r] \left(Y^{(1,\theta)}[\theta, \phi]^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - \right.$$

$$2 \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] +$$

$$\left. \left. \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right) \right)$$

Sum[Tcontra[[a]] mcontra[[b]] econtra[[c]] Rcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Tcontra[[a]] mcontra[[b]] econtra[[c]] Rcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Tcontra[[a]] mcontra[[b]] econtra[[c]] Rcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \left(\left(\sqrt{\frac{1}{2\ell + 2\ell^2}} h_{1m}^{(1,\theta)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] \right. \right. \right.$$

$$\left. \left(Y^{(\theta,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) + 2 Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] \right) + \right.$$

$$\left. \left. Y^{(1,\theta)}[\theta, \phi]^2 Y^{(2,\theta)}[\theta, \phi] \right) \right) / \left(r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) \right)$$

Sum[Tcontra[[a]] mcontra[[b]] econtra[[c]] Rcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{2\ell + 2\ell^2}} \text{Csc}[\theta]$$

$$h_{1e}^{(1,\theta)}[t, r] \left(\text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right.$$

$$Y^{(1,\theta)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - Y^{(2,\theta)}[\theta, \phi] \right) \left. \right)$$

Sum[Tcontra[[a]] mcontra[[b]] econtra[[c]] Rcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta]^3 \left(h_{te}^{(1,\theta)}[t, r] + r h_{te}^{(1,1)}[t, r] \right)$$

$$\left(\text{Cot}[\theta] Y^{(\theta,1)}[\theta, \phi]^3 - Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \text{Sin}[\theta]^2 Y^{(1,\theta)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right.$$

$$Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(Y^{(\theta,2)}[\theta, \phi] - \text{Sin}[\theta]^2 Y^{(2,\theta)}[\theta, \phi] \right) \left. \right)$$

■ R_{0mmr}

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] Rcontra[[d]] R[ss][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] Rcontra[[d]] R[sp][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \left(-r \sqrt{\frac{1}{\ell + \ell^2}} \text{Cot}[\theta] \text{Csc}[\theta]^3 Y^{(\theta,1)}[\theta, \phi]^3 h_2^{(\theta,1)}[t, r] + \right. \\
Y^{(1,\theta)}[\theta, \phi]^2 \left(h_1[t, r] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - \right. \\
\left. r \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] h_2^{(\theta,1)}[t, r] Y^{(1,1)}[\theta, \phi] \right) - \\
\left. \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] h_2[t, r] \left(\text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \right. \\
\left. Y^{(1,\theta)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - Y^{(2,\theta)}[\theta, \phi] \right) \right) + \\
\left. \text{Csc}[\theta] Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(-2 \text{Csc}[\theta] h_1[t, r] Y^{(1,1)}[\theta, \phi] + \right. \right. \\
\left. r \sqrt{\frac{1}{\ell + \ell^2}} h_2^{(\theta,1)}[t, r] \left(-\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right) + \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 \\
\left. \left(r \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] h_2^{(\theta,1)}[t, r] Y^{(1,1)}[\theta, \phi] + h_1[t, r] \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right) \right) \Bigg)$$

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] Rcontra[[d]] R[h1][t, r] Tol][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$-\frac{Y[\theta, \phi] \text{hol}^{(1,\theta)}[t, r]}{r}$$

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] Rcontra[[d]] R[h1m][t, r] T1m][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$-\frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{2\ell + 2\ell^2}} \text{Csc}[\theta] \\
h1m^{(1,\theta)}[t, r] \left(\text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\
\left. Y^{(1,\theta)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - Y^{(2,\theta)}[\theta, \phi] \right) \right)$$

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] Rcontra[[d]] R[h1e][t, r] T1e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$-\frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{2\ell + 2\ell^2}} h1e^{(1,\theta)}[t, r] \\
\left(Y^{(1,\theta)}[\theta, \phi]^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - 2 \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] \right. \\
\left. Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right)$$

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] Rcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{2 r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)}$$

$$\left(\text{hte}^{(1,\theta)}[t, r] + r \text{hte}^{(1,1)}[t, r] \right) \left(\sqrt{\ell(1+\ell)} Y[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) + \right.$$

$$\frac{1}{2} \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta]^2 \left(4 Y^{(1,\theta)}[\theta, \phi]^2 \left(Y^{(\theta,2)}[\theta, \phi] + \text{Cos}[\theta] \text{Sin}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - 8 Y^{(\theta,1)}[\theta, \phi] \right.$$

$$\left. \left. Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + 4 Y^{(\theta,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right) \right)$$

■ R_{0mme}

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] econtra[[d]] R[ss][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] econtra[[d]] R[sp][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}}$$

$$\left(-h_1[t, r] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) - \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] h_2[t, r] \right.$$

$$\left(Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,3)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,1)}[\theta, \phi] + Y^{(2,1)}[\theta, \phi] \right) + \right.$$

$$\left. Y^{(\theta,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 \left(2 \text{Cot}[\theta] Y^{(\theta,2)}[\theta, \phi] + Y^{(1,\theta)}[\theta, \phi] - Y^{(1,2)}[\theta, \phi] \right) - \right.$$

$$\left. \left. \text{Cot}[\theta] Y^{(2,\theta)}[\theta, \phi] - Y^{(3,\theta)}[\theta, \phi] \right) \right)$$

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] econtra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] econtra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \frac{1}{\sqrt{2} r} \text{hot}^{(1,\theta)}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] econtra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] econtra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r} \sqrt{\frac{1}{2\ell + 2\ell^2}} h1e^{(1,\theta)}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

Sum[Tcontra[[a]] mcontra[[b]] mcontra[[c]] econtra[[d]] R[h2e[t, r] T2e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{2r} \left(-\sqrt{\ell(1+\ell)} + 2\sqrt{\frac{1}{\ell + \ell^2}} \right) h2e^{(1,\theta)}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

■ R_{0m0m}

Sum[Tcontra[[a]] mcontra[[b]] Tcontra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$-\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \left(r Y[\theta, \phi] h_0^{(\theta,1)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) + \right. \\ \left. h_0[t, r] \left(Y^{(1,\theta)}[\theta, \phi]^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - 2 \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right) \right)$$

Sum[Tcontra[[a]] mcontra[[b]] Tcontra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$-\frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \left(Y[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) h_1^{(1,\theta)}[t, r] + \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] \right. \\ \left. h_2^{(1,\theta)}[t, r] \left(\text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(1,\theta)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - Y^{(2,\theta)}[\theta, \phi] \right) \right) \right)$$

Sum[Tcontra[[a]] mcontra[[b]] Tcontra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Tcontra[[a]] mcontra[[b]] Tcontra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$-\frac{Y[\theta, \phi] hot^{(2,\theta)}[t, r]}{\sqrt{2}}$$

Sum[Tcontra[[a]] mcontra[[b]] Tcontra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

```
Sum[Tcontra[[a]] mcontra[[b]] Tcontra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Tcontra[[a]] mcontra[[b]] Tcontra[[c]] mcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
-  $\frac{1}{2 \left( \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)}$ 
hte(2,0)[t, r]  $\left( \sqrt{\ell(1+\ell)} Y[\theta, \phi] \left( \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right) + \frac{1}{2} \sqrt{\frac{1}{\ell + \ell^2}} \right.$ 
 $\text{Csc}[\theta]^2 \left( 4 Y^{(1,0)}[\theta, \phi]^2 \left( Y^{(0,2)}[\theta, \phi] + \text{Cos}[\theta] \text{Sin}[\theta] Y^{(1,0)}[\theta, \phi] \right) - 8 Y^{(0,1)}[\theta, \phi] \right.$ 
 $\left. \left. Y^{(1,0)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + 4 Y^{(0,1)}[\theta, \phi]^2 \left( 2 \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] + Y^{(2,0)}[\theta, \phi] \right) \right) \right)$ 
```

■ R_{reem}

```
Sum[Rcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Rcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Rcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Rcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Rcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
 $\frac{1}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}} \sqrt{\frac{1}{2\ell + 2\ell^2}} \text{h1m}[t, r] \left( 2 \text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + \right.$ 
 $\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \left( \text{Csc}[\theta]^2 Y^{(0,3)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,1)}[\theta, \phi] + Y^{(2,1)}[\theta, \phi] \right) +$ 
 $Y^{(1,0)}[\theta, \phi] \left( \text{Csc}[\theta]^2 \left( -2 \text{Cot}[\theta] Y^{(0,2)}[\theta, \phi] - \text{Cos}[2\theta] Y^{(1,0)}[\theta, \phi] + Y^{(1,2)}[\theta, \phi] \right) + \right.$ 
 $\left. \left. \text{Cot}[\theta] Y^{(2,0)}[\theta, \phi] + Y^{(3,0)}[\theta, \phi] \right) \right)$ 
```

```
Sum[Rcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Rcontra[[a]] econtra[[b]] econtra[[c]] mcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

■ R_{rmrm}

```
Sum[Rcontra[[a]] mcontra[[b]] Rcontra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Rcontra[[a]] mcontra[[b]] Rcontra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
0
```

```
Sum[Rcontra[[a]] mcontra[[b]] Rcontra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \left(r Y[\theta, \phi] \text{hol}^{(\theta,1)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) - \right. \\ \left. \text{hol}[t, r] \left(Y^{(1,\theta)}[\theta, \phi]^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - 2 \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right) \right)$$

```
Sum[Rcontra[[a]] mcontra[[b]] Rcontra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
- \frac{Y[\theta, \phi] \left( 2 \text{hot}^{(\theta,1)}[t, r] + r \text{hot}^{(\theta,2)}[t, r] \right)}{\sqrt{2} r}
```

```
Sum[Rcontra[[a]] mcontra[[b]] Rcontra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} 2 \sqrt{\frac{1}{2 \ell + 2 \ell^2}} \text{Csc}[\theta]^3 \left(h1m[t, r] + r h1m^{(\theta,1)}[t, r] \right) \\ \left(\text{Cot}[\theta] Y^{(\theta,1)}[\theta, \phi]^3 - Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \text{Sin}[\theta]^2 Y^{(1,\theta)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\ \left. Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(Y^{(\theta,2)}[\theta, \phi] - \text{Sin}[\theta]^2 Y^{(2,\theta)}[\theta, \phi] \right) \right)$$

```
Sum[Rcontra[[a]] mcontra[[b]] Rcontra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
  {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify
```

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] \left(h1e[t, r] + r h1e^{(\theta,1)}[t, r] \right) \\ \left(Y^{(1,\theta)}[\theta, \phi]^2 \left(\text{Csc}[\theta] Y^{(\theta,2)}[\theta, \phi] + \text{Cos}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - 2 \text{Csc}[\theta] Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + \right. \\ \left. \text{Csc}[\theta] Y^{(\theta,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right)$$

$$\begin{aligned}
& \text{Sum}[\text{Rcontra}[[a]] \text{mcontra}[[b]] \text{Rcontra}[[c]] \text{mcontra}[[d]] \text{R}[\text{hte}[\mathbf{t}, \mathbf{r}] \text{T2e}[[a, b, c, d]], \\
& \quad \{a, 1, \text{dim}\}, \{b, 1, \text{dim}\}, \{c, 1, \text{dim}\}, \{d, 1, \text{dim}\}] // \text{FullSimplify} \\
& - \frac{1}{2 r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \\
& \quad \left(2 \text{hte}^{(\theta,1)}[\mathbf{t}, \mathbf{r}] + r \text{hte}^{(\theta,2)}[\mathbf{t}, \mathbf{r}] \right) \left(\sqrt{\ell(1+\ell)} Y[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) + \right. \\
& \quad \left. \frac{1}{2} \sqrt{\frac{1}{\ell+\ell^2}} \text{Csc}[\theta]^2 \left(4 Y^{(1,\theta)}[\theta, \phi]^2 \left(Y^{(\theta,2)}[\theta, \phi] + \text{Cos}[\theta] \text{Sin}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) - 8 Y^{(\theta,1)}[\theta, \right. \right. \\
& \quad \left. \left. \phi \right) Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + 4 Y^{(\theta,1)}[\theta, \phi]^2 \left(2 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right) \left. \right)
\end{aligned}$$

■ R_{\{term\}}

$$\begin{aligned}
& \text{Sum}[\text{Rcontra}[[a]] \text{econtra}[[b]] \text{Rcontra}[[c]] \text{mcontra}[[d]] \text{R}[\text{ss}[[a, b, c, d]], \\
& \quad \{a, 1, \text{dim}\}, \{b, 1, \text{dim}\}, \{c, 1, \text{dim}\}, \{d, 1, \text{dim}\}] // \text{FullSimplify} \\
& 0
\end{aligned}$$

$$\begin{aligned}
& \text{Sum}[\text{Rcontra}[[a]] \text{econtra}[[b]] \text{Rcontra}[[c]] \text{mcontra}[[d]] \text{R}[\text{sp}[[a, b, c, d]], \\
& \quad \{a, 1, \text{dim}\}, \{b, 1, \text{dim}\}, \{c, 1, \text{dim}\}, \{d, 1, \text{dim}\}] // \text{FullSimplify} \\
& 0
\end{aligned}$$

$$\begin{aligned}
& \text{Sum}[\text{Rcontra}[[a]] \text{econtra}[[b]] \text{Rcontra}[[c]] \text{mcontra}[[d]] \text{R}[\text{hol}[\mathbf{t}, \mathbf{r}] \text{Tot}[[a, b, c, d]], \\
& \quad \{a, 1, \text{dim}\}, \{b, 1, \text{dim}\}, \{c, 1, \text{dim}\}, \{d, 1, \text{dim}\}] // \text{FullSimplify} \\
& - \frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \\
& \quad \text{Csc}[\theta] \text{hol}[\mathbf{t}, \mathbf{r}] \left(\text{Cot}[\theta] \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^3 - \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\
& \quad \left. Y^{(1,\theta)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - Y^{(2,\theta)}[\theta, \phi] \right) \right) \\
& 0
\end{aligned}$$

$$\begin{aligned}
& \text{Sum}[\text{Rcontra}[[a]] \text{econtra}[[b]] \text{Rcontra}[[c]] \text{mcontra}[[d]] \text{R}[\text{h1m}[\mathbf{t}, \mathbf{r}] \text{T1m}[[a, b, c, d]], \\
& \quad \{a, 1, \text{dim}\}, \{b, 1, \text{dim}\}, \{c, 1, \text{dim}\}, \{d, 1, \text{dim}\}] // \text{FullSimplify} \\
& \frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \\
& \quad \sqrt{\frac{1}{2\ell+2\ell^2}} \left(\text{h1m}[\mathbf{t}, \mathbf{r}] + r \text{h1m}^{(\theta,1)}[\mathbf{t}, \mathbf{r}] \right) \left(4 \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] + \right. \\
& \quad \text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - 3 \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] - Y^{(2,\theta)}[\theta, \phi] \right) + \\
& \quad \left. Y^{(1,\theta)}[\theta, \phi]^2 \left(-\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi] \right) \right)
\end{aligned}$$

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] mcontra[[d]] R[hle[t, r] T1e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} 2 \sqrt{\frac{1}{2\ell + 2\ell^2}} \text{Csc}[\theta]^3 \left(\text{hle}[t, r] + r \text{hle}^{(0,1)}[t, r] \right) \\ \left(\text{Cot}[\theta] Y^{(0,1)}[\theta, \phi]^3 - Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \text{Sin}[\theta]^2 Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\ \left. Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(Y^{(0,2)}[\theta, \phi] - \text{Sin}[\theta]^2 Y^{(2,0)}[\theta, \phi] \right) \right)$$

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] mcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \frac{1}{r \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta]^3 \left(2 \text{hte}^{(0,1)}[t, r] + r \text{hte}^{(0,2)}[t, r] \right) \\ \left(\text{Cot}[\theta] Y^{(0,1)}[\theta, \phi]^3 - Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \text{Sin}[\theta]^2 Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\ \left. Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(Y^{(0,2)}[\theta, \phi] - \text{Sin}[\theta]^2 Y^{(2,0)}[\theta, \phi] \right) \right)$$

■ R_{rere}

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] econtra[[d]] R[ss][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] econtra[[d]] R[sp][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] econtra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} \\ \left(r Y[\theta, \phi] \text{hol}^{(0,1)}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right) + \right. \\ \left. \text{hol}[t, r] \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi] \left(Y^{(0,1)}[\theta, \phi] \left(-\text{Csc}[\theta]^2 Y^{(0,2)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,0)}[\theta, \phi] \right) - \right. \right. \right. \\ \left. \left. 2 Y^{(1,0)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] \right) - Y^{(1,0)}[\theta, \phi]^2 Y^{(2,0)}[\theta, \phi] \right) \right)$$

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] econtra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \frac{Y[\theta, \phi] \left(2 \text{hot}^{(0,1)}[t, r] + r \text{hot}^{(0,2)}[t, r] \right)}{\sqrt{2} r}$$

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] econtra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2 \right)} 2 \sqrt{\frac{1}{2\ell + 2\ell^2}} \text{Csc}[\theta]^3 \left(\text{h1m}[t, r] + r \text{h1m}^{(0,1)}[t, r] \right) \\ \left(\text{Cot}[\theta] Y^{(0,1)}[\theta, \phi]^3 - Y^{(0,1)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \text{Sin}[\theta]^2 Y^{(1,0)}[\theta, \phi]^2 Y^{(1,1)}[\theta, \phi] + \right. \\ \left. Y^{(0,1)}[\theta, \phi] Y^{(1,0)}[\theta, \phi] \left(Y^{(0,2)}[\theta, \phi] - \text{Sin}[\theta]^2 Y^{(2,0)}[\theta, \phi] \right) \right)$$

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] econtra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{1}{r^2 \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \left(h1e[t, r] + r h1e^{(\theta,1)}[t, r] \right) \\ \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] \left(Y^{(\theta,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) + \right. \right. \\ \left. \left. 2 Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] \right) + Y^{(1,\theta)}[\theta, \phi]^2 Y^{(2,\theta)}[\theta, \phi] \right)$$

Sum[Rcontra[[a]] econtra[[b]] Rcontra[[c]] econtra[[d]] R[hTe[t, r] T2e][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \frac{1}{2 r \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right)} \\ \left(2 hTe^{(\theta,1)}[t, r] + r hTe^{(\theta,2)}[t, r] \right) \left(\sqrt{\ell (1 + \ell)} Y[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2 \right) + \right. \\ \left. 2 \sqrt{\frac{1}{\ell + \ell^2}} \left(\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi] \left(Y^{(\theta,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,2)}[\theta, \phi] - \text{Cot}[\theta] Y^{(1,\theta)}[\theta, \phi] \right) + \right. \right. \right. \\ \left. \left. 2 Y^{(1,\theta)}[\theta, \phi] Y^{(1,1)}[\theta, \phi] \right) + Y^{(1,\theta)}[\theta, \phi]^2 Y^{(2,\theta)}[\theta, \phi] \right) \right)$$

■ R_{rmem}

Sum[Rcontra[[a]] mcontra[[b]] econtra[[c]] mcontra[[d]] R[ss][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Rcontra[[a]] mcontra[[b]] econtra[[c]] mcontra[[d]] R[sp][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

0

Sum[Rcontra[[a]] mcontra[[b]] econtra[[c]] mcontra[[d]] R[hol[t, r] Tol][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$\frac{\text{hol}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}}{r^2}$$

Sum[Rcontra[[a]] mcontra[[b]] econtra[[c]] mcontra[[d]] R[hot[t, r] Tot][[a, b, c, d]],
 {a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify

$$- \frac{1}{\sqrt{2} r} \text{hot}^{(\theta,1)}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

**Sum[Rcontra[[a]] mcontra[[b]] econtra[[c]] mcontra[[d]] R[h1m[t, r] T1m][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify**

$$\frac{1}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}} \sqrt{\frac{1}{2\ell + 2\ell^2}} \text{Csc}[\theta] \text{h1m}[t, r] \\ \left(Y^{(1,\theta)}[\theta, \phi] \left(\text{Csc}[\theta]^2 Y^{(\theta,3)}[\theta, \phi] + \text{Cot}[\theta] Y^{(1,1)}[\theta, \phi] + Y^{(2,1)}[\theta, \phi] \right) + \right. \\ \left. Y^{(\theta,1)}[\theta, \phi] \left(\text{Csc}[\theta]^2 \left(2 \text{Cot}[\theta] Y^{(\theta,2)}[\theta, \phi] + Y^{(1,\theta)}[\theta, \phi] - Y^{(1,2)}[\theta, \phi] \right) - \right. \right. \\ \left. \left. \text{Cot}[\theta] Y^{(2,\theta)}[\theta, \phi] - Y^{(3,\theta)}[\theta, \phi] \right) \right)$$

**Sum[Rcontra[[a]] mcontra[[b]] econtra[[c]] mcontra[[d]] R[h1e[t, r] T1e][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify**

$$-\frac{1}{r^2} \sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \text{h1e}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

**Sum[Rcontra[[a]] mcontra[[b]] econtra[[c]] mcontra[[d]] R[hte[t, r] T2e][[a, b, c, d]],
{a, 1, dim}, {b, 1, dim}, {c, 1, dim}, {d, 1, dim}] // FullSimplify**

$$\frac{1}{2r} \left(-\sqrt{\ell(1+\ell)} + 2 \sqrt{\frac{1}{\ell + \ell^2}} \right) \text{hte}^{(\theta,1)}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

■ R_{rkrk}

R[ss][[3, 2, 3, 2]] // FullSimplify

0

R[sp][[3, 2, 3, 2]] // FullSimplify

0

R[hol[t, r] Tol][[3, 2, 3, 2]] // FullSimplify

$$r Y[\theta, \phi] \text{hol}^{(\theta,1)}[t, r] - \text{hol}[t, r] Y^{(2,\theta)}[\theta, \phi]$$

R[hot[t, r] Tot][[3, 2, 3, 2]] // FullSimplify

$$-\frac{r Y[\theta, \phi] \left(2 \text{hot}^{(\theta,1)}[t, r] + r \text{hot}^{(\theta,2)}[t, r] \right)}{\sqrt{2}}$$

R[h1m[t, r] T1m][[3, 2, 3, 2]] // FullSimplify

$$\sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \text{Csc}[\theta] \left(\text{h1m}[t, r] + r \text{h1m}^{(\theta,1)}[t, r] \right) \left(\text{Cot}[\theta] Y^{(\theta,1)}[\theta, \phi] - Y^{(1,1)}[\theta, \phi] \right)$$

R[h1e[t, r] T1e][[3, 2, 3, 2]] // FullSimplify

$$\sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \left(\text{h1e}[t, r] + r \text{h1e}^{(\theta,1)}[t, r] \right) Y^{(2,\theta)}[\theta, \phi]$$

R[hte[t, r] T2e][[3, 2, 3, 2]] // FullSimplify

$$-\frac{1}{2} r \left(2 \text{hte}^{(\theta,1)}[t, r] + r \text{hte}^{(\theta,2)}[t, r] \right) \left(\sqrt{\ell(1+\ell)} Y[\theta, \phi] + 2 \sqrt{\frac{1}{\ell+\ell^2}} Y^{(2,\theta)}[\theta, \phi] \right)$$

■ R_{0r0r} Analysis

R[ss][[4, 2, 4, 2]] // FullSimplify

0

R[sp][[4, 2, 4, 2]] // FullSimplify

0

R[hol[t, r] Tol][[4, 2, 4, 2]] // FullSimplify

$$r \sin[\theta]^2 Y[\theta, \phi] \text{hol}^{(\theta,1)}[t, r] - \text{hol}[t, r] \left(Y^{(\theta,2)}[\theta, \phi] + \cos[\theta] \sin[\theta] Y^{(1,\theta)}[\theta, \phi] \right)$$

$$\frac{(r Y[\theta, \phi] \text{hol}^{(\theta,1)}[t, r] - \text{hol}[t, r] Y^{(2,\theta)}[\theta, \phi])}{r^2} + \frac{1}{r^2 \sin[\theta]^2} (r \sin[\theta]^2 Y[\theta, \phi] \text{hol}^{(\theta,1)}[t, r] - \text{hol}[t, r] (Y^{(\theta,2)}[\theta, \phi] + \cos[\theta] \sin[\theta] Y^{(1,\theta)}[\theta, \phi])) // FullSimplify$$

$$\frac{1}{r^2}$$

$$(2 r Y[\theta, \phi] \text{hol}^{(\theta,1)}[t, r] - \text{hol}[t, r] (\csc[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \cot[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi]))$$

R[hot[t, r] Tot][[4, 2, 4, 2]] // FullSimplify

$$-\frac{1}{\sqrt{2}} r \sin[\theta]^2 Y[\theta, \phi] (2 \text{hot}^{(\theta,1)}[t, r] + r \text{hot}^{(\theta,2)}[t, r])$$

$$-\frac{r Y[\theta, \phi] (2 \text{hot}^{(\theta,1)}[t, r] + r \text{hot}^{(\theta,2)}[t, r])}{\sqrt{2} r^2}$$

$$(r \sin[\theta]^2 Y[\theta, \phi] (2 \text{hot}^{(\theta,1)}[t, r] + r \text{hot}^{(\theta,2)}[t, r])) / (\sqrt{2} r^2 \sin[\theta]^2) // FullSimplify$$

$$-\frac{\sqrt{2} Y[\theta, \phi] (2 \text{hot}^{(\theta,1)}[t, r] + r \text{hot}^{(\theta,2)}[t, r])}{r}$$

R[h1m[t, r] T1m][[4, 2, 4, 2]] // FullSimplify

$$-\sqrt{2} \sqrt{\frac{1}{\ell+\ell^2}} (h1m[t, r] + r h1m^{(\theta,1)}[t, r]) (\cos[\theta] Y^{(\theta,1)}[\theta, \phi] - \sin[\theta] Y^{(1,1)}[\theta, \phi])$$

$$\sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \csc[\theta] (h1m[t, r] + r h1m^{(\theta,1)}[t, r]) \frac{(\cot[\theta] Y^{(\theta,1)}[\theta, \phi] - Y^{(1,1)}[\theta, \phi])}{r^2} -$$

$$\sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} (h1m[t, r] + r h1m^{(\theta,1)}[t, r])$$

$$\frac{(\cos[\theta] Y^{(\theta,1)}[\theta, \phi] - \sin[\theta] Y^{(1,1)}[\theta, \phi])}{r^2 \sin[\theta]^2} // \text{FullSimplify}$$

θ

R[h1e[t, r] T1e][[4, 2, 4, 2]] // FullSimplify

$$\sqrt{\frac{1}{2\ell + 2\ell^2}} (h1e[t, r] + r h1e^{(\theta,1)}[t, r]) (2 Y^{(\theta,2)}[\theta, \phi] + \sin[2\theta] Y^{(1,\theta)}[\theta, \phi])$$

$$\sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} (h1e[t, r] + r h1e^{(\theta,1)}[t, r]) \frac{Y^{(2,\theta)}[\theta, \phi]}{r^2} + \frac{1}{r^2 \sin[\theta]^2} \sqrt{\frac{1}{2\ell + 2\ell^2}}$$

$$(h1e[t, r] + r h1e^{(\theta,1)}[t, r]) (2 Y^{(\theta,2)}[\theta, \phi] + \sin[2\theta] Y^{(1,\theta)}[\theta, \phi]) // \text{FullSimplify}$$

$$\frac{1}{r^2} \sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} (h1e[t, r] + r h1e^{(\theta,1)}[t, r])$$

$$(\csc[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \cot[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi])$$

R[h2e[t, r] T2e][[4, 2, 4, 2]] // FullSimplify

$$-\frac{1}{2} r (2 h2e^{(\theta,1)}[t, r] + r h2e^{(\theta,2)}[t, r])$$

$$\left(\sqrt{\ell(1+\ell)} \sin[\theta]^2 Y[\theta, \phi] + \sqrt{\frac{1}{\ell + \ell^2}} (2 Y^{(\theta,2)}[\theta, \phi] + \sin[2\theta] Y^{(1,\theta)}[\theta, \phi]) \right)$$

$$-\frac{1}{2} r (2 h2e^{(\theta,1)}[t, r] + r h2e^{(\theta,2)}[t, r]) \frac{\left(\sqrt{\ell(1+\ell)} Y[\theta, \phi] + 2 \sqrt{\frac{1}{\ell + \ell^2}} Y^{(2,\theta)}[\theta, \phi] \right)}{r^2} -$$

$$\frac{1}{2} r (2 h2e^{(\theta,1)}[t, r] + r h2e^{(\theta,2)}[t, r]) \frac{1}{r^2 \sin[\theta]^2} \left(\sqrt{\ell(1+\ell)} \sin[\theta]^2 Y[\theta, \phi] + \right.$$

$$\left. \sqrt{\frac{1}{\ell + \ell^2}} (2 Y^{(\theta,2)}[\theta, \phi] + \sin[2\theta] Y^{(1,\theta)}[\theta, \phi]) \right) // \text{FullSimplify}$$

$$-\frac{1}{r} (2 h2e^{(\theta,1)}[t, r] + r h2e^{(\theta,2)}[t, r])$$

$$\left(\sqrt{\ell(1+\ell)} Y[\theta, \phi] + \sqrt{\frac{1}{\ell + \ell^2}} (\csc[\theta]^2 Y^{(\theta,2)}[\theta, \phi] + \cot[\theta] Y^{(1,\theta)}[\theta, \phi] + Y^{(2,\theta)}[\theta, \phi]) \right)$$

// θ

■ R_{0r0r}

$$\frac{1}{r^2} \left(2 r Y[\theta, \phi] \text{hol}^{(\theta,1)}[t, r] - \text{hol}[t, r] \left(-\ell (\ell + 1) Y[\theta, \phi] \right) \right)$$

$$\frac{1}{r^2} \left(2 r Y_{\ell, m}[\theta, \phi] \text{h}^{\theta, \ell}{}^{(\theta,1)}[t, r] - \text{h}^{\theta, \ell}[t, r] \left(-\ell (\ell + 1) Y_{\ell, m}[\theta, \phi] \right) \right)$$

$$- \frac{1}{r} \sqrt{2} Y_{\ell, m}[\theta, \phi] \left(2 \alpha \text{hte}^{(\theta,1)}[t, r] + r \alpha \text{hte}^{(\theta,2)}[t, r] \right)$$

$$- \frac{\sqrt{2} Y_{\ell, m}[\theta, \phi] \left(2 \text{hot}^{(\theta,1)}[t, r] + r \text{hot}^{(\theta,2)}[t, r] \right)}{r}$$

$$\frac{1}{r^2} \sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \left(\text{h1e}[t, r] + r \text{h1e}^{(\theta,1)}[t, r] \right) \left(-\ell (\ell + 1) Y[\theta, \phi] \right)$$

$$\beta \frac{1}{r^2} \sqrt{\frac{2}{\ell + \ell^2}} \left(\text{h1e}[t, r] + r \text{h1e}^{(\theta,1)}[t, r] \right) \left(-\ell (\ell + 1) Y[\theta, \phi] \right)$$

$$\frac{1}{r^2} \left(2 r Y[\theta, \phi] D \left[\left(a \text{hte}[t, r] + b r \text{hte}^{(\theta,1)}[t, r] \right), r \right] - \right. \\ \left. \left(a \text{hte}[t, r] + b r \text{hte}^{(\theta,1)}[t, r] \right) \left(-\ell (\ell + 1) Y[\theta, \phi] \right) \right) // \text{FullSimplify}$$

$$\frac{1}{r^2} Y[\theta, \phi] \left(a \ell (1 + \ell) \text{hte}[t, r] + r \left((2 a + b (2 + \ell + \ell^2)) \text{hte}^{(\theta,1)}[t, r] + 2 b r \text{hte}^{(\theta,2)}[t, r] \right) \right)$$

$$\%1 /. a \rightarrow \left(\frac{2}{\ell (\ell + 1)} \right)^{\frac{1}{2}} \beta - \left((\ell - 1) \frac{(\ell + 2)}{2 \ell (\ell + 1)} \right)^{\frac{1}{2}} \gamma /.$$

$$b \rightarrow \left(\frac{2}{\ell (\ell + 1)} \right)^{\frac{1}{2}} \beta + \left(\frac{2}{\ell (\ell + 1) (\ell + 2) (\ell - 1)} \right)^{\frac{1}{2}} \gamma // \text{FullSimplify}$$

$$\frac{1}{\sqrt{2} r^2 \sqrt{\ell (1 + \ell)}} Y[\theta, \phi] \left(\ell (1 + \ell) \left(2 \sqrt{\ell (1 + \ell)} \sqrt{\frac{1}{\ell + \ell^2}} \beta - \sqrt{-2 + \ell + \ell^2} \gamma \right) \text{hte}[t, r] + \right.$$

$$2 r \left(\left(-\sqrt{-2 + \ell + \ell^2} \gamma + \sqrt{\ell (1 + \ell)} \left(\sqrt{\frac{1}{\ell + \ell^2}} (4 + \ell + \ell^2) \beta + (2 + \ell + \ell^2) \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) \right) \right)$$

$$\left. \text{hte}^{(\theta,1)}[t, r] + 2 r \sqrt{\ell (1 + \ell)} \left(\sqrt{\frac{1}{\ell + \ell^2}} \beta + \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) \text{hte}^{(\theta,2)}[t, r] \right)$$

Assuming [$\ell \in \text{Integers} \ \&\& \ell > 0 \ \&\& \beta \in \text{Reals} \ \&\& \beta > 0 \ \&\& \gamma \in \text{Reals} \ \&\& \gamma > 0$,

$$\begin{aligned}
 & \text{FullSimplify}\left[\frac{1}{\sqrt{2} \, r^2 \sqrt{\ell (1+\ell)}} Y[\theta, \phi] \right. \\
 & \left(\ell (1+\ell) \left(2 \sqrt{\ell (1+\ell)} \sqrt{\frac{1}{\ell + \ell^2}} \beta - \sqrt{-2 + \ell + \ell^2} \gamma \right) \text{hte}[\mathbf{t}, \mathbf{r}] + 2 r \left(\left(-\sqrt{-2 + \ell + \ell^2} \gamma + \right. \right. \right. \\
 & \left. \left. \left. \sqrt{\ell (1+\ell)} \left(\sqrt{\frac{1}{\ell + \ell^2}} (4 + \ell + \ell^2) \beta + (2 + \ell + \ell^2) \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) \right) \text{hte}^{(\theta,1)}[\mathbf{t}, \right. \right. \\
 & \left. \left. \left. r\right] + 2 r \sqrt{\ell (1+\ell)} \left(\sqrt{\frac{1}{\ell + \ell^2}} \beta + \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) \text{hte}^{(\theta,2)}[\mathbf{t}, \mathbf{r}] \right) \right] \\
 & \left(Y[\theta, \phi] \right. \\
 & \left(\ell (1+\ell) \left(2 \beta - \sqrt{-2 + \ell + \ell^2} \gamma \right) \text{hte}[\mathbf{t}, \mathbf{r}] + \frac{1}{\sqrt{-2 + \ell + \ell^2}} 2 r \left(\left(\sqrt{-2 + \ell + \ell^2} (4 + \ell + \ell^2) \beta + 4 \gamma \right) \right. \right. \\
 & \left. \left. \left. \text{hte}^{(\theta,1)}[\mathbf{t}, \mathbf{r}] + 2 r \left(\sqrt{-2 + \ell + \ell^2} \beta + \gamma \right) \text{hte}^{(\theta,2)}[\mathbf{t}, \mathbf{r}] \right) \right) \right) / \left(\sqrt{2} \, r^2 \sqrt{\ell (1+\ell)} \right) \\
 & \mathbf{U} = \frac{1}{\sqrt{2} \, r^2 \sqrt{\ell (1+\ell)}} Y[\theta, \phi] \\
 & \left(\ell (1+\ell) \left(2 \sqrt{\ell (1+\ell)} \sqrt{\frac{1}{\ell + \ell^2}} \beta - \sqrt{-2 + \ell + \ell^2} \gamma \right) \text{hte}[\mathbf{t}, \mathbf{r}] + 2 r \left(\left(-\sqrt{-2 + \ell + \ell^2} \gamma + \right. \right. \right. \\
 & \left. \left. \left. \sqrt{\ell (1+\ell)} \left(\sqrt{\frac{1}{\ell + \ell^2}} (4 + \ell + \ell^2) \beta + (2 + \ell + \ell^2) \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) \right) \right) \right. \\
 & \left. \left. \left. \text{hte}^{(\theta,1)}[\mathbf{t}, \mathbf{r}] + 2 r \sqrt{\ell (1+\ell)} \left(\sqrt{\frac{1}{\ell + \ell^2}} \beta + \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) \text{hte}^{(\theta,2)}[\mathbf{t}, \mathbf{r}] \right) \right) \right]; \\
 & \text{Coefficient}[\mathbf{V}, \text{hte}^{(\theta,2)}[\mathbf{t}, \mathbf{r}]] // \text{FullSimplify} \\
 & -2 \sqrt{2} \left(\sqrt{\frac{1}{\ell + \ell^2}} \beta + \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) Y[\theta, \phi] \\
 & \text{Coefficient}[\mathbf{U}, \text{hte}^{(\theta,2)}[\mathbf{t}, \mathbf{r}]] // \text{FullSimplify} \\
 & 2 \sqrt{2} \left(\sqrt{\frac{1}{\ell + \ell^2}} \beta + \sqrt{\frac{1}{\ell (-2 - \ell + 2 \ell^2 + \ell^3)}} \gamma \right) Y[\theta, \phi] \\
 & - \frac{\sqrt{2} \, Y[\theta, \phi] (2 \text{hot}^{(\theta,1)}[\mathbf{t}, \mathbf{r}] + r \text{hot}^{(\theta,2)}[\mathbf{t}, \mathbf{r}])}{r}
 \end{aligned}$$

$$-\frac{1}{r}\sqrt{2}\gamma[\theta, \phi] \left(2\alpha \text{hte}^{(\theta,1)}[t, r] + r\alpha D[\text{hte}[t, r], \{r, 2\}]\right) // \text{FullSimplify}$$

$$-\frac{\sqrt{2}\alpha\gamma[\theta, \phi] \left(2\text{hte}^{(\theta,1)}[t, r] + r\text{hte}^{(\theta,2)}[t, r]\right)}{r}$$

$$\%4 /. \alpha \rightarrow 2 \left(\frac{1}{\ell(\ell+1)}\right)^{\frac{1}{2}} \beta + 2 \left(\frac{1}{\ell(\ell+1)(\ell+2)(\ell-1)}\right)^{\frac{1}{2}} \gamma // \text{FullSimplify}$$

$$-\frac{1}{r}2\sqrt{2} \left(\sqrt{\frac{1}{\ell+\ell^2}}\beta + \sqrt{\frac{1}{\ell(-2-\ell+2\ell^2+\ell^3)}}\gamma\right) \gamma[\theta, \phi] \left(2\text{hte}^{(\theta,1)}[t, r] + r\text{hte}^{(\theta,2)}[t, r]\right)$$

$$V = -\frac{1}{r}2\sqrt{2} \left(\sqrt{\frac{1}{\ell+\ell^2}}\beta + \sqrt{\frac{1}{\ell(-2-\ell+2\ell^2+\ell^3)}}\gamma\right) \gamma[\theta, \phi] \left(2\text{hte}^{(\theta,1)}[t, r] + r\text{hte}^{(\theta,2)}[t, r]\right);$$

$$\frac{1}{r^2}\sqrt{2}\sqrt{\frac{1}{\ell+\ell^2}} \left(\beta \text{hte}[t, r] + \beta r \text{hte}^{(\theta,1)}[t, r]\right) (-\ell(\ell+1)\gamma[\theta, \phi])$$

$$-\frac{1}{r^2}\sqrt{2}\ell(1+\ell)\sqrt{\frac{1}{\ell+\ell^2}}\gamma[\theta, \phi] \left(\beta \text{hte}[t, r] + r\beta \text{hte}^{(\theta,1)}[t, r]\right)$$

$$W = -\frac{1}{r^2}\sqrt{2}\ell(1+\ell)\sqrt{\frac{1}{\ell+\ell^2}}\gamma[\theta, \phi] \left(\beta \text{hte}[t, r] + r\beta \text{hte}^{(\theta,1)}[t, r]\right);$$

$$\text{Coefficient}[(U+V+W), \text{hte}^{(\theta,1)}[t, r]] // \text{FullSimplify}$$

$$-\frac{1}{r\ell(1+\ell)}$$

$$\sqrt{2} \left(\sqrt{\ell(1+\ell)}\sqrt{-2+\ell+\ell^2} - (-1+\ell)\ell(1+\ell)(2+\ell)\sqrt{\frac{1}{\ell(-2-\ell+2\ell^2+\ell^3)}}\right) \gamma[\theta, \phi]$$

$$\text{Coefficient}[(U+V+W), \text{hte}^{(\theta,2)}[t, r]] // \text{FullSimplify}$$

0

$$\text{Coefficient}[(U+V+W), \text{hte}[t, r]] // \text{FullSimplify}$$

$$-\frac{\sqrt{\ell(1+\ell)}\sqrt{-2+\ell+\ell^2}\gamma[\theta, \phi]}{\sqrt{2}r^2}$$

$$\text{Assuming}[\ell \in \text{Integers} \&\& \ell > 0, \text{FullSimplify}[U+V+W]]$$

$$-\frac{\sqrt{(-1+\ell)\ell(1+\ell)(2+\ell)}\gamma \text{hte}[t, r]\gamma[\theta, \phi]}{\sqrt{2}r^2}$$

$$\text{Assuming}[\ell \in \text{Integers} \&\& \ell > 0, \text{FullSimplify}\left[-\frac{1}{r\ell(1+\ell)}\right.$$

$$\left.\sqrt{2} \left(\sqrt{\ell(1+\ell)}\sqrt{-2+\ell+\ell^2} - (-1+\ell)\ell(1+\ell)(2+\ell)\sqrt{\frac{1}{\ell(-2-\ell+2\ell^2+\ell^3)}}\right) \gamma[\theta, \phi]\right]$$

0

Assuming[$\ell \in \text{Integers} \ \&\& \ell > 0$, FullSimplify[Coefficient[U, hte^(0,1)[t, r]]]] // FullSimplify

$$\frac{\sqrt{2} \left((4 + \ell + \ell^2) \beta + \frac{4\gamma}{\sqrt{-2 + \ell + \ell^2}} \right) Y[\theta, \phi]}{r \sqrt{\ell (1 + \ell)}}$$

Assuming[$\ell \in \text{Integers} \ \&\& \ell > 0$, FullSimplify[Coefficient[V, hte^(0,1)[t, r]]]] // FullSimplify

$$-\frac{4\sqrt{2} \left(\frac{\beta}{\sqrt{\ell (1 + \ell)}} + \frac{\gamma}{\sqrt{(-1 + \ell) \ell (1 + \ell) (2 + \ell)}} \right) Y[\theta, \phi]}{r}$$

Assuming[$\ell \in \text{Integers} \ \&\& \ell > 0$, FullSimplify[Coefficient[W, hte^(0,1)[t, r]]]] // FullSimplify

$$-\frac{\sqrt{2} \sqrt{\ell (1 + \ell)} \beta Y[\theta, \phi]}{r}$$

Assuming[$\ell \in \text{Integers} \ \&\& \ell > 0$, FullSimplify[Coefficient[U + V + W, hte^(0,1)[t, r]]]]

0

Assuming[$\ell \in \text{Integers} \ \&\& \ell > 0$, FullSimplify[U + V + W]]

$$-\frac{\sqrt{(-1 + \ell) \ell (1 + \ell) (2 + \ell)} \gamma \text{hte}[t, r] Y[\theta, \phi]}{\sqrt{2} r^2}$$

$$R_{\{0r0r\}} = -\frac{\sqrt{(-1 + \ell) \ell (1 + \ell) (2 + \ell)} \gamma \text{hte}[t, r] Y[\theta, \phi]}{\sqrt{2} r^2}.$$

■ R_{0r0e}

econtra[[3]] invg[[4, 4]] R[ss][[4, 2, 4, 3]] // FullSimplify

0

econtra[[3]] invg[[4, 4]] R[sp][[4, 2, 4, 3]] // FullSimplify

0

econtra[[3]] invg[[4, 4]] R[hol[t, r] Tol][[4, 2, 4, 3]] // FullSimplify

$$\frac{\text{hol}[t, r] Y^{(1,0)}[\theta, \phi]^2}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}}$$

econtra[[3]] invg[[4, 4]] R[hot[t, r] Tot][[4, 2, 4, 3]] // FullSimplify

$$-\frac{\text{hot}^{(0,1)}[t, r] Y^{(1,0)}[\theta, \phi]^2}{\sqrt{2} r \sqrt{\text{Csc}[\theta]^2 Y^{(0,1)}[\theta, \phi]^2 + Y^{(1,0)}[\theta, \phi]^2}}$$

econtra[[3]] invg[[4, 4]] R[h1m[t, r] T1m][[4, 2, 4, 3]] // FullSimplify

$$\left(\sqrt{\frac{1}{2\ell + 2\ell^2}} \operatorname{Csc}[\theta] h1m[t, r] Y^{(1, \theta)}[\theta, \phi] \right. \\ \left. \left(2 Y^{(\theta, 1)}[\theta, \phi] + \operatorname{Csc}[\theta]^2 Y^{(\theta, 3)}[\theta, \phi] + \operatorname{Cot}[\theta] Y^{(1, 1)}[\theta, \phi] + Y^{(2, 1)}[\theta, \phi] \right) \right] / \\ \left(r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2} \right) \\ \left(\sqrt{\frac{1}{2\ell + 2\ell^2}} \operatorname{Csc}[\theta] h1m[t, r] Y^{(1, \theta)}[\theta, \phi] \left(2 Y^{(\theta, 1)}[\theta, \phi] - \ell(\ell + 1) Y^{(\theta, 1)}[\theta, \phi] \right) \right) / \\ \left(r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2} \right)$$

econtra[[3]] invg[[4, 4]] R[h1e[t, r] T1e][[4, 2, 4, 3]] // FullSimplify

$$- \frac{\sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} h1e[t, r] Y^{(1, \theta)}[\theta, \phi]^2}{r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2}}$$

econtra[[3]] invg[[4, 4]] R[hte[t, r] T2e][[4, 2, 4, 3]] // FullSimplify

$$\frac{\left(-\sqrt{\ell(1 + \ell)} + 2\sqrt{\frac{1}{\ell + \ell^2}} \right) hte^{(\theta, 1)}[t, r] Y^{(1, \theta)}[\theta, \phi]^2}{2 r \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2}}$$

econtra[[4]] invg[[3, 3]] R[ss][[3, 2, 3, 4]] // FullSimplify

0

econtra[[4]] invg[[3, 3]] R[sp][[3, 2, 3, 4]] // FullSimplify

0

econtra[[4]] invg[[3, 3]] R[hol[t, r] Tol][[3, 2, 3, 4]] // FullSimplify

$$\frac{\operatorname{Csc}[\theta]^2 hol[t, r] Y^{(\theta, 1)}[\theta, \phi]^2}{r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2}}$$

econtra[[4]] invg[[3, 3]] R[hot[t, r] Tot][[3, 2, 3, 4]] // FullSimplify

$$- \frac{\operatorname{Csc}[\theta]^2 hot^{(\theta, 1)}[t, r] Y^{(\theta, 1)}[\theta, \phi]^2}{\sqrt{2} r \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2}}$$

econtra[[4]] invg[[3, 3]] R[h1m[t, r] T1m][[3, 2, 3, 4]] // FullSimplify

$$\left(\sqrt{\frac{1}{2\ell + 2\ell^2}} \operatorname{Csc}[\theta] h1m[t, r] Y^{(\theta, 1)}[\theta, \phi] \right. \\ \left(\operatorname{Csc}[\theta]^2 \left(2 \operatorname{Cot}[\theta] Y^{(\theta, 2)}[\theta, \phi] + \operatorname{Cos}[2\theta] Y^{(1, \theta)}[\theta, \phi] - Y^{(1, 2)}[\theta, \phi] \right) - \right. \\ \left. \left. \operatorname{Cot}[\theta] Y^{(2, \theta)}[\theta, \phi] - Y^{(3, \theta)}[\theta, \phi] \right) \right) / \left(r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2} \right)$$

$$\left(\sqrt{\frac{1}{2\ell+2\ell^2}} \operatorname{Csc}[\theta] \operatorname{h1m}[t, r] Y^{(\theta,1)}[\theta, \phi] \left(-2 Y^{(1,\theta)}[\theta, \phi] + \ell(\ell+1) Y^{(1,\theta)}[\theta, \phi] \right) \right) /$$

$$\left(r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} \right)$$

`econtra[[4]] invg[[3, 3]] R[h1e[t, r] T1e][[3, 2, 3, 4]] // FullSimplify`

$$- \frac{\sqrt{2} \sqrt{\frac{1}{\ell+\ell^2}} \operatorname{Csc}[\theta]^2 \operatorname{h1e}[t, r] Y^{(\theta,1)}[\theta, \phi]^2}{r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}}$$

`econtra[[4]] invg[[3, 3]] R[h2e[t, r] T2e][[3, 2, 3, 4]] // FullSimplify`

$$\left(\left(-\sqrt{\ell(1+\ell)} + 2 \sqrt{\frac{1}{\ell+\ell^2}} \right) \operatorname{Csc}[\theta]^2 \operatorname{h2e}^{(\theta,1)}[t, r] Y^{(\theta,1)}[\theta, \phi]^2 \right) /$$

$$\left(2 r \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} \right)$$

■ Analysis of R_{0r0e}

$$\left(\frac{\operatorname{hol}[t, r] Y^{(1,\theta)}[\theta, \phi]^2}{r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}} + \frac{\operatorname{Csc}[\theta]^2 \operatorname{hol}[t, r] Y^{(\theta,1)}[\theta, \phi]^2}{r^2 \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}} \right) //$$

`FullSimplify`

$$\frac{\operatorname{hol}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}}{r^2}$$

`%31 /. hol[t, r] -> (a hte[t, r] + b r hte^{(\theta,1)}[t, r]) // FullSimplify`

$$\frac{1}{r^2} (a \operatorname{hte}[t, r] + b r \operatorname{hte}^{(\theta,1)}[t, r]) \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

$$\left(-\frac{\operatorname{hot}^{(\theta,1)}[t, r] Y^{(1,\theta)}[\theta, \phi]^2}{\sqrt{2} r \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}} - \frac{\operatorname{Csc}[\theta]^2 \operatorname{hot}^{(\theta,1)}[t, r] Y^{(\theta,1)}[\theta, \phi]^2}{\sqrt{2} r \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}} \right) //$$

`FullSimplify`

$$- \frac{1}{\sqrt{2} r} \operatorname{hot}^{(\theta,1)}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}$$

$$- \frac{1}{\sqrt{2} r} \alpha \operatorname{hte}^{(\theta,1)}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} //$$

$$\left(\left(\sqrt{\frac{1}{2\ell+2\ell^2}} \text{Csc}[\theta] \text{h1m}[t, r] Y^{(1,\theta)}[\theta, \phi] \left(2Y^{(\theta,1)}[\theta, \phi] - \ell(\ell+1) Y^{(\theta,1)}[\theta, \phi] \right) \right) / \right. \\ \left. \left(r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} \right) + \right. \\ \left. \left(\sqrt{\frac{1}{2\ell+2\ell^2}} \text{Csc}[\theta] \text{h1m}[t, r] Y^{(\theta,1)}[\theta, \phi] \left(-2Y^{(1,\theta)}[\theta, \phi] + \ell(\ell+1) Y^{(1,\theta)}[\theta, \phi] \right) \right) / \right. \\ \left. \left(r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} \right) \right) // \text{FullSimplify}$$

0

$$- \frac{\sqrt{2} \sqrt{\frac{1}{\ell+\ell^2}} \text{Csc}[\theta]^2 \text{h1e}[t, r] Y^{(\theta,1)}[\theta, \phi]^2}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}} - \\ \frac{\sqrt{2} \sqrt{\frac{1}{\ell+\ell^2}} \text{h1e}[t, r] Y^{(1,\theta)}[\theta, \phi]^2}{r^2 \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}} // \text{FullSimplify} \\ - \frac{1}{r^2} \sqrt{2} \sqrt{\frac{1}{\ell+\ell^2}} \text{h1e}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} \\ - \frac{1}{r^2} \sqrt{2} \sqrt{\frac{1}{\ell+\ell^2}} \beta \text{hte}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} //$$

Assuming[$\ell \in \text{Integers} \ \&\& \ \ell > 0$,

$$\text{FullSimplify} \left[\left(\left(-\sqrt{\ell(1+\ell)} + 2\sqrt{\frac{1}{\ell+\ell^2}} \right) \text{Csc}[\theta]^2 \text{hte}^{(\theta,1)}[t, r] Y^{(\theta,1)}[\theta, \phi]^2 \right) / \right. \\ \left(2r \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} \right) + \\ \left(\left(-\sqrt{\ell(1+\ell)} + 2\sqrt{\frac{1}{\ell+\ell^2}} \right) \text{hte}^{(\theta,1)}[t, r] Y^{(1,\theta)}[\theta, \phi]^2 \right) / \\ \left(2r \sqrt{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2} \right) \right] \\ - \frac{1}{2r} (-2 + \ell + \ell^2) \text{hte}^{(\theta,1)}[t, r] \sqrt{\frac{\text{Csc}[\theta]^2 Y^{(\theta,1)}[\theta, \phi]^2 + Y^{(1,\theta)}[\theta, \phi]^2}{\ell + \ell^2}}$$

$$\begin{aligned}
& -\frac{1}{2r}(-2+\ell+\ell^2) \gamma \operatorname{hte}^{(\theta,1)}[t, r] \sqrt{\frac{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2}{\ell + \ell^2}} - \\
& \frac{1}{r^2} \sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \beta \operatorname{hte}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2} - \\
& \frac{1}{\sqrt{2} r} \alpha \operatorname{hte}^{(\theta,1)}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2} + \frac{1}{r^2} \\
& (a \operatorname{hte}[t, r] + b r \operatorname{hte}^{(\theta,1)}[t, r]) \sqrt{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2} // \text{FullSimplify} \\
& \frac{1}{2r^2} \left(2 \left(a - \sqrt{2} \sqrt{\frac{1}{\ell + \ell^2}} \beta \right) \operatorname{hte}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2} + \right. \\
& \left. r \operatorname{hte}^{(\theta,1)}[t, r] \left((2b - \sqrt{2} \alpha) \sqrt{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2} - \right. \right. \\
& \left. \left. (-2 + \ell + \ell^2) \gamma \sqrt{\frac{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2}{\ell + \ell^2}} \right) \right)
\end{aligned}$$

$$\%33 /. a \rightarrow \left(\frac{2}{\ell(\ell+1)} \right)^{\frac{1}{2}} \beta - \left((\ell-1) \frac{(\ell+2)}{2\ell(\ell+1)} \right)^{\frac{1}{2}} \gamma /.$$

$$b \rightarrow \left(\frac{2}{\ell(\ell+1)} \right)^{\frac{1}{2}} \beta + \left(\frac{2}{\ell(\ell+1)(\ell+2)(\ell-1)} \right)^{\frac{1}{2}} \gamma /.$$

$$\alpha \rightarrow 2 \left(\frac{1}{\ell(\ell+1)} \right)^{\frac{1}{2}} \beta + 2 \left(\frac{1}{\ell(\ell+1)(\ell+2)(\ell-1)} \right)^{\frac{1}{2}} \gamma // \text{FullSimplify}$$

$$\begin{aligned}
& -\frac{1}{2r^2} \gamma \left(\sqrt{2 - \frac{4}{\ell} + \frac{4}{1+\ell}} \operatorname{hte}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2} + \right. \\
& \left. r(-2 + \ell + \ell^2) \operatorname{hte}^{(\theta,1)}[t, r] \sqrt{\frac{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2}{\ell + \ell^2}} \right)
\end{aligned}$$

Assuming[$\ell \in \text{Integers} \ \&\& \ \ell > 0 \ \&\& \ \gamma \in \text{Reals} \ \&\& \ \gamma > 0 \ \&\& \ \beta \in \text{Reals} \ \&\& \ \beta > 0$, FullSimplify[

$$\text{Coefficient}\left[-\frac{1}{2r^2} \gamma \left(\sqrt{2 - \frac{4}{\ell} + \frac{4}{1+\ell}} \operatorname{hte}[t, r] \sqrt{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2} + \right.$$

$$\left. r(-2 + \ell + \ell^2) \operatorname{hte}^{(\theta,1)}[t, r] \sqrt{\frac{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2}{\ell + \ell^2}} \right), \operatorname{hte}^{(\theta,1)}[t, r] \Big] \Big]$$

$$-\frac{(-2 + \ell + \ell^2) \gamma \sqrt{\frac{\operatorname{Csc}[\theta]^2 \Upsilon^{(\theta,1)}[\theta, \phi]^2 + \Upsilon^{(1,\theta)}[\theta, \phi]^2}{\ell + \ell^2}}}{2r}$$

$$-\gamma \frac{(-2 + \ell + \ell^2)}{2r(\ell(\ell+1))^{\frac{1}{2}}}$$

Assuming $[\ell \in \text{Integers} \ \&\& \ell > 0 \ \&\& \gamma \in \text{Reals} \ \&\& \gamma > 0 \ \&\& \beta \in \text{Reals} \ \&\& \beta > 0, \text{FullSimplify}[$

$$\text{Coefficient}\left[-\frac{1}{2 r^2} \gamma \left(\sqrt{2 - \frac{4}{\ell} + \frac{4}{1 + \ell}} \text{hte}[t, r] \sqrt{\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2} + \right.$$

$$\left. r (-2 + \ell + \ell^2) \text{hte}^{(\theta, 1)}[t, r] \sqrt{\frac{\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2}{\ell + \ell^2}}\right], \text{hte}[t, r] \]]$$

$$-\frac{1}{2 r^2} \gamma \sqrt{\left(2 - \frac{4}{\ell} + \frac{4}{1 + \ell}\right) \left(\text{Csc}[\theta]^2 Y^{(\theta, 1)}[\theta, \phi]^2 + Y^{(1, \theta)}[\theta, \phi]^2\right)}$$
