Supplementary Materials to "A comparison of tests for the one-way ANOVA problem for functional data"

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This supporting material contains the R codes of the programs, which allow to perform the tests for the one-way ANOVA problem for functional data considered in the paper, and the tables, which contain empirical sizes and powers of those tests obtained in remaining simulations S1 and S2 respectively, and in models M1–M2 and M3–M7 respectively; the results of multiple comparisons of tests in simulations S1 and S2 and models M1–M7 with $n_i = 20, 30, i = 1, 2, 3$; the estimates of the parameters of the L^2N, L^2B, FN, FB and GPF tests for the Canadian temperature data and the orthosis data for $[a, b] = [0, 1]$ and $[a, b] = [0.8, 1]$. We also present the figures, which depict simulated examples corresponding with models M1–M6.

1 R codes

In this section, we describe the implementation of the tests for the one-way ANOVA problem for functional data in the R program. We use the R package `fda`. This package is described in Ramsay et al. (2009). In R, the Fourier system is created by the function `create.fourier.basis`. In order to convert raw data into a functional object the functions `smooth.basisPar` and `Data2fd` can be used.

The following function `BIC` implements the Bayesian Information Criterion to select the optimal value of K in a basis function representation of the stochastic processes $X_{ij}(t)$ given in (4). The argument `x` is a data frame or matrix of data, whose each row is a discretized version of a function. The argument `maxK` is the maximum value of K given in (4). K has to be odd, which is dictated by the function `create.fourier.basis` used to create the Fourier system. Its default value is equal to 101.

```
BIC = function(x, maxK = 101){
  n = nrow(x)      ## the number of observations
  p = ncol(x)      ## the number of time points
  x = as.matrix(x)
  if(p <= maxK){if(p%%2==1) maxK = p-2 else maxK = p-1} else maxK = maxK
  v = matrix(rep(0, n*length(seq(3, maxK, 2))), nrow = n)
  for(i in seq(3, maxK, 2)){
    fbasis = create.fourier.basis(c(0, p), i)
    fdata = smooth.basisPar(1:p, t(x), fbasis)
    v[,(i-1)/2] = log(p*(1-fdata$df/p)^2*fdata$gcv/2)+i*log(p)/p
  }
  K = rep(0, n)
  for(j in 1:n) K[j] = 2*which(v[j,] == min(v[j,]))+1
  temp = table(as.vector(K))
  return(max(as.numeric(names(temp)[temp == max(temp)])))
}
```

To implement the LH, R, P and W tests, we can use the well known function `manova` for the MANOVA tests and the functions described above. Therefore, we do not present the detailed codes for these tests. We also do not present the code of the FL test implementation, since it is available on the website "`http://orfe.princeton.edu/~jqfan/papers/pub/hanova.s`" of Professor Jianqing Fan. Below, however, we give the code of the simulation, which calculates the 0.05 upper quantile of the distribution of the FL test statistic T_{HANOVA}. The function `FL.statistic` calculates the values of the statistic T_{AN} given by (4) in Fan and Lin (1998), whose distribution is the same as the distribution of T_{HANOVA}. Table 1 contains the 0.05 upper quantiles of the distribution of the FL test statistic for some T^*, which were used in simulations and examples of the present paper.

```
FL.statistic = function(T){
  x = rnorm(T)
  FL.star = max((cumsum(x^2)-(1:T))*(1/sqrt(2*(1:T))))
  logT = log(log(T))
  FL.stat = sqrt(2*logT)*FL.star - (2*logT+0.5*log(logT)-0.5*log(4*pi))
}

# simulation for T = 25
set.seed(1234)
T = 25
nr = 1000000
simulation = replicate(nr, FL.statistic(T))
```

```
sort(simulation)[nr*0.95] # 0.05 upper quantile = 3.776794
```

Table 1: $\alpha = 0.05$ upper quantiles of the distribution of the FL test statistic based on 1,000,000 simulations.

T^*	0.05 upper quantile
25	3.776794
51	3.820904
60	3.835702
64	3.839860
66	3.842256
72	3.849700
83	3.842888
88	3.847345
99	3.852353
100	3.843317

Other tests are performed by the function `fanova.test` given below. The notations in the program are consistent with or similar to those used in the paper. The argument `x` is a data frame or matrix of data, whose each row is a discretized version of a function, and its first column contains group labels. The argument `test` is a number associated with a test, which we would like to perform. The following coding is used: 1 – the FP test, 2 – the CH test, 3 – the CS test, 4 – the L^{2}N test, 5 – the L^{2}B test, 6 – the L^{2}b test, 7 – the FN test, 8 – the FB test, 9 – the Fb test, 10 – the GPF test. The default value of this argument is 1 for the FP test. For some of the tests, we need to give the last argument `params`, which is a parameter or contains parameters of the tests. For the FP test, we need to give a number of permutation replicates `R` (its default value is 1000), and specify if we want to choose optimal values of K given in (4) by BIC (`bic = T` – default value) or not (`bic = F`). In the second case, K given in (4) is equal to the last argument `maxK`, which is the same as in the function `BIC`. We can omit BIC in choosing optimal value of K in (4), when the functions are observed on a large grid of design time points. For the CH and CS tests, the argument `params` is a number of discretized artificial trajectories for each process $Z_i(t)$ appearing in the limit statistic V given in (3) (in our simulations and examples, `params` was always equal to 2000). For the L^{2}b and Fb tests, the argument `params` is a number of bootstrap replicates (in our simulations and examples, `params` was always equal to 10,000). As outputs, we obtain value of test statistic and p-value of a test. Moreover, for the L^{2}N, L^{2}B, FN, FB and GPF tests, we also obtain the estimates of the parameters of these tests. To perform the CH and CS tests, we have to load the R package `MASS`. The part of the code for the L^{2}N, L^{2}B, L^{2}b, FN, FB, Fb and GPF tests is based on the Matlab codes from the website "`http://www.stat.nus.edu.sg/~zhangjt/books/Chapman/FANOVA.htm`" of Professor Jin-Ting Zhang.

```
fanova.test = function(x, test = 1, params = list(R = 1000, bic = T, maxK = 101)){
  n = nrow(x)                ## the number of observations
  p = ncol(x)-1              ## the number of time points
  aflag = x[,1]              ## the vector of group labels
  aflag0 = unique(aflag)     ## the vector of unique group labels
  k = length(aflag0)         ## the number of groups
  x = as.matrix(x[,2:(p+1)])
  vnog = rep(0, k)           ## the vector of the numbers of observations in groups
  for(i in 1:k) vnog[i] = nrow(x[aflag==aflag0[i],])
  if(test==1){
    if(params$bic==T) K = BIC(x, params$maxK) else K = params$maxK
    fbasis = create.fourier.basis(c(0, p), K)
    lfdata = vector('list', k)
    mfdata = matrix(rep(0, K*n), nrow = K)
    for(i in 1:k){
      lfdata[[i]] = Data2fd(1:p, t(x[aflag==aflag0[i],]), fbasis)$coefs
      mfdata[,aflag==aflag0[i]] = lfdata[[i]]
    }
    a = 0
    b = 0
    c = 0
    for(i in 1:k){
      a = a + sum(t(lfdata[[i]])%*%lfdata[[i]])/vnog[i]
      for(j in 1:k) b = b + sum(t(lfdata[[i]])%*%(lfdata[[j]]))
      c = c + sum((lfdata[[i]])^2)
    }
    b = b/n
    t0 = (a-b)/(c-a)
    stat = t0*(n-k)/(k-1)  ## the value of the test statistic (F) for raw data
    r = rep(0, params$R)
    for(i in 1:params$R){
      K = 1:n
      z = vector('list', k)
      for(j in 1:k){
        k1 = sample(K, size = vnog[j], replace = F)
        z[[j]] = mfdata[,k1]
        for(l in 1:vnog[j]) K = K[-which(K==k1[l])]
```

```
    }
    a = 0
    for(m in 1:k) a = a + sum(t(z[[m]])%*%z[[m]])/vnog[m]
    r[i] = (a-b)/(c-a)
  }
  pvalue = mean(r>=t0)
}
if(test==2){
  stat = 0      ## the value of the test statistic (Vn) for raw data
  for(i in 1:(k-1)){
    for(j in (i+1):k){
      stat = stat + vnog[i]*sum((colMeans(x[aflag==aflag0[i],])
                    -colMeans(x[aflag==aflag0[j],]))^2)
    }
  }
  gamma = var(x)
  Z = vector('list', k)
  for(i in 1:k)  Z[[i]] = mvrnorm(params, rep(0, nrow(gamma)), gamma)
  V = rep(0, params)
  for(m in 1:params){
    for(i in 1:(k-1)){
      for(j in (i+1):k){
        V[m] = V[m] + sum((Z[[i]][m,]-sqrt(vnog[i]/vnog[j])*Z[[j]][m,])^2)
      }
    }
  }
  pvalue = mean(V>stat)
}
if(test==3){
  stat = 0      ## the value of the test statistic (Vn) for raw data
  for(i in 1:(k-1)){
    for(j in (i+1):k){
      stat = stat + vnog[i]*sum((colMeans(x[aflag==aflag0[i],])
                    -colMeans(x[aflag==aflag0[j],]))^2)
    }
  }
  gammas = vector('list', k)
  for(i in 1:k) gammas[[i]] = var(x[aflag==aflag0[i],])
  Z = vector('list', k)
```

```
  for(i in 1:k) Z[[i]] = mvrnorm(params, rep(0, nrow(gammas[[i]])), gammas[[i]])
  V = rep(0, params)
  for(m in 1:params){
    for(i in 1:(k-1)){
      for(j in (i+1):k){
        V[m] = V[m] + sum((Z[[i]][m,]-sqrt(vnog[i]/vnog[j])*Z[[j]][m,])^2)
      }
    }
  }
  pvalue = mean(V>stat)
}
if(test==4|test==5|test==6|test==7|test==8|test==9|test==10){
  ## computing pointwise SSR, SSE
  mu0 = colMeans(x)     ## pooled sample mean function
  vmu = matrix(rep(0, k*p), ncol = p)
  z = matrix(rep(0, n*p), ncol = p)
  SSR = 0
  SSE = 0
  for(i in 1:k){
    xi = x[aflag==aflag0[i],]
    mui = colMeans(xi)
    zi = xi - as.matrix(rep(1, vnog[i]))%*%mui
    vmu[i,] = mui
    if(i==1) {z[1:vnog[i],] = zi} else
             {z[(cumsum(vnog)[i-1]+1):cumsum(vnog)[i],] = zi}
    SSR = SSR + vnog[i]*(mui - mu0)^2
    SSE = SSE + colSums(zi^2)
  }
  if(n>p) gamma = t(z)%*%z/(n-k) else gamma = z%*%t(z)/(n-k)
  A = sum(diag(gamma))
  B = sum(diag(gamma%*%gamma))
  A2N = A^2
  B2N = B
  A2B = (n-k)*(n-k+1)/(n-k-1)/(n-k+2)*(A^2-2*B/(n-k+1))
  B2B = (n-k)^2/(n-k-1)/(n-k+2)*(B-A^2/(n-k))
  if(test==4|test==5|test==6) stat = sum(SSR)
  if(test==7|test==8|test==9) stat = sum(SSR)/sum(SSE)*(n-k)/(k-1)
  if(test==6|test==9){
    btstatL2b = rep(0, params)
```

```
  btstatFb = rep(0, params)
  for(ii in 1:params){
    btvmu = matrix(rep(0, k*p), ncol = p)
    btz = matrix(rep(0, n*p), ncol = p)
    for(i in 1:k){
      xi = x[aflag==aflag0[i],]
      btflag = floor(runif(vnog[i])*(vnog[i]-1))+1
      btxi = xi[btflag,]
      btmui = colMeans(btxi)
      btzi = btxi - as.matrix(rep(1, vnog[i]))%*%btmui
      if(i==1) {btz[1:vnog[i],] = btzi} else
                 {btz[(cumsum(vnog)[i-1]+1):cumsum(vnog)[i],] = btzi}
      btmui = btmui-vmu[i,]
      btvmu[i,] = btmui
    }
    btmu0 = vnog%*%btvmu/n
    btSSR = 0
    for(i in 1:k) btSSR = btSSR + vnog[i]*(btvmu[i,]-btmu0)^2
    btstatL2b[ii] = sum(btSSR)
    if(n>p) btgamma = t(btz)%*%btz/(n-k) else btgamma = btz%*%t(btz)/(n-k)
      btA = sum(diag(btgamma))
      btstatFb[ii] = sum(btSSR)/btA/(k-1)
  }
}
if(test==4){
  betaL2N = B2N/A
  kappaL2N = A2N/B2N
  pvalue = 1-pchisq(stat/betaL2N, (k-1)*kappaL2N)
  test_st_pval_par = list(test_statistic = stat, pvalue = pvalue,
                          betaL2N = betaL2N, dL2N = (k-1)*kappaL2N)
}
if(test==5){
  betaL2B = B2B/A
  kappaL2B = A2B/B2B
  pvalue = 1-pchisq(stat/betaL2B, (k-1)*kappaL2B)
  test_st_pval_par = list(test_statistic = stat, pvalue = pvalue,
                          betaL2B = betaL2B, dL2B = (k-1)*kappaL2B)
}
if(test==6) pvalue = mean(btstatL2b>=stat)
```

```
    if(test==7){
      kappaFN = A2N/B2N
      pvalue = 1-pf(stat, (k-1)*kappaFN, (n-k)*kappaFN)
      test_st_pval_par = list(test_statistic = stat, pvalue = pvalue,
                              d1FN = (k-1)*kappaFN, d2FN = (n-k)*kappaFN)
    }
    if(test==8){
      kappaFB = A2B/B2B
      pvalue = 1-pf(stat, (k-1)*kappaFB, (n-k)*kappaFB)
      test_st_pval_par = list(test_statistic = stat, pvalue = pvalue,
                              d1FB = (k-1)*kappaFB, d2FB = (n-k)*kappaFB)
    }
    if(test==9) pvalue = mean(btstatFb>=stat)
    if(test==10){
      z = z/(as.matrix(rep(1, n))%*% sqrt(colSums(z^2)))
      if(n>=p) z=t(z)%*%z else z=z%*%t(z)
      stat = mean(SSR/SSE*(n-k)/(k-1))
      betaGPF = (sum(diag(z%*%z))/p^2/(k-1))/((n-k)/(n-k-2))
      dGPF = ((n-k)/(n-k-2))^2/(sum(diag(z%*%z))/p^2/(k-1))
      pvalue = 1-pchisq(stat/betaGPF,dGPF)
      test_st_pval_par = list(test_statistic = stat, pvalue = pvalue,
                              betaGPF = betaGPF, dGPF = dGPF)
    }
  }
  if(test==1|test==2|test==3|test==6|test==9){
    return(list(test_statistic = stat, pvalue = pvalue))}
    else {return(test_st_pval_par)}
}
```

2 Tables and figures

In Tables 2 and 3, empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in simulation S1 with $n_i = 20, 30, i = 1, 2, 3$ are presented. Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in simulation S2 with $n_i = 20, 30, i = 1, 2, 3$ are given in Tables 4 and 5. In Tables 6–8 and 12–14, empirical sizes (as percentages) of all tests obtained in models M1 and M2 are given. Empirical powers (as percentages) of all tests obtained in models M3–M7 are presented in Tables 9–11 and 15–17. Tables 18 and 19 contain the results of multiple comparisons of tests in simulations S1 and S2 and models M1–M7 with $n_i = 20, 30, i = 1, 2, 3$. The estimates of the parameters of the L^2N, L^2B, FN, FB and GPF tests for the Canadian temperature data and the orthosis data for $[a, b] = [0, 1]$ and $[a, b] = [0.8, 1]$ are presented in Table 20. Figures 1 and 2 depict simulated examples corresponding with models M1–M6 with $n_i = 30, i = 1, 2, 3$, $\sigma_1 = 0.2/25$ in the normal and the Wiener cases.

Table 2: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in simulation S1 with $n_i = 20, i = 1, 2, 3$.

Data set	Test														
	FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
CBF	4	3	14	3	3	3	3	4	5	3	4	4	3	2	13
ChlorineConcentration	5	9	21	10	9	5	5	5	5	5	5	5	4	9	27
CinC_ECG_torso	5	2	21	0	0	3	5	5	5	5	4	5	2	5	44
Cricket_X	3	16	41	0	4	1	4	5	5	3	5	5	0	2	12
Cricket_Y	4	24	46	0	7	3	3	4	5	3	4	4	1	4	14
Haptics	8	2	10	3	3	6	6	7	8	6	7	7	5	6	24
Non-Invasive Thorax1	2	20	39	0	5	1	1	1	1	1	1	1	1	1	24
Non-Invasive Thorax2	2	14	32	0	6	1	3	3	3	2	2	3	0	2	28
Swedish Leaf	5	14	30	2	11	5	6	6	6	6	6	6	3	4	15
Symbols	7	6	20	1	3	4	5	5	10	5	5	5	3	6	33
Synthetic Control	3	4	13	4	4	3	3	3	3	3	3	3	2	3	7
uWaveGestureLibrary_X	6	7	14	1	5	5	6	6	7	5	6	6	3	7	25
uWaveGestureLibrary_Y	8	8	15	4	4	8	7	8	8	8	8	8	5	8	27
uWaveGestureLibrary_Z	10	2	21	2	9	9	10	10	10	10	10	10	7	9	32
Mean	5	9	24	2	5	4	5	5	6	5	5	5	3	5	23

Table 3: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in simulation S1 with $n_i = 30, i = 1, 2, 3$.

Data set	Test														
	FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
CBF	8	0	11	1	0	5	4	6	7	5	6	6	4	5	14
ChlorineConcentration	5	3	15	4	4	5	4	5	5	3	5	5	3	4	26
CinC_ECG_torso	5	6	20	3	5	5	5	5	5	5	5	5	3	5	30
Symbols	8	6	21	0	1	7	8	8	8	8	8	8	6	6	25
Synthetic Control	2	5	20	5	5	1	1	2	2	1	2	2	0	1	7
uWaveGestureLibrary_X	5	6	15	0	3	4	5	5	5	4	5	5	4	5	25
uWaveGestureLibrary_Y	7	7	25	2	5	5	6	7	8	7	7	7	5	8	19
uWaveGestureLibrary_Z	9	5	22	2	4	8	8	9	9	8	9	9	8	9	25
Mean	6	5	19	2	3	5	5	6	6	5	6	6	4	5	21

Table 4: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in simulation S2 with $n_i = 20, i = 1, 2, 3$.

Data set	Test														
	FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
CBF	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
ChlorineConcentration	6	35	59	27	32	6	6	7	7	6	7	7	4	9	27
CinC_ECG_torso	9	41	68	22	32	7	6	8	9	6	7	8	5	10	47
Cricket_X	61	47	72	43	49	57	56	57	61	56	56	58	54	61	90
Cricket_Y	82	33	77	31	36	80	80	81	83	80	81	82	78	86	94
Haptics	66	64	82	54	59	65	65	70	71	65	66	70	55	78	92
Non-Invasive Thorax1	99	99	100	60	98	99	99	99	99	99	99	99	99	100	100
Non-Invasive Thorax2	100	99	99	49	97	100	100	100	100	100	100	100	100	100	100
Swedish Leaf	98	99	100	77	98	98	97	98	99	97	98	99	97	99	100
Symbols	100	97	99	47	93	100	100	100	100	100	100	100	100	100	100
Synthetic Control	100	98	99	96	98	100	100	100	100	100	100	100	100	100	100
uWaveGestureLibrary_X	100	55	75	24	52	100	99	100	100	99	100	100	99	100	100
uWaveGestureLibrary_Y	98	56	79	39	56	97	96	98	98	97	98	98	93	98	100
uWaveGestureLibrary_Z	93	50	77	38	50	93	93	93	93	93	93	93	93	94	95
Mean	79	70	85	51	68	79	78	79	80	78	79	80	77	81	89

Table 5: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in simulation S2 with $n_i = 30, i = 1, 2, 3$.

Data set	Test														
	FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
CBF	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
ChlorineConcentration	3	40	74	35	39	3	3	4	4	4	3	3	1	10	33
CinC_ECG_torso	12	86	96	56	78	12	9	12	13	8	12	12	7	12	53
Cricket_X	68	85	95	83	84	72	70	73	76	71	73	74	66	78	95
Cricket_Y	94	96	97	90	96	94	94	94	94	93	94	94	91	95	97
Haptics	81	70	86	63	69	77	75	80	81	75	79	80	73	90	94
Non-Invasive Thorax1	100	100	100	55	100	100	100	100	100	100	100	100	100	100	100
Non-Invasive Thorax2	100	99	100	54	99	100	100	100	100	100	100	100	100	100	100
Swedish Leaf	100	100	100	87	100	100	100	100	100	100	100	100	100	100	100
Symbols	100	100	100	64	99	100	100	100	100	100	100	100	100	100	100
Synthetic Control	100	95	96	95	95	100	100	100	100	100	100	100	100	100	100
uWaveGestureLibrary_X	98	71	89	61	72	98	98	98	98	98	98	98	98	98	99
uWaveGestureLibrary_Y	100	65	80	39	57	100	99	100	100	99	100	100	99	100	100
uWaveGestureLibrary_Z	94	56	79	37	51	94	94	94	95	94	94	94	94	94	96
Mean	82	83	92	66	81	82	82	83	83	82	82	83	81	84	91

Table 6: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in models M1 and M2 with $n_i = 10, i = 1, 2, 3$ in the normal case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M1	σ_1	5	8	22	3	6	0	0	0	4	0	0	4	0	3	15
	σ_2	4	5	11	2	2	1	1	1	6	1	1	3	1	2	15
	σ_3	4	7	22	5	6	0	0	0	4	0	0	3	0	2	13
	σ_4	3	8	25	4	6	0	0	0	4	0	0	4	0	0	18
	σ_5	4	9	26	8	7	0	0	1	4	0	0	4	0	3	23
	σ_6	4	4	23	6	5	2	0	2	3	1	2	3	0	2	16
	σ_7	3	5	19	3	5	0	0	0	3	0	0	1	0	0	13
	Mean	4	7	21	4	5	0	0	1	4	0	0	3	0	2	16
M2	σ_1	3	6	24	3	6	0	0	0	4	0	0	4	0	3	15
	σ_2	4	5	11	2	2	1	1	1	6	1	1	3	1	2	15
	σ_3	3	6	22	4	5	0	0	0	4	0	0	3	0	2	13
	σ_4	3	8	27	4	6	0	0	0	4	0	0	4	0	0	18
	σ_5	4	7	23	6	5	0	0	1	4	0	0	4	0	3	23
	σ_6	4	4	21	6	5	2	0	2	3	1	2	3	0	2	16
	σ_7	2	6	23	5	5	0	0	0	3	0	0	1	0	0	13
	Mean	3	6	22	4	5	0	0	1	4	0	0	3	0	2	16
Total mean		4	6	21	4	5	0	0	1	4	0	0	3	0	2	16

Table 7: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in models M1 and M2 with $n_i = 20, i = 1, 2, 3$ in the normal case.

Model		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M1	σ_1	3	10	25	7	8	2	1	3	4	1	2	4	1	3	10
	σ_2	9	6	16	6	5	4	3	4	11	3	4	10	1	7	12
	σ_3	4	3	18	4	3	1	1	2	6	1	1	5	1	3	7
	σ_4	3	7	21	5	6	3	2	3	5	2	3	4	1	5	10
	σ_5	4	3	15	3	3	0	0	0	2	0	0	1	0	1	7
	σ_6	5	7	15	7	7	1	1	2	3	1	1	2	1	1	8
	σ_7	4	4	10	4	3	0	0	1	6	0	0	5	0	3	6
	Mean	5	6	17	5	5	2	1	2	5	1	2	4	1	3	9
M2	σ_1	4	10	19	7	7	2	1	3	4	1	2	4	1	3	10
	σ_2	9	6	16	6	5	4	3	4	11	3	4	10	1	7	12
	σ_3	6	3	20	4	3	1	1	2	6	1	1	5	1	3	7
	σ_4	3	6	20	4	5	3	2	3	5	2	3	4	1	5	10
	σ_5	4	4	14	4	4	0	0	0	2	0	0	1	0	1	7
	σ_6	4	6	15	7	6	1	1	2	3	1	1	2	1	1	8
	σ_7	3	4	9	4	3	0	0	1	6	0	0	5	0	3	6
	Mean	5	6	16	5	5	2	1	2	5	1	2	4	1	3	9
Total mean		5	6	17	5	5	2	1	2	5	1	2	4	1	3	9

Table 8: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in models M1 and M2 with $n_i = 30, i = 1, 2, 3$ in the normal case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M1	σ_1	5	3	13	3	3	3	3	3	5	2	3	5	1	3	10
	σ_2	1	0	16	0	0	0	0	1	2	0	1	2	0	0	5
	σ_3	6	5	21	6	5	2	2	4	4	2	4	4	1	3	13
	σ_4	5	3	18	3	3	4	3	4	6	3	4	6	2	3	9
	σ_5	6	6	22	4	5	3	3	3	4	3	3	3	1	4	7
	σ_6	6	4	22	6	6	1	0	1	3	1	1	2	0	3	9
	σ_7	4	2	14	3	2	2	2	2	4	2	2	4	1	3	4
	Mean	5	3	18	4	3	2	2	3	4	2	3	4	1	3	8
M2	σ_1	4	3	14	3	3	3	2	3	5	2	3	5	1	3	10
	σ_2	1	0	16	0	0	0	0	1	2	0	1	2	0	0	5
	σ_3	6	5	21	6	5	2	2	4	4	2	4	4	1	3	13
	σ_4	6	4	18	4	4	4	3	4	6	3	4	6	2	3	9
	σ_5	5	6	21	4	5	3	3	3	4	3	3	3	1	4	7
	σ_6	5	4	23	7	6	1	1	1	3	1	1	2	0	3	9
	σ_7	5	3	15	4	3	2	1	2	4	2	2	4	1	3	4
	Mean	5	4	18	4	4	2	2	3	4	2	3	4	1	3	8
Total mean		5	3	18	4	4	2	2	3	4	2	3	4	1	3	8

Table 9: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in models M3–M7 with $n_i = 10, i = 1, 2, 3$ in the normal case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M3	σ_1	100	100	100	79	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	66	86	47	57	100	99	100	100	100	100	100	94	100	100
	σ_3	75	31	54	30	34	43	30	50	70	41	45	67	6	55	93
	σ_4	39	22	50	14	17	13	8	17	34	11	15	33	0	22	68
	σ_5	19	9	32	8	8	0	0	5	18	0	2	16	0	8	44
	σ_6	16	12	32	9	12	3	1	4	15	2	4	12	1	8	41
	σ_7	9	6	33	5	6	4	2	6	15	4	5	11	0	7	28
	Mean	51	35	55	27	33	38	34	40	50	37	39	48	29	43	68
M4	σ_1	100	100	100	81	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	99	100	52	97	100	100	100	100	100	100	100	100	100	100
	σ_3	100	78	99	39	65	100	100	100	100	100	100	100	100	100	100
	σ_4	100	40	84	20	32	99	99	100	100	100	100	100	99	100	100
	σ_5	100	34	71	24	28	91	89	96	99	90	95	99	53	93	100
	σ_6	87	30	59	22	28	67	59	73	91	63	70	87	25	76	96
	σ_7	60	26	63	25	25	36	27	41	65	33	39	63	4	52	86
	Mean	92	58	82	38	54	85	82	87	94	84	86	93	69	89	97
M5	σ_1	100	100	100	61	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	98	100	46	92	100	100	100	100	100	100	100	100	100	100
	σ_3	100	55	87	36	53	100	100	100	100	100	100	100	100	100	100
	σ_4	99	43	77	34	43	98	94	99	99	98	98	99	76	99	99
	σ_5	87	30	54	27	30	62	57	68	87	59	63	85	25	72	100
	σ_6	65	26	56	21	24	39	30	46	65	34	44	62	8	50	98
	σ_7	49	29	53	24	25	21	16	29	47	18	24	43	2	29	84
	Mean	86	54	75	36	52	74	71	77	85	73	76	84	59	79	97
M6	σ_1	100	100	100	54	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	55	84	40	52	100	100	100	100	100	100	100	100	100	100
	σ_3	80	27	56	27	26	56	50	63	79	55	59	78	19	66	97
	σ_4	28	14	39	11	13	6	3	7	25	4	5	19	0	14	67
	σ_5	21	11	33	8	10	7	6	9	17	7	9	17	0	10	58
	σ_6	24	17	43	15	16	1	1	8	21	1	5	20	0	13	55
	σ_7	7	13	26	6	10	0	0	0	8	0	0	8	0	8	32
	Mean	51	34	54	23	32	39	37	41	50	38	40	49	31	44	73
M7	σ_1	100	100	100	62	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	64	98	100	100	100	100	100	100	100	100	100	100
	σ_3	100	79	97	49	74	100	100	100	100	100	100	100	100	100	100
	σ_4	100	55	88	44	52	99	99	99	100	99	99	100	92	99	100
	σ_5	97	44	68	45	45	86	76	91	98	81	89	96	45	92	100
	σ_6	77	37	60	32	36	49	41	57	76	48	51	73	12	59	100
	σ_7	64	34	59	31	32	25	21	31	54	24	27	51	3	40	93
	Mean	91	64	82	47	62	80	77	83	90	79	81	89	65	84	99
Total mean		74	49	70	34	47	63	60	66	74	62	64	73	50	68	87

Table 10: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in models M3–M7 with $n_i = 20, i = 1, 2, 3$ in the normal case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M3	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	98	92	98	79	88	96	95	96	97	95	96	97	90	97	100
	σ_4	75	61	83	52	56	51	44	55	65	47	52	63	30	54	94
	σ_5	35	30	55	29	30	23	20	23	31	21	23	31	14	27	58
	σ_6	25	21	42	20	20	14	7	14	19	8	14	18	4	16	41
	σ_7	21	18	38	14	17	14	7	14	17	9	13	16	5	12	35
	Mean	65	60	74	56	59	57	53	57	61	54	57	61	49	58	75
M4	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_5	100	100	100	89	98	100	100	100	100	100	100	100	100	100	100
	σ_6	100	94	99	81	91	100	99	100	100	100	100	100	98	100	100
	σ_7	97	84	98	68	75	97	96	97	99	97	97	99	89	96	100
	Mean	100	97	100	91	95	100	99	100	100	100	100	100	98	99	100
M5	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	95	100	100	100	100	100	100	100	100	100	100	100
	σ_5	100	95	100	81	94	100	100	100	100	100	100	100	99	100	100
	σ_6	94	84	96	63	78	87	80	87	95	83	87	95	76	89	100
	σ_7	83	71	84	61	65	71	64	74	80	66	74	80	50	73	99
	Mean	97	93	97	86	91	94	92	94	96	93	94	96	89	95	100
M6	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	98	95	100	83	92	96	96	97	99	96	97	99	94	98	100
	σ_4	79	59	80	49	56	63	59	66	79	61	64	77	44	69	99
	σ_5	46	43	67	38	37	27	21	32	40	21	31	39	12	35	78
	σ_6	33	22	49	18	23	20	12	24	33	13	20	32	6	21	64
	σ_7	27	24	48	23	23	16	10	21	29	13	20	29	2	22	49
	Mean	69	63	78	59	62	60	57	63	69	58	62	68	51	64	84
M7	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	99	100	100	100	100	100	100	100	100	100	100	100
	σ_5	100	100	100	93	100	100	100	100	100	100	100	100	100	100	100
	σ_6	98	90	98	77	86	98	98	98	98	98	98	98	94	97	100
	σ_7	94	84	97	74	83	88	87	89	93	87	89	93	80	92	100
	Mean	99	96	99	92	96	98	98	98	99	98	98	99	96	98	100
Total mean		86	82	89	77	80	82	80	82	85	80	82	85	77	83	92

Table 11: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in models M3–M7 with $n_i = 30, i = 1, 2, 3$ in the normal case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M3	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	99	100	100	100	100	100	100	100	100	100	100	100
	σ_4	89	82	96	73	78	81	78	84	86	80	83	86	71	84	99
	σ_5	60	46	75	41	43	42	33	44	54	34	44	52	24	49	89
	σ_6	37	35	62	32	33	28	25	30	37	23	28	36	20	36	64
	σ_7	23	22	45	24	24	17	16	17	22	16	17	21	12	19	42
	Mean	73	69	83	67	68	67	65	68	71	65	67	71	61	70	85
M4	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_5	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_7	100	100	100	95	98	100	100	100	100	100	100	100	100	100	100
	Mean	100	100	100	99	100	100	100	100	100	100	100	100	100	100	100
M5	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_5	100	100	100	99	99	100	100	100	100	100	100	100	100	100	100
	σ_6	100	98	100	93	96	100	100	100	100	100	100	100	100	100	100
	σ_7	99	89	100	84	87	96	90	96	98	92	96	97	84	96	100
	Mean	100	98	100	97	97	99	99	99	100	99	99	100	98	99	100
M6	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	96	82	97	71	77	90	87	92	96	87	91	96	80	91	100
	σ_5	72	62	88	56	60	61	53	63	70	55	62	70	44	67	97
	σ_6	57	53	72	47	49	37	30	40	48	33	38	46	25	39	83
	σ_7	32	24	50	20	23	16	11	18	26	12	17	25	6	20	64
	Mean	80	74	87	71	73	72	69	73	77	70	73	77	65	74	92
M7	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_5	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_7	100	99	100	97	99	99	99	99	99	98	99	99	98	99	100
	Mean	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
Total mean		90	88	94	87	88	88	86	88	90	87	88	89	85	89	95

Table 12: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in models M1 and M2 with $n_i = 10, i = 1, 2, 3$ in the Wiener case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M1	σ_1	7	8	23	4	7	6	7	9	9	7	7	7	5	9	20
	σ_2	4	5	13	2	3	4	6	6	10	9	5	5	2	7	28
	σ_3	5	6	22	4	5	4	6	7	9	8	5	5	4	5	24
	σ_4	5	8	30	4	7	4	8	7	8	7	6	6	2	7	20
	σ_5	5	8	28	5	7	4	9	8	9	8	7	7	3	10	19
	σ_6	7	6	25	4	5	6	9	9	9	9	7	8	4	9	23
	σ_7	5	7	23	6	6	4	6	5	7	6	5	5	3	6	19
	Mean	5	7	23	4	6	5	7	7	9	8	6	6	3	8	22
M2	σ_1	7	4	22	2	3	6	8	9	9	7	7	7	5	9	20
	σ_2	4	3	14	4	4	4	6	6	10	9	5	5	2	7	28
	σ_3	5	7	24	4	5	4	6	7	9	8	5	5	4	5	24
	σ_4	5	8	30	4	7	4	7	7	8	7	6	6	2	7	20
	σ_5	5	8	28	5	7	4	8	8	9	8	7	7	3	10	19
	σ_6	7	6	25	4	5	6	8	9	9	9	7	8	4	9	23
	σ_7	5	7	23	6	6	4	6	5	7	6	5	5	3	6	19
	Mean	5	6	24	4	5	5	7	7	9	8	6	6	3	8	22
Total mean		5	7	24	4	6	5	7	7	9	8	6	6	3	8	22

Table 13: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in models M1 and M2 with $n_i = 20, i = 1, 2, 3$ in the Wiener case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^{2}N	L^{2}B	L^{2}b	FN	FB	Fb	GPF	FL
M1	σ_1	5	10	24	7	8	3	5	4	8	4	4	4	3	4	17
	σ_2	5	6	18	6	5	6	7	7	7	7	6	6	3	4	19
	σ_3	1	3	21	3	3	1	2	1	4	1	1	1	1	1	18
	σ_4	1	7	20	5	6	1	3	3	4	2	1	2	1	5	15
	σ_5	1	4	16	4	4	1	2	1	3	2	1	1	1	2	19
	σ_6	6	5	17	6	5	6	6	6	6	6	6	6	6	7	20
	σ_7	6	4	10	4	3	6	8	8	10	8	7	7	6	7	21
	Mean	4	6	18	5	5	3	5	4	6	4	4	4	3	4	18
M2	σ_1	5	7	15	6	7	3	5	4	8	4	4	4	3	4	17
	σ_2	5	3	13	3	2	6	7	7	7	7	6	6	3	4	19
	σ_3	1	3	21	3	3	1	2	1	4	1	1	1	1	1	18
	σ_4	1	7	20	5	6	1	3	3	4	2	1	2	1	5	15
	σ_5	1	4	16	4	4	1	2	1	3	2	1	1	1	2	19
	σ_6	6	5	17	6	5	6	6	6	6	6	6	6	6	7	20
	σ_7	6	4	10	4	3	6	8	8	10	8	7	7	6	7	21
	Mean	4	5	16	4	4	3	5	4	6	4	4	4	3	4	18
Total mean		4	5	17	5	5	3	5	4	6	4	4	4	3	4	18

Table 14: Empirical sizes (as percentages) of all tests for the one-way ANOVA problem obtained in models M1 and M2 with $n_i = 30, i = 1, 2, 3$ in the Wiener case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^{2}N	L^{2}B	L^{2}b	FN	FB	Fb	GPF	FL
M1	σ_1	6	3	13	3	3	6	7	7	7	7	7	7	7	6	17
	σ_2	6	0	15	0	0	5	5	6	6	5	5	5	5	7	13
	σ_3	12	5	21	6	6	9	12	12	12	12	12	12	8	10	19
	σ_4	5	3	18	3	3	4	4	5	6	5	5	5	3	7	11
	σ_5	6	6	19	4	5	5	6	6	6	5	6	6	5	7	15
	σ_6	5	5	22	7	6	4	4	5	5	4	4	4	4	5	12
	σ_7	3	2	14	3	2	3	3	3	3	3	3	3	3	4	19
	Mean	6	3	17	4	4	5	6	6	6	6	6	6	5	7	15
M2	σ_1	6	4	17	2	4	6	7	7	7	7	7	7	7	6	17
	σ_2	6	2	16	1	2	5	5	6	6	5	5	5	5	7	13
	σ_3	12	6	22	6	6	9	12	12	12	12	12	12	8	10	19
	σ_4	5	3	18	3	3	4	4	5	6	5	5	5	3	7	11
	σ_5	6	6	19	4	5	5	6	6	6	5	6	6	5	7	15
	σ_6	5	5	22	7	6	4	4	5	5	4	4	4	4	5	12
	σ_7	3	2	14	3	2	3	3	3	3	3	3	3	3	4	19
	Mean	6	4	18	4	4	5	6	6	6	6	6	6	5	7	15
Total mean		6	4	18	4	4	5	6	6	6	6	6	6	5	7	15

Table 15: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in models M3–M6 with $n_i = 10, i = 1, 2, 3$ in the Wiener case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M3	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	92	86	99	51	82	90	91	92	94	91	87	88	70	100	100
	σ_3	25	36	71	25	33	24	23	26	32	26	23	25	10	80	92
	σ_4	7	14	54	13	15	5	8	9	13	9	8	9	3	43	65
	σ_5	12	4	33	9	6	9	11	13	14	12	11	12	6	21	43
	σ_6	5	12	33	9	10	2	6	5	6	7	3	5	2	12	33
	σ_7	6	5	37	8	5	6	6	7	8	9	6	6	2	13	26
	Mean	35	37	61	31	36	34	35	36	38	36	34	35	28	53	66
M4	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	66	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	54	100	100	100	100	100	100	100	100	100	100	100
	σ_4	92	100	100	38	100	89	90	94	95	90	86	88	71	100	100
	σ_5	60	100	100	43	100	54	59	63	73	62	51	53	32	100	100
	σ_6	31	98	100	48	94	27	31	33	43	34	28	30	20	100	100
	σ_7	8	94	100	47	87	7	11	11	20	12	6	6	4	99	100
	Mean	70	99	100	57	97	68	70	72	76	71	67	68	61	100	100
M5	σ_1	100	100	100	69	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	68	100	100	100	100	100	100	100	100	100	100	100
	σ_3	93	100	100	48	97	92	93	94	94	93	92	92	79	99	100
	σ_4	49	99	100	41	96	47	53	53	60	53	50	51	37	83	100
	σ_5	32	74	98	32	69	30	35	36	38	35	31	33	24	57	100
	σ_6	15	67	92	41	60	16	20	19	22	20	18	18	12	40	99
	σ_7	15	54	86	34	48	12	17	19	20	19	15	15	10	25	98
	Mean	58	85	97	48	81	57	60	60	62	60	58	58	52	72	100
M6	σ_1	100	100	100	74	100	100	100	100	100	100	100	100	100	100	100
	σ_2	86	100	100	51	99	82	86	88	89	90	82	82	71	100	100
	σ_3	31	80	100	38	69	27	32	33	35	32	28	31	22	60	100
	σ_4	7	41	74	21	32	4	8	8	13	8	7	7	1	20	99
	σ_5	11	29	69	29	31	10	13	15	17	14	12	12	8	21	84
	σ_6	12	15	55	18	19	12	13	13	13	12	12	12	8	20	68
	σ_7	4	18	51	19	19	4	5	5	8	6	5	5	1	9	43
	Mean	36	55	78	36	53	34	37	37	39	37	35	36	30	47	85
Total mean		50	69	84	43	67	48	50	51	54	51	49	49	43	68	88

Table 16: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in models M3–M6 with $n_i = 20, i = 1, 2, 3$ in the Wiener case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^2N	L^2B	L^2b	FN	FB	Fb	GPF	FL
M3	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	50	100	100	100	100	47	50	51	55	50	48	49	39	99	100
	σ_4	20	93	98	76	86	18	21	20	21	22	19	19	15	78	86
	σ_5	8	67	87	58	66	13	12	14	14	13	10	11	5	52	50
	σ_6	5	40	68	29	32	4	5	5	5	4	5	5	3	24	40
	σ_7	9	34	63	26	27	8	8	10	10	10	8	8	8	19	30
	Mean	42	76	88	70	73	41	42	43	44	43	41	42	39	67	72
M4	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_5	98	100	100	100	100	98	98	98	98	98	98	98	95	100	100
	σ_6	87	100	100	100	100	85	86	87	91	86	82	82	74	100	100
	σ_7	48	100	100	100	100	45	46	47	50	46	45	45	39	100	100
	Mean	90	100	100	100	100	90	90	90	91	90	89	89	87	100	100
M5	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	90	100	100	100	100	91	91	93	93	92	89	90	85	100	100
	σ_5	57	100	100	100	100	53	59	60	61	57	54	57	49	95	100
	σ_6	42	100	100	100	100	41	44	44	44	45	44	44	38	77	100
	σ_7	27	100	100	100	100	27	28	29	32	28	27	27	25	51	100
	Mean	74	100	100	100	100	73	75	75	76	75	73	74	71	89	100
M6	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	99	100	100	100	100	99	99	99	99	99	99	99	99	99	100
	σ_3	51	100	100	100	100	48	54	53	56	52	50	51	47	88	100
	σ_4	30	100	100	100	100	30	32	31	33	32	31	31	25	45	100
	σ_5	17	100	100	91	99	18	17	18	19	19	17	18	16	27	99
	σ_6	12	97	100	85	93	10	14	14	15	14	13	14	11	25	93
	σ_7	9	85	96	65	78	9	9	9	10	9	9	9	8	13	79
	Mean	45	97	99	92	96	45	46	46	47	46	46	46	44	57	96
Total mean		63	93	97	90	92	62	63	64	65	63	62	63	60	78	92

Table 17: Empirical powers (as percentages) of all tests for the one-way ANOVA problem obtained in models M3–M6 with $n_i = 30, i = 1, 2, 3$ in the Wiener case.

Model		Test														
		FP	LH	R	P	W	CH	CS	L^{2}N	L^{2}B	L^{2}b	FN	FB	Fb	GPF	FL
M3	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	94	100	100	100	100	92	92	92	93	93	91	91	89	100	100
	σ_4	33	100	100	100	100	30	31	33	36	32	31	31	28	97	98
	σ_5	14	97	100	87	96	12	14	15	16	14	13	13	11	69	86
	σ_6	16	78	94	68	76	16	18	18	18	18	18	18	16	50	55
	σ_7	6	56	83	50	51	6	6	6	7	6	6	6	5	28	37
	Mean	52	90	97	86	89	51	52	52	53	52	51	51	50	78	82
M4	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_5	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_7	90	100	100	100	100	91	89	90	92	90	90	90	87	100	100
	Mean	99	100	100	100	100	99	98	99	99	99	99	99	98	100	100
M5	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_4	99	100	100	100	100	99	99	99	100	100	99	99	99	100	100
	σ_5	87	100	100	100	100	87	87	89	89	87	88	88	85	100	100
	σ_6	60	100	100	100	100	60	62	62	63	63	62	62	60	91	100
	σ_7	46	100	100	100	100	44	45	48	49	47	44	45	43	72	100
	Mean	85	100	100	100	100	84	85	85	86	85	85	85	84	95	100
M6	σ_1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	σ_3	84	100	100	100	100	82	84	85	86	81	82	82	81	99	100
	σ_4	37	100	100	100	100	34	37	36	36	36	35	36	34	68	100
	σ_5	26	100	100	100	100	25	26	27	27	27	25	25	24	43	100
	σ_6	13	100	100	100	100	12	14	14	14	13	13	13	11	24	100
	σ_7	17	100	100	98	99	18	18	19	19	18	18	18	17	29	93
	Mean	54	100	100	100	100	53	54	54	55	54	53	53	52	66	99
Total mean		72	98	99	97	97	72	72	73	73	72	72	72	71	85	95

Table 18: Results of the Nemenyi post hoc test for the results of the simulations S1 and S2, the models M1 and M2, and the models M3–M7 with $n_i = 20, i = 1, 2, 3$. The critical values of this test given in (7) for the results of the simulation S1, the simulation S2, the models M1 and M2 in each case, the models M3–M7 in the normal case, and the models M3–M6 in the Wiener case with $n_i = 20, i = 1, 2, 3$ are equal to 5.732225, 5.732225, 5.732225, 3.625378 and 4.053295 respectively.

Empirical size (S1)			Empirical power (S2)		
Test	Ranks mean	Homogeneous groups	Test	Ranks mean	Homogeneous groups
R	14.50	a	FL	12.75	a
FL	14.36	a	GPF	11.39	a b
L^2B	9.75	a b	L^2B	10.14	a b c
LH	9.46	a b	FB	9.64	a b c
L^2N	8.46	b c	FP	9.18	a b c
FB	8.46	b c	L^2N	9.07	a b c
FP	8.43	b c d	R	8.71	a b c
FN	7.89	b c d	FN	8.61	a b c d
W	7.18	b c d	CH	7.86	a b c d
CS	6.96	b c d	L^2b	6.86	b e d
GPF	6.96	b c d	CS	6.75	b c d
L^2b	6.61	b c d	Fb	5.82	b c d
CH	5.04	b c d	LH	5.64	c d
P	3.21	c d	W	4.61	c d
Fb	2.71	d	P	2.96	d

Empirical size (M1, M2 in normal case)			Empirical power (M3–M7 in normal case)		
Test	Ranks mean	Homogeneous groups	Test	Ranks mean	Homogeneous groups
R	15.00	a	FL	11.14	a
FL	13.86	a	R	10.51	a b
L^2B	11.00	a b	FP	9.96	a b c
LH	10.89	a b	L^2B	9.76	a b c d
P	10.46	a b	FB	9.54	a b c d
W	9.86	a b	L^2N	8.14	a b c d e
FB	9.71	a b c	GPF	8.10	a b c d e
FP	9.61	a b c	FN	7.80	a b c d e
GPF	7.71	b c d	CH	7.53	a b c d e
L^2N	5.75	b c d	LH	7.46	b c d e
CH	4.00	c d	L^2b	6.86	c d e
FN	4.00	c d	W	6.56	c d e
CS	2.86	d	CS	6.30	d e
L^2b	2.86	d	Fb	5.19	e
Fb	2.43	d	P	5.16	e

Empirical size (M1, M2 in Wiener case)			Empirical power (M3–M6 in Wiener case)		
Test	Ranks mean	Homogeneous groups	Test	Ranks mean	Homogeneous groups
FL	14.64	a	R	11.63	a
R	14.29	a	LH	11.39	a
L^2B	10.96	a b	FL	11.00	a b
CS	9.00	a b c	W	10.98	a b
P	8.36	b c	P	10.64	a b c
LH	8.18	b c	GPF	9.55	a b c d
L^2b	7.57	b c	L^2B	8.05	a b c d e
GPF	7.50	b c	L^2N	7.21	b c d e f
W	7.32	b c	L^2b	6.96	b c d e f
L^2N	7.21	b c	CS	6.61	c d e f
FB	5.93	b c	FP	5.91	d e f
FN	5.50	b c	FB	5.80	d e f
FP	5.18	c	FN	5.36	e f
CH	4.57	c	CH	5.13	e f
Fb	3.79	c	Fb	3.77	f

Table 19: Results of the Nemenyi post hoc test for the results of the simulations S1 and S2, the models M1 and M2, and the models M3–M7 with $n_i = 30, i = 1, 2, 3$. The critical values of this test given in (7) for the results of the simulation S1, the simulation S2, the models M1 and M2 in each case, the models M3–M7 in the normal case, and the models M3–M6 in the Wiener case with $n_i = 30, i = 1, 2, 3$ are equal to 7.583021, 5.732225, 5.732225, 3.625378 and 4.053295 respectively.

Empirical size (S1)				
Test	Ranks mean	Homogeneous groups		
FL	14.75	a		
R	14.25	a	b	
L^2B	10.00	a	b	c
FP	9.63	a	b	c
L^2N	9.25	a	b	c
FN	9.25	a	b	c
FB	9.25	a	b	c
GPF	7.44	a	b	c
LH	7.13		b	c
CS	6.31			c
L^2b	6.00			c
CH	5.88			c
W	4.44			c
P	3.25			c
Fb	3.19			c

Empirical power (S2)			
Test	Ranks mean	Homogeneous groups	
FL	11.50	a	
GPF	9.64	a	
L^2B	9.57	a	b
R	9.39	a	b
L^2N	8.64	a	b
FB	8.46	a	b
FN	8.25	a	b
FP	8.18	a	b
CH	8.07	a	b
LH	7.36	a	b
CS	7.18	a	b
L^2b	7.18	a	b
W	6.50	a	b
Fb	6.21	a	b
P	3.86		b

Empirical size (M1, M2 in normal case)							
Test	Ranks mean	Homogeneous groups					
R	15.00	a					
FL	13.71	a	b				
FP	11.75	a	b	c			
L^2B	10.57	a	b	c	d		
FB	9.71	a	b	c	d		
P	8.82		b	c	d	e	
W	8.21		b	c	d	e	
LH	8.14		b	c	d	e	
GPF	6.61			c	d	e	f
L^2N	6.57			c	d	e	f
FN	6.57			c	d	e	f
CH	5.14				d	e	f
L^2b	3.86					e	f
CS	3.75					e	f
Fb	1.57						f

Empirical power (M3–M7 in normal case)				
Test	Ranks mean	Homogeneous groups		
FL	10.04	a		
R	9.87	a	b	
FP	9.47	a	b	c
L^2B	9.00	a	b	c
FB	8.76	a	b	c
GPF	8.27	a	b	c
L^2N	8.01	a	b	c
LH	7.93	a	b	c
FN	7.76	a	b	c
CH	7.49	a	b	c
W	7.14	a	b	c
CS	6.84	a	b	c
L^2b	6.84	a	b	c
P	6.34		b	c
Fb	6.23			c

Empirical size (M1, M2 in Wiener case)						
Test	Ranks mean	Homogeneous groups				
R	14.75	a				
FL	14.25	a	b			
GPF	10.29	a	b	c		
L^2B	9.79	a	b	c	d	
L^2N	9.36	a	b	c	d	e
FP	8.64		b	c	d	e
FN	7.93			c	d	e
FB	7.93			c	d	e
CS	7.43			c	d	e
L^2b	7.14			c	d	e
Fb	5.36			c	d	e
CH	5.21			c	d	e
P	4.18				d	e
LH	4.00					e
W	3.75					e

Empirical power (M3–M6 in Wiener case)							
Test	Ranks mean	Homogeneous groups					
R	10.86	a					
LH	10.75	a					
W	10.59	a	b				
P	10.45	a	b	c			
FL	10.21	a	b	c	d		
GPF	9.39	a	b	c	d	e	
L^2B	8.23	a	b	c	d	e	f
L^2N	7.32	a	b	c	d	e	f
L^2b	6.96	a	b	c	d	e	f
CS	6.55		b	c	d	e	f
FP	6.45			c	d	e	f
FB	6.18				d	e	f
FN	6.02					e	f
CH	5.55					e	f
Fb	4.48						f

Table 20: The estimates of the parameters of the L^2N, L^2B, FN, FB and GPF tests for the Canadian temperature data and the orthosis data for $[a, b] = [0, 1]$ and $[a, b] = [0.8, 1]$.

Test	Canadian temp. data	Orthosis data $[0, 1]$	Orthosis data $[0.8, 1]$
L^2N	$(\hat{\beta}, \hat{d}) = (6235.9, 3.005)$	$(\hat{\beta}, \hat{d}) = (2404.2, 15.755)$	$(\hat{\beta}, \hat{d}) = (481.45, 8.2828)$
L^2B	$(\hat{\beta}, \hat{d}) = (5773.9, 3.121)$	$(\hat{\beta}, \hat{d}) = (2001.0, 18.763)$	$(\hat{\beta}, \hat{d}) = (433.16, 9.0394)$
FN	$(\hat{d}_1, \hat{d}_2) = (3.005, 48.09)$	$(\hat{d}_1, \hat{d}_2) = (15.76, 189.1)$	$(\hat{d}_1, \hat{d}_2) = (8.2828, 99.393)$
FB	$(\hat{d}_1, \hat{d}_2) = (3.121, 49.94)$	$(\hat{d}_1, \hat{d}_2) = (18.76, 225.2)$	$(\hat{d}_1, \hat{d}_2) = (9.0394, 108.47)$
GPF	$(\hat{\beta}_w, \hat{d}_w) = (0.271, 3.931)$	$(\hat{\beta}_w, \hat{d}_w) = (0.046, 23.10)$	$(\hat{\beta}_w, \hat{d}_w) = (0.1132, 9.358)$

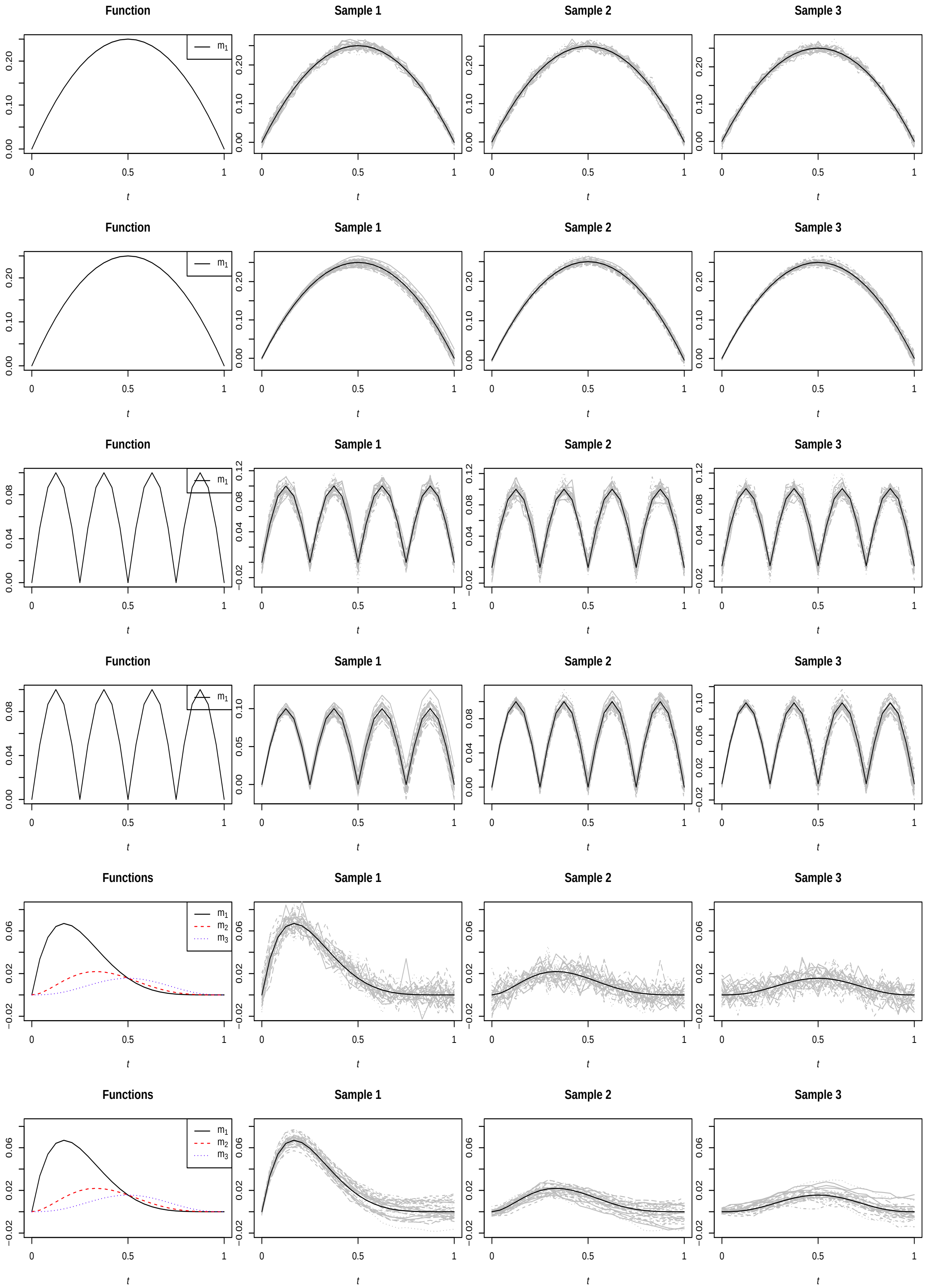

Figure 1: Artificial examples from models M1 (rows 1–2), M2 (rows 3–4) and M3 (rows 5–6) with $n_i = 30, i = 1, 2, 3$ and $\sigma_1 = 0.2/25$. In odd rows the errors are normal variables, and in even rows they are Wiener processes. The plots depict the trajectories (gray lines) and the m_i functions (black curve).

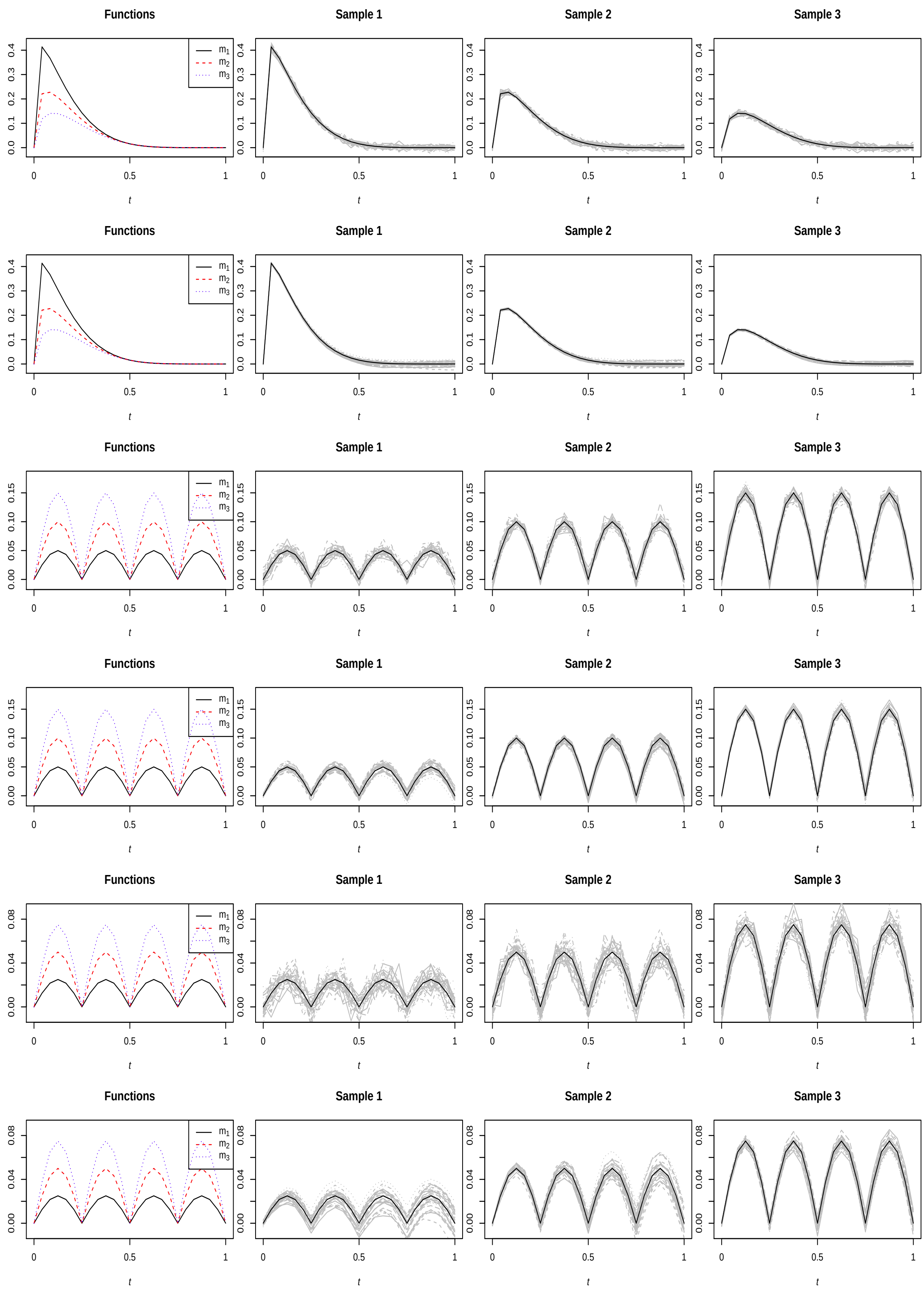

Figure 2: Artificial examples from models M4 (rows 1–2), M5 (rows 3–4) and M6 (rows 5–6) with $n_i = 30, i = 1, 2, 3$ and $\sigma_1 = 0.2/25$. In odd rows the errors are normal variables, and in even rows they are Wiener processes. The plots depict the trajectories (gray lines) and the m_i functions (black curve).