Supplementary Material:

Modelling and Diagnostic Tests for Poisson and Negative-binomial Count Time Series

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Abbreviations and Acronyms

(B)NB = (bivariate) negative binomial
(B)Poi = (bivariate) Poisson
CLT = central limit theorem
i. i. d. = independent and identically distributed
pgf = probability generating function
(I)INAR = (iterated-thinning) integer-valued autoregressive

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S.1 Proof of Equations (3) and (5)

According to Kocherlakota & Kocherlakota (2014),

$$\mu_{(r,s)} \;=\; \mu_{1,(r)}\,\mu_{2,(s)} \sum_{i=0}^{\min\{r,s\}} \binom{r}{i}\binom{s}{i}\, i!\, \left(\frac{\lambda_0}{(\lambda_0+\lambda_1)(\lambda_0+\lambda_2)}\right)^i$$

holds for the BPoi-distribution. But since the summand for $i=0$ in "$\sum_{i=0}^{\min\{r,s\}}\ldots$" equals 1, we can rewrite this formula as stated in (3).

According to Kocherlakota & Kocherlakota (2014), the BNB's joint factorial moments $\mu_{(k,l)}$ are given by

$$\mu_{(k,l)} \;=\; \mu_{1,(k)}\,\mu_{2,(l)} \sum_{i=0}^{\min\{k,l\}} \frac{\binom{n+k+l-i-1}{l-i}\binom{k}{i}}{\binom{n+l-1}{l}} \left(\frac{\pi_0\,(1-\pi_\bullet)}{(\pi_1+\pi_0)(\pi_2+\pi_0)}\right)^i. \tag{S.1}$$

Recall the Chu–Vandermonde identity

$$\binom{x+y}{l} \;=\; \sum_{i=0}^{l}\binom{x}{i}\binom{y}{l-i} \quad \text{for any } x,y\in\mathbb{R} \text{ and } l\in\mathbb{N},$$

as well as the well-known transformation

$$\binom{-y}{l} \;=\; (-1)^l\binom{y+l-1}{l}.$$

Using these equations, we derive

$$\begin{aligned}
\binom{x+y+l-1}{l} &= (-1)^l\binom{-x-y}{l} = (-1)^l\sum_{i=0}^{l}\binom{-x}{l-i}\binom{-y}{i}\\
&= (-1)^l\sum_{i=0}^{l}(-1)^{l-i}\binom{x+l-i-1}{l-i}\binom{-y}{i}\\
&= \sum_{i=0}^{l}(-1)^i\binom{x+l-i-1}{l-i}\binom{-y}{i},
\end{aligned}$$

where in the second-last step, we used a back transformation. By choosing $x=n+k$ and $y=-k$, we get

$$\binom{n+l-1}{l} \;=\; \sum_{i=0}^{\min\{k,l\}}(-1)^i\binom{n+k+l-i-1}{l-i}\binom{k}{i}. \tag{S.2}$$

Applying Equation (S.2) to (S.1), we get

$$\begin{aligned}
\mu_{(k,l)}(h)-\mu_{1,(k)}\,\mu_{2,(l)} &= \mu_{1,(k)}\,\mu_{2,(l)}\sum_{i=0}^{\min\{k,l\}}\frac{\binom{n+k+l-i-1}{l-i}\binom{k}{i}}{\binom{n+l-1}{l}}\left(\left(\frac{\pi_0\,(1-\pi_\bullet)}{(\pi_1+\pi_0)(\pi_2+\pi_0)}\right)^i-(-1)^i\right)\\
&= \mu_{1,(k)}\,\mu_{2,(l)}\sum_{i=1}^{\min\{k,l\}}\frac{\binom{n+k+l-i-1}{l-i}\binom{k}{i}}{\binom{n+l-1}{l}}\left(\left(\frac{\pi_0\,(1-\pi_\bullet)}{(\pi_1+\pi_0)(\pi_2+\pi_0)}\right)^i-(-1)^i\right).
\end{aligned}$$

This completes the proof of Equation (5).

S.2 Proof of Equations (9) and (16)

For $(X_t, X_{t-h}) \sim \text{BPoi}\big(\lambda(h),\, \lambda(h),\, \lambda_0(h)\big)$ with $\lambda(h) = \big(1-\rho(h)\big)\,\mu$ and $\lambda_0(h) = \rho(h)\,\mu$, we have $\frac{\lambda_0(h)}{(\lambda_0(h)+\lambda(h))^2} = \rho(h)/\mu$, so (9) immediately follows from (3).

For $(X_t, X_{t-h}) \sim \text{BNB}\big(n,\, \pi(h),\, \pi(h),\, \pi_0(h)\big)$ with $\pi(h), \pi_0(h)$ being of the form

$$\pi(h) = \frac{1-\rho(h)}{n/\mu+1-\rho(h)} \quad\text{and}\quad \pi_0(h) = \frac{n/(n+\mu)-1+\rho(h)}{n/\mu+1-\rho(h)},$$

we have

$$\pi_0(h)+\pi(h) \;=\; \frac{n/(n+\mu)}{n/\mu+1-\rho(h)}$$

and

$$1-\pi_0(h)-2\,\pi(h) \;=\; \frac{n/\mu-n/(n+\mu)}{n/\mu+1-\rho(h)} \;=\; \frac{n^2/\big(\mu\,(n+\mu)\big)}{n/\mu+1-\rho(h)}.$$

Thus, it follows that

$$\begin{aligned}
\frac{\pi_0(h)\,\big(1-\pi_0(h)-2\,\pi(h)\big)}{\big(\pi_0(h)+\pi(h)\big)^2} \;&=\; \frac{n^2\,\big(n/(n+\mu)-1+\rho(h)\big)}{\mu\,(n+\mu)\big(n/(n+\mu)\big)^2}\\
&=\; \frac{n+\mu}{\mu}\Big(\frac{n}{n+\mu}-1+\rho(h)\Big) \;=\; (1+n/\mu)\,\rho(h)-1.
\end{aligned}$$

Insertion into (5) together with the binomial sum formula yields

$$\begin{aligned}
\mu_{(k,l)}(h)-\mu_{(k)}\,\mu_{(l)} \;&=\; \mu_{(k)}\,\mu_{(l)} \sum_{i=1}^{\min\{k,l\}} \frac{\binom{n+k+l-i-1}{l-i}\binom{k}{i}}{\binom{n+l-1}{l}} \Big(\big((1+\tfrac{n}{\mu})\,\rho(h)-1\big)^i-(-1)^i\Big)\\
=\; \mu_{(k)}\,\mu_{(l)} &\sum_{i=1}^{\min\{k,l\}} \frac{\binom{n+k+l-i-1}{l-i}\binom{k}{i}}{\binom{n+l-1}{l}} \Big(\sum_{j=0}^{i} \tbinom{i}{j}\,(1+\tfrac{n}{\mu})^j\,\rho(h)^j\,(-1)^{i-j} \;-\; (-1)^i\Big) \qquad \text{(S.3)}\\
=\; \mu_{(k)}\,\mu_{(l)} &\sum_{i=1}^{\min\{k,l\}} \frac{\binom{n+k+l-i-1}{l-i}\binom{k}{i}}{\binom{n+l-1}{l}} \sum_{j=1}^{i} \tbinom{i}{j}\,(1+\tfrac{n}{\mu})^j\,\rho(h)^j\,(-1)^{i-j}.
\end{aligned}$$

In the particular case of the NB-IINAR(1) model according to Definition 2, we have $n/\mu = \alpha(1-\rho)$ and $\rho(h) = \rho^h$, which immediately yields (16).

S.3 Proof of Equation (15)

Al-Osh & Aly (1992) showed that

$$\begin{aligned}
\text{pgf}_{X_h}(z) \;&=\; \Big(\tfrac{\alpha(1-\rho)}{\alpha(1-\rho)+(1-\rho^h)\,(1-z)}\Big)^n \cdot \text{pgf}_{X_0}\Big(1-\tfrac{\alpha(1-\rho)\rho^h\,(1-z)}{\alpha(1-\rho)+(1-\rho^h)\,(1-z)}\Big) && \text{(S.4)}\\
&\underset{h\to\infty}{\longrightarrow}\; \Big(\tfrac{\alpha(1-\rho)}{\alpha(1-\rho)+1-z}\Big)^n. && \text{(S.5)}
\end{aligned}$$

Using the law of total expectation, we obtain the joint bivariate pgf of (X_t, X_{t-h}) as

$$\begin{aligned}
\mathrm{pgf}_{X_h,X_0}(1-z_1,1-z_2) \;&=\; E\big[(1-z_2)^{X_0}\, E[(1-z_1)^{X_h}\mid X_0]\big]\\
&\overset{(S.4)}{=} \Big(\tfrac{\alpha(1-\rho)}{\alpha(1-\rho)+(1-\rho^h)\,z_1}\Big)^n\cdot E\bigg[(1-z_2)^{X_0}\,\Big(1-\tfrac{\alpha(1-\rho)\rho^h\,z_1}{\alpha(1-\rho)+(1-\rho^h)\,z_1}\Big)^{X_0}\bigg]\\
&= \Big(\tfrac{\alpha(1-\rho)}{\alpha(1-\rho)+(1-\rho^h)\,z_1}\Big)^n\cdot \mathrm{pgf}_X\Big(1-z_2-\tfrac{\alpha(1-\rho)\rho^h\,z_1(1-z_2)}{\alpha(1-\rho)+(1-\rho^h)\,z_1}\Big)\\
&\overset{(S.5)}{=} \Big(\tfrac{\alpha(1-\rho)}{\alpha(1-\rho)+(1-\rho^h)\,z_1}\Big)^n\cdot\left(\tfrac{\alpha(1-\rho)}{\alpha(1-\rho)+z_2+\frac{\alpha(1-\rho)\rho^h\,z_1(1-z_2)}{\alpha(1-\rho)+(1-\rho^h)\,z_1}}\right)^n\\
&= \left(\tfrac{\alpha^2(1-\rho)^2}{\big(\alpha(1-\rho)+(1-\rho^h)\,z_1\big)\,\big(\alpha(1-\rho)+z_2\big)+\alpha(1-\rho)\rho^h\,z_1(1-z_2)}\right)^n\\
&= \left(\tfrac{\alpha^2(1-\rho)}{\alpha^2(1-\rho)+\alpha\,(z_1+z_2)+\frac{1-\rho^h}{1-\rho}\,z_1z_2-\alpha\rho^h\,z_1z_2}\right)^n\\
&= \left(\tfrac{\alpha^2(1-\rho)}{\alpha(1+\alpha-\alpha\rho)-\alpha\,(1-z_1)(1-z_2)+\frac{1-\rho^h}{1-\rho}\,(1+\alpha-\alpha\rho)\,z_1z_2}\right)^n.
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{pgf}_{X_h,X_0}(z_1,z_2) \;&=\; \left(\tfrac{\alpha^2(1-\rho)/(1+\alpha-\alpha\rho)}{\alpha-\alpha/(1+\alpha-\alpha\rho)\,z_1z_2+\frac{1-\rho^h}{1-\rho}\,(1-z_1-z_2+z_1z_2)}\right)^n\\
&= \left(\tfrac{\alpha^2(1-\rho)/(1+\alpha-\alpha\rho)}{\alpha+\frac{1-\rho^h}{1-\rho}-\frac{1-\rho^h}{1-\rho}\,(z_1+z_2)-\big(\alpha/(1+\alpha-\alpha\rho)-\frac{1-\rho^h}{1-\rho}\big)\,z_1z_2}\right)^n,
\end{aligned}$$

which completes the proof of (15).

S.4 Proof of Theorem 1

To derive the asymptotics of $\hat{\mathrm{T}}_{(r,s)}$ from (21), we define the function $\tau(u,v,w)=\frac{u}{v\,w}$ such that $\hat{\mathrm{T}}_{(r,s)}=\tau\big(\overline{X_{(r)}},\overline{X_{(s)}},\overline{X_{(r-s)}}\big)$ and $\mathrm{T}_{(r,s)}=\tau\big(\mu_{(r)},\mu_{(r-s)},\mu_{(s)}\big)$. Then, (19) implies the asymptotic normality of $\sqrt{T}\,\big(\hat{\mathrm{T}}_{(r,s)}-\mathrm{T}_{(r,s)}\big)$. To derive the asymptotic variance, we need the gradient of τ, which is given by

$$\mathrm{grad}\,\tau(u,v,w) \;=\; \left(\frac{1}{v\,w},-\frac{u}{v^2\,w},-\frac{u}{v\,w^2}\right).$$

Evaluated in $(\mu_{(r)},\mu_{(r-s)},\mu_{(s)})$, this leads to $\mathbf{D}=\mathrm{grad}\,\tau\big(\mu_{(r)},\mu_{(r-s)},\mu_{(s)}\big)$ as

$$\mathbf{D} \;=\; \left(\tfrac{1}{\mu_{(r-s)}\mu_{(s)}},-\tfrac{\mu_{(r)}}{\mu_{(r-s)}^2\mu_{(s)}},-\tfrac{\mu_{(r)}}{\mu_{(r-s)}\mu_{(s)}^2}\right).$$

Then, we calculate $\sigma^2_{\hat{\mathrm{T}}_{(r,s)}} = \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top$ with $\boldsymbol{\Sigma}$ following (22):

$$\begin{aligned}
\sigma^2_{\hat{\mathrm{T}}_{(r,s)}} &= \big(\tfrac{1}{\mu_{(r-s)}\mu_{(s)}}\big)^2\,\sigma_{(r,r)} \;-\; 2\,\tfrac{1}{\mu_{(r-s)}\mu_{(s)}}\,\tfrac{\mu_{(r)}}{\mu^2_{(r-s)}\mu_{(s)}}\,\sigma_{(r,r-s)}\\
&\quad + \big(\tfrac{\mu_{(r)}}{\mu^2_{(r-s)}\mu_{(s)}}\big)^2\,\sigma_{(r-s,r-s)} \;-\; 2\,\tfrac{1}{\mu_{(r-s)}\mu_{(s)}}\,\tfrac{\mu_{(r)}}{\mu_{(r-s)}\mu^2_{(s)}}\,\sigma_{(r,s)}\\
&\quad + \big(\tfrac{\mu_{(r)}}{\mu_{(r-s)}\mu^2_{(s)}}\big)^2\,\sigma_{(s,s)} \;+\; 2\,\tfrac{\mu_{(r)}}{\mu^2_{(r-s)}\mu_{(s)}}\,\tfrac{\mu_{(r)}}{\mu_{(r-s)}\mu^2_{(s)}}\,\sigma_{(r-s,s)}\\
&= \big(\tfrac{\mu_{(r)}}{\mu_{(r-s)}\mu_{(s)}}\big)^2\Big(A_{r,r} - 2A_{r,r-s} + A_{r-s,r-s} - 2A_{r,s} + A_{s,s} + 2A_{r-s,s}\Big).
\end{aligned}$$

For the bias correction according to (20), we derive the Hessian $\mathbf{H}_\tau(u,v,w)$, and evaluate it as $\mathbf{H} := \mathbf{H}_\tau\big(\mu_{(r)}, \mu_{(r-s)}, \mu_{(s)}\big)$. We get

$$\mathbf{H}_\tau(u,v,w) \;=\; \begin{pmatrix} 0 & -\frac{1}{v^2w} & -\frac{1}{vw^2}\\ -\frac{1}{v^2w} & \frac{2u}{v^3w} & \frac{u}{v^2w^2}\\ -\frac{1}{vw^2} & \frac{u}{v^2w^2} & \frac{2u}{vw^3} \end{pmatrix} \;=\; \frac{u}{vw}\begin{pmatrix} 0 & -\frac{1}{uv} & -\frac{1}{uw}\\ -\frac{1}{uv} & \frac{2}{v^2} & \frac{1}{vw}\\ -\frac{1}{uw} & \frac{1}{vw} & \frac{2}{w^2} \end{pmatrix},$$

hence

$$\mathbf{H} \;=\; \frac{\mu_{(r)}}{\mu_{(r-s)}\mu_{(s)}}\begin{pmatrix} 0 & -\frac{1}{\mu_{(r)}\mu_{(r-s)}} & -\frac{1}{\mu_{(r)}\mu_{(s)}}\\ -\frac{1}{\mu_{(r)}\mu_{(r-s)}} & \frac{2}{\mu^2_{(r-s)}} & \frac{1}{\mu_{(r-s)}\mu_{(s)}}\\ -\frac{1}{\mu_{(r)}\mu_{(s)}} & \frac{1}{\mu_{(r-s)}\mu_{(s)}} & \frac{2}{\mu^2_{(s)}} \end{pmatrix}.$$

So (20) leads to the approximate mean

$$\begin{aligned}
\mathrm{T}_{(r,s)} \;+\; \tfrac{1}{T}\,\tfrac{\mu_{(r)}}{\mu_{(r-s)}\mu_{(s)}}\Big(&\tfrac{1}{\mu^2_{(r-s)}}\,\sigma_{(r-s,r-s)} + \tfrac{1}{\mu^2_{(s)}}\,\sigma_{(s,s)}\\
&-\tfrac{1}{\mu_{(r)}\,\mu_{(r-s)}}\,\sigma_{(r,r-s)} - \tfrac{1}{\mu_{(r)}\,\mu_{(s)}}\,\sigma_{(r,s)} + \tfrac{1}{\mu_{(r-s)}\mu_{(s)}}\,\sigma_{(r-s,s)}\Big),
\end{aligned}$$

which completes the proof of Theorem 1.

S.5 Proof of Corollary 1

We have to compute $\sigma_{(k,l)}$ according (22), which implies the expression for $A_{k,l} = \sigma_{(k,l)}/(\mu_{(k)}\,\mu_{(l)})$. In (i), we consider the Poi-null. Plugging-in (9) into (22), we get

$$\begin{aligned}
\sigma_{(k,l)} &= \mu_{(k,l)}(0) - \mu_{(k)}\,\mu_{(l)} \;+\; 2\sum_{h=1}^{\infty}\big(\mu_{(k,l)}(h) - \mu_{(k)}\,\mu_{(l)}\big)\\
&= \mu_{(k)}\,\mu_{(l)}\sum_{i=1}^{\min\{k,l\}}\binom{k}{i}\binom{l}{i}\frac{i!}{\mu^i}\Big(1 + 2\sum_{h=1}^{\infty}\rho(h)^i\Big).
\end{aligned}$$

For the Poi-INAR(1) model, we have $\rho(h) = \rho^h$, leading to

$$1 + 2\sum_{h=1}^{\infty}\rho(h)^i \;=\; 1 + 2\sum_{h=1}^{\infty}(\rho^i)^h \;=\; \tfrac{1+\rho^i}{1-\rho^i},$$

which completes the proof of part (i).

For the NB-null considered in part (ii), we first insert the general expression (S.3) into (22) and get

$$\sigma_{(k,l)} \;=\; \mu_{(k)}\,\mu_{(l)} \sum_{i=1}^{\min\{k,l\}} \frac{\binom{n+k+l-i-1}{l-i}\binom{k}{i}}{\binom{n+l-1}{l}} \sum_{j=1}^{i} \tbinom{i}{j}\,(1+\tfrac{n}{\mu})^j\,(-1)^{i-j}\,\Big(1+2\,\textstyle\sum_{h=1}^{\infty}\rho(h)^j\Big).$$

In the particular case of the NB-IINAR(1) model according to Definition 2, we have $n/\mu = \alpha(1-\rho)$ and $\rho(h)=\rho^h$. So using the same argument as for the Poi-INAR(1) model, we complete the proof of Corollary 1.

S.6 Proof of Example 2

For the second-order statistic $\hat{\mathrm{T}}_{(2,1)}$ as considered by Kyriakoussis et al. (1998), Theorem 1 implies

$$\mu_{\hat{\mathrm{T}}_{(2,1)}} \;=\; \mathrm{T}_{(2,1)}\Big(1-\tfrac{1}{T}A_{1,1}\Big), \qquad \sigma^2_{\hat{\mathrm{T}}_{(2,1)}} \;=\; \mathrm{T}^2_{(2,1)}\Big(A_{2,2}+4A_{1,1}-4A_{2,1}\Big).$$

For the Poi-null, we use Corollary 1 (i) to compute

$$A_{1,1} \;=\; \tfrac{1}{\mu}\,\tfrac{1+\rho}{1-\rho}, \qquad A_{2,1} \;=\; 2\,A_{1,1}, \qquad A_{2,2} \;=\; 4\,A_{1,1}+\tfrac{2}{\mu^2}\,\tfrac{1+\rho^2}{1-\rho^2}.$$

Utilizing that $\mathrm{T}_{(2,1)}=1$ yields the formulae given in Example 2 (i).
For the NB-null, we get from Corollary 1 (ii) that

$$A_{1,1} \;=\; \tfrac{1+\alpha(1-\rho)}{n}\,\tfrac{1+\rho}{1-\rho}, \qquad A_{2,1} \;=\; 2\,A_{1,1}, \qquad A_{2,2} \;=\; 4\,A_{1,1}+\tfrac{2(1+\alpha(1-\rho))^2}{n(n+1)}\,\tfrac{1+\rho^2}{1-\rho^2}.$$

This is used to complete the proof of Example 2.

S.7 Proof of Theorems 2 and 3

To derive the asymptotics of the Stein–Chen statistic (26) for i. i. d. Poi-counts, Aleksandrov et al. (2022, Eq. (11)) defined the vectors

$$\boldsymbol{Y}_t \;=\; \Big(X_t-\mu,\ X_t\,f(X_t)-\mu\cdot\phi_1,\ f(X_t+1)-\phi_1\Big)^\top,$$

where $\mu=E[X_t]$ and $\phi_i := E[f(X_t+i)]$. They showed that

$$\sqrt{T}\,\overline{\boldsymbol{Y}} \ \xrightarrow{\mathrm{d}}\ \mathrm{N}(\mathbf{0},\boldsymbol{\Sigma}) \quad \text{with } \boldsymbol{\Sigma}=(\sigma_{i,j})_{i,j=1\ldots,3},$$

where $\overline{\boldsymbol{Y}} = \big(\overline{X}-\mu, \overline{X\,f(X)}-\mu\cdot\phi_1, \overline{f(X+1)}-\phi_1\big)^\top$, and where

$$\begin{aligned}
&\sigma_{11}=\mu, \qquad \sigma_{12}=\big(\phi_1+\mu\,(\phi_2-\phi_1)\big)\,\sigma_{11}, \qquad \sigma_{13}=(\phi_2-\phi_1)\,\sigma_{11},\\
&\sigma_{22}=\mu\phi_{11}(0)+\mu^2\phi_{22}(0)-\mu^2\phi_1^2, \quad \sigma_{23}=\mu(\phi_{12}(0)-\phi_1^2), \quad \sigma_{33}=\phi_{11}(0)-\phi_1^2,
\end{aligned} \tag{S.6}$$

with $\phi_{ij}(k) = E\Big[f(X_t+i)f(X_{t+k}+j)\Big]$, see Lemma 1 and Eq. (14) in Aleksandrov et al. (2022).

From now on, consider the choice $f(x) = \exp(-x)$, and let $\psi(u) = \exp\big(\mu\,(e^u-1)\big)$ denote the Poisson's mgf (recall Table 1). Then, the $\phi_i, \phi_{ij}(k)$ required for (S.6) are

$$\begin{aligned}
\phi_{11}(0) &= E\Big[\big(e^{-X_t-1}\big)^2\Big] = e^{-2}\psi(-2), & \phi_1 &= E\Big[e^{-X_t-1}\Big] = e^{-1}\psi(-1),\\
\phi_{12}(0) &= E\Big[e^{-X_t-1}e^{-X_t-2}\Big] = e^{-1}\phi_{11}(0), & \phi_2 &= E\Big[e^{-X_t-2}\Big] = e^{-1}\phi_1,\\
\phi_{22}(0) &= E\Big[\big(e^{-X_t-2}\big)^2\Big] = e^{-2}\phi_{11}(0).
\end{aligned}$$

Insertion into (S.6) yields

$$\begin{aligned}
&\sigma_{11} = \mu, \quad \sigma_{12} = e^{-1}\psi(-1)\big(1+\mu\,(e^{-1}-1)\big)\,\sigma_{11}, \quad \sigma_{13} = e^{-1}\psi(-1)\,(e^{-1}-1)\,\sigma_{11},\\
&\sigma_{22} = e^{-2}\mu\,(1+e^{-2}\mu)\,\psi(-2) - e^{-2}\mu^2\,\psi(-1)^2, \qquad\qquad \text{(S.7)}\\
&\sigma_{23} = e^{-2}\mu\big(e^{-1}\psi(-2)-\psi(-1)^2\big), \quad \sigma_{33} = e^{-2}\big(\psi(-2)-\psi(-1)^2\big).
\end{aligned}$$

In Theorem 1 of Aleksandrov et al. (2022), it was shown that the (asymptotic) bias-corrected mean and the variance of the statistic (26) are generally given by

$$\mu_{\hat{\mathrm{T}}_f^{\mathrm{Poi}}} \;=\; 1+\frac{1}{T}\,\frac{\mu\,\sigma_{33}-\sigma_{23}}{\mu\,\phi_1^2}, \qquad \sigma^2_{\hat{\mathrm{T}}_f^{\mathrm{Poi}}} \;=\; \frac{1}{\mu^2\,\phi_1^2}\Big(\sigma_{22}-2\mu\,\sigma_{23}+\mu^2\,\sigma_{33}\Big) - \frac{\sigma_{11}}{\mu^2}.$$

Insertion of (S.7) yields

$$\mu_{\hat{\mathrm{T}}_{\exp}^{\mathrm{Poi}}} \;=\; 1+\frac{1}{T}\,\frac{e^{-2}\mu\,(1-e^{-1})\,\psi(-2)}{\mu\cdot e^{-2}\psi(-1)^2} \;=\; 1+\frac{1}{T}\,\frac{\psi(-2)}{\psi(-1)^2}\,(1-e^{-1})$$

as well as

$$\begin{aligned}
\sigma^2_{\hat{\mathrm{T}}_{\exp}^{\mathrm{Poi}}} \;&=\; \frac{1}{\mu^2\cdot e^{-2}\psi(-1)^2}\Big(e^{-2}\mu\,(1+e^{-2}\mu)\,\psi(-2) - e^{-2}\mu^2\,\psi(-1)^2\\
&\qquad -2\mu\cdot e^{-2}\mu\big(e^{-1}\psi(-2)-\psi(-1)^2\big) + \mu^2\cdot e^{-2}\big(\psi(-2)-\psi(-1)^2\big)\Big) - \frac{1}{\mu}\\
&=\; \frac{\psi(-2)}{\psi(-1)^2}\Big(\frac{1}{\mu}+e^{-2}-2e^{-1}+1\Big) - \frac{1}{\mu}\\
&=\; \frac{\psi(-2)}{\psi(-1)^2}\Big(\frac{1}{\mu}+(1-e^{-1})^2\Big) - \frac{1}{\mu}.
\end{aligned}$$

Using that $\psi(-2)/\psi(-1)^2 = e^{\mu(e^{-2}-1)}/e^{2\mu(e^{-1}-1)} = e^{\mu(1-e^{-1})^2}$, this completes the proof of part (i) of Theorem 2.

For proving part (ii) of Theorem 2, we first analyze the mgf $\psi(u) = E[e^{uX}]$ of $\mathrm{NB}(n,\pi)$ with $\pi = \frac{n}{n+\mu}$, which is given by (recall Table 1)

$$\psi(u) \;=\; \Big(\frac{\pi}{1-(1-\pi)e^u}\Big)^n \;=\; \Big(\frac{n}{n+\mu\,(1-e^u)}\Big)^n. \qquad \text{(S.8)}$$

Its derivatives compute as

$$
\begin{aligned}
\psi'(u) \quad &= \quad \psi(u)\,\frac{n(1-\pi)e^u}{1-(1-\pi)e^u} \;=\; \psi(u)\,\frac{n\mu\, e^u}{n+\mu\,(1-e^u)}, && \text{(S.9)}\\
\psi''(u) \quad &= \quad \psi'(u)\,\frac{n(1-\pi)e^u}{1-(1-\pi)e^u} + \psi(u)\,\Big(\frac{n\,(1-\pi)e^u}{1-(1-\pi)e^u} + \frac{n\,(1-\pi)^2e^{2u}}{\big(1-(1-\pi)e^u\big)^2}\Big)\\
&\overset{\text{(S.9)}}{=} \quad \psi(u)\,\Big(\frac{n\,(1-\pi)e^u}{1-(1-\pi)e^u} + \frac{n\,(n+1)(1-\pi)^2e^{2u}}{\big(1-(1-\pi)e^u\big)^2}\Big) && \text{(S.10)}\\
&= \quad \psi(u)\,\frac{n\,(1-\pi)e^u\big(1+n\,(1-\pi)e^u\big)}{\big(1-(1-\pi)e^u\big)^2} \;=\; \psi(u)\,\frac{n\,\mu\, e^u\big(n+\mu+n\mu\, e^u\big)}{\big(n+\mu\,(1-e^u)\big)^2},
\end{aligned}
$$

confirming the results provided by Table 2. As a consequence, $E\big[X\,\exp(-X)\big] = \psi'(-1) = \psi(-1)\,\frac{n\,\mu\, e^{-1}}{n+\mu\,(1-e^{-1})}$ such that statistic $\mathrm{T}_f^{\mathrm{Poi}}$ from (27) takes the value

$$
\mathrm{T}_f^{\mathrm{Poi}} \;=\; \frac{E\big[X\,\exp(-X)\big]}{E[X]\cdot e^{-1}\,E\big[\exp(-X)\big]} \;=\; \frac{n}{n+\mu\,(1-e^{-1})},
$$

as stated in part (ii) of Theorem 2. Furthermore, for the vector $\boldsymbol{Z}_t := \big(X_t,\ X_t\,e^{-X_t},\ e^{-X_t-1}\big)^\top$, it follows that

$$
\mu_{\boldsymbol{Z}} := E[\boldsymbol{Z}_t] \;=\; \Big(E[X_t],\ E\big[X_t\,e^{-X_t}\big],\ E\big[\,e^{-X_t-1}\big]\Big)^\top \;=\; \Big(\mu,\ \psi'(-1),\ e^{-1}\psi(-1)\Big)^\top.
$$

Like before, let $\boldsymbol{Y}_t := \boldsymbol{Z}_t - \boldsymbol{\mu}_{\boldsymbol{Z}}$. Then, applying the CLT (17), we conclude that

$$
\sqrt{T}\,\overline{\boldsymbol{Y}} \;\overset{\mathrm{d}}{\to}\; \mathrm{N}(\boldsymbol{0},\boldsymbol{\Sigma}) \text{ with } \boldsymbol{\Sigma} = \big(\sigma_{ij}\big)_{i,j=1,2,3}, \tag{S.11}
$$

where we calculate

$$
\begin{aligned}
\sigma_{11} \;&=\; E[X_t^2] - E[X_t]^2 \;=\; \sigma^2 = \psi''(0) - \mu^2,\\
\sigma_{12} \;&=\; E[X_t^2e^{-X_t}] - E[X_t]E[X_te^{-X_t}] \;=\; \psi''(-1) - \mu\,\psi'(-1),\\
\sigma_{13} \;&=\; E[X_te^{-X_t-1}] - E[X_t]E[e^{-X_t-1}] \;=\; e^{-1}\psi'(-1) - e^{-1}\mu\,\psi(-1),\\
\sigma_{22} \;&=\; E[X_t^2e^{-2X_t}] - E[X_te^{-X_t}]^2 \;=\; \psi''(-2) - \psi'(-1)^2,\\
\sigma_{23} \;&=\; E[X_te^{-2X_t-1}] - E[X_te^{-X_t}]E[e^{-X_t-1}] \;=\; e^{-1}\psi'(-2) - e^{-1}\psi'(-1)\,\psi(-1),\\
\sigma_{33} \;&=\; E[e^{-2X_t-2}] - E[e^{-X_t-1}]^2 \;=\; e^{-2}\psi(-2) - e^{-2}\psi(-1)^2.
\end{aligned}
$$

For part (ii) of Theorem 2, we need to derive the asymptotics of $\hat{\mathrm{T}}_{\exp}^{\mathrm{Poi}}$ from (26) under the NB-null, which follows from applying the Delta method to $\tau(\overline{\boldsymbol{Z}})$ with $\tau(u,v,w) = \frac{v}{u\,w}$, also see Appendix A.2 in Aleksandrov et al. (2022). Except exchanging $u \leftrightarrow v$, this is the same function as in Appendix S.4. Thus, gradient and Hessian are known as

$$
\operatorname{grad}\tau(u,v,w) \;=\; \Big(-\frac{v}{u^2\,w},\frac{1}{u\,w},-\frac{v}{u\,w^2}\Big)
$$

and

$$\mathbf{H}_\tau(u,v,w) \;=\; \frac{v}{uw}\begin{pmatrix} \frac{2}{u^2} & -\frac{1}{uv} & \frac{1}{uw} \\ -\frac{1}{uv} & 0 & -\frac{1}{vw} \\ \frac{1}{uw} & -\frac{1}{vw} & \frac{2}{w^2} \end{pmatrix},$$

respectively. Evaluating $\mathbf{D} := \operatorname{grad}\tau\big(\boldsymbol{\mu}_{\boldsymbol{Z}}\big)$, we get

$$\mathbf{D} \;=\; \tfrac{e}{\mu\,\psi(-1)}\Big(-\tfrac{\psi'(-1)}{\mu},\; 1,\; -\tfrac{e\,\psi'(-1)}{\psi(-1)}\Big), \tag{S.12}$$

which further simplifies in the NB-case by using (S.9). We calculate the variance of $\sqrt{T}\,\big(\hat{\mathrm{T}}^{\mathrm{Poi}}_{\exp} - \mathrm{T}^{\mathrm{Poi}}_{\exp}\big)$ as $\sigma^2_{\hat{\mathrm{T}}^{\mathrm{Poi}}_{\exp}} = \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top$, with $\boldsymbol{\Sigma}$ taken from Equation (S.11):

$$\begin{aligned}
\sigma^2_{\hat{\mathrm{T}}_f} \;=\;& \Big(\tfrac{e}{\mu\,\psi(-1)}\Big)^2\Bigg(\Big(\tfrac{\psi'(-1)}{\mu}\Big)^2\sigma_{11} - 2\Big(\tfrac{\psi'(-1)}{\mu}\Big)\sigma_{12} + \sigma_{22} + 2\Big(\tfrac{\psi'(-1)}{\mu}\Big)\Big(\tfrac{e\,\psi'(-1)}{\psi(-1)}\Big)\sigma_{13} \\
&-2\Big(\tfrac{e\,\psi'(-1)}{\psi(-1)}\Big)\sigma_{23} + \Big(\tfrac{e\,\psi'(-1)}{\psi(-1)}\Big)^2\sigma_{33}\Bigg) \\
=\;& \Big(\tfrac{e}{\mu\,\psi(-1)}\Big)^2\Bigg(\Big(\tfrac{\psi'(-1)}{\mu}\Big)^2\sigma^2 - 2\Big(\tfrac{\psi'(-1)}{\mu}\Big)\big(\psi''(-1) - \mu\,\psi'(-1)\big) \\
&+\psi''(-2) - \psi'(-1)^2 + 2\Big(\tfrac{\psi'(-1)^2}{\psi(-1)\mu}\Big)\big(\psi'(-1) - \mu\,\psi(-1)\big) \\
&-2\Big(\tfrac{\psi'(-1)}{\psi(-1)}\Big)\big(\psi'(-2) - \psi'(-1)\,\psi(-1)\big) + \Big(\tfrac{\psi'(-1)}{\psi(-1)}\Big)^2\big(\psi(-2) - \psi(-1)^2\big)\Bigg) \\
=\;& \Big(\tfrac{e}{\mu\,\psi(-1)}\Big)^2\Bigg(\Big(\tfrac{\psi'(-1)}{\mu}\Big)^2\sigma^2 - 2\Big(\tfrac{\psi'(-1)}{\mu}\Big)\psi''(-1) + \psi''(-2) \\
&+2\Big(\tfrac{\psi'(-1)^2}{\psi(-1)\mu}\Big)\psi'(-1) - 2\Big(\tfrac{\psi'(-1)}{\psi(-1)}\Big)\psi'(-2) + \Big(\tfrac{\psi'(-1)}{\psi(-1)}\Big)^2\psi(-2)\Bigg) \\
=\;& \Big(\tfrac{e}{\mu\,\psi(-1)}\Big)^2\Big(\tfrac{\psi'(-1)^2\,\sigma^2}{\mu^2} - 2\,\tfrac{\psi'(-1)\,\psi''(-1)}{\mu} + \psi''(-2) + 2\,\tfrac{\psi'(-1)^3}{\psi(-1)\mu} \\
&-2\,\tfrac{\psi'(-1)\,\psi'(-2)}{\psi(-1)} + \tfrac{\psi'(-1)^2\,\psi(-2)}{\psi(-1)^2}\Big).
\end{aligned}$$

In the NB-case, this can be further simplified by using that $\sigma^2 = \frac{(n+\mu)\mu}{n}$.

$$\begin{aligned}
\sigma^2_{\hat{\mathrm{T}}^{\mathrm{Poi}}_{\exp}} \;=\;& \Big(\tfrac{e}{\mu\,\psi(-1)}\Big)^2\Big(\tfrac{\psi'(-1)^2(n+\mu)}{n\,\mu} + \tfrac{2}{\mu}\,\tfrac{\psi'(-1)}{\psi(-1)}\big(\psi'(-1)^2 - \psi(-1)\psi''(-1)\big) \\
&+\tfrac{\psi'(-1)^2}{\psi(-1)^2}\,\psi(-2) - 2\,\tfrac{\psi'(-1)}{\psi(-1)}\,\psi'(-2) + \psi''(-2)\Big).
\end{aligned}$$

To compute the approximate bias correction according to (20), we first need to evaluate

$\mathbf{H} := \mathbf{H}_\tau(\boldsymbol{\mu}_{\boldsymbol{Z}})$. We get

$$\mathbf{H} \;=\; \tfrac{e}{\mu\,\psi(-1)} \begin{pmatrix} \frac{2\psi'(-1)}{\mu^2} & -\frac{1}{\mu} & \frac{e\psi'(-1)}{\mu\psi(-1)} \\ -\frac{1}{\mu} & 0 & -\frac{e}{\psi(-1)} \\ \frac{e\psi'(-1)}{\mu\psi(-1)} & -\frac{e}{\psi(-1)} & \frac{2e^2\psi'(-1)}{\psi(-1)^2} \end{pmatrix}.$$

This leads to the bias correction

$$\begin{aligned}
&\tfrac{1}{T}\left(\tfrac{1}{2}\,h_{11}\,\sigma_{11} + \tfrac{1}{2}\,h_{33}\,\sigma_{33} + h_{12}\,\sigma_{12} + h_{13}\,\sigma_{13} + h_{23}\,\sigma_{23}\right)\\
&= \tfrac{1}{T}\,\tfrac{e}{\mu\,\psi(-1)}\left(\tfrac{\psi'(-1)\,\sigma_{11}}{\mu^2} - \tfrac{\sigma_{12}}{\mu} + \tfrac{e\,\psi'(-1)\,\sigma_{13}}{\mu\psi(-1)} - \tfrac{e\,\sigma_{23}}{\psi(-1)} + \tfrac{e^2\psi'(-1)\,\sigma_{33}}{\psi(-1)^2}\right)\\
&= \tfrac{1}{T}\,\tfrac{e}{\mu\,\psi(-1)}\left(\tfrac{\psi'(-1)\,\sigma^2}{\mu^2} - \tfrac{\psi''(-1)-\mu\,\psi'(-1)}{\mu} + \tfrac{\psi'(-1)\,\left(\psi'(-1)-\mu\,\psi(-1)\right)}{\mu\psi(-1)}\right.\\
&\qquad \left. - \tfrac{\psi'(-2)-\psi'(-1)\,\psi(-1)}{\psi(-1)} + \tfrac{\psi'(-1)\,\left(\psi(-2)-\psi(-1)^2\right)}{\psi(-1)^2}\right)\\
&= \tfrac{1}{T}\,\tfrac{e}{\mu\,\psi(-1)}\left(\tfrac{\psi'(-1)\,\sigma^2}{\mu^2} - \tfrac{\psi''(-1)}{\mu} + \tfrac{\psi'(-1)^2}{\mu\psi(-1)} - \tfrac{\psi'(-2)}{\psi(-1)} + \tfrac{\psi'(-1)\psi(-2)}{\psi(-1)^2}\right).
\end{aligned}$$

In the NB-case, it further simplifies by using that $\sigma^2 = \frac{(n+\mu)\mu}{n}$ to

$$\begin{aligned}
\mu_{\hat{\mathrm{T}}^{\mathrm{Poi}}_{\exp}} \;&=\; \tfrac{n}{n+\mu\,(1-e^{-1})} + \tfrac{1}{T}\,\tfrac{e}{\mu\,\psi(-1)}\,\cdot\\
&\left(\tfrac{\psi'(-1)(n+\mu)}{n\,\mu} + \tfrac{\psi'(-1)^2}{\mu\psi(-1)} - \tfrac{\psi''(-1)}{\mu} - \tfrac{\psi'(-2)}{\psi(-1)} + \tfrac{\psi'(-1)\psi(-2)}{\psi(-1)^2}\right).
\end{aligned}$$

This completes the proof of part (ii) of Theorem 2.

Next, we derive the asymptotics for (29) if $f(x) = \exp(-x)$. Note that we can rewrite

$$\mathrm{T}^{\mathrm{NB}}_{\exp} \;=\; \frac{n_0 + E[X]}{E[X]}\,\frac{E\big[X\,\exp(-X)\big]}{n_0\,E\big[\exp(-X-1)\big] + e^{-1}\,E\big[X\,\exp(-X)\big]}.$$

Thus, for being able to apply the Delta method to $\sqrt{T}\,\overline{\boldsymbol{Y}}$, we define the function $\tau(u,v,w) = \frac{(n_0+u)v}{u(n_0 w+e^{-1}v)}$ such that

$$\mathrm{T}^{\mathrm{NB}}_{\exp} = \tau\big(\mu,\ \mu\,\phi_1,\ \phi_1\big) \;=\; \frac{(n_0+\mu)\,\mu\,\phi_1}{\mu\,\left(n_0\,\phi_1 + e^{-1}\mu\,\phi_1\right)} \;=\; \frac{n_0+\mu}{n_0 + e^{-1}\mu}$$

under the Poi(μ)-assumption being considered in part (i) in Theorem 3. Then, (19) implies the asymptotic normality of $\sqrt{T}\,\big(\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp} - \mathrm{T}^{\mathrm{NB}}_{\exp}\big)$. To derive the asymptotic variance, we need the gradient of τ, which is given by

$$\operatorname{grad}\tau(u,v,w) \;=\; \left(-\frac{e n_0 v}{u^2(v+e n_0 w)},\ \frac{e^2 n_0 w(n_0+u)}{u(v+e n_0 w)^2},\ -\frac{e^2 n_0 v(n_0+u)}{u(v+e n_0 w)^2}\right)^{\top}.$$

Evaluated in $(\mu,\ \mu\,\phi_1,\ \phi_1)$, this leads to $\mathbf{D} = \operatorname{grad}\boldsymbol{\tau}(\mu,\ \mu\,\phi_1,\ \phi_1)$ as

$$\mathbf{D} \ = \ \left(-\frac{en_0}{\mu(en_0+\mu)},\ \frac{e^2n_0(n_0+\mu)}{\mu\phi_1(en_0+\mu)^2},\ -\frac{e^2n_0(n_0+\mu)}{\phi_1(en_0+\mu)^2}\right).$$

Then, we calculate $\sigma^2_{\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp}} = \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top$ as:

$$\begin{aligned}
\sigma^2_{\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp}} &= d_1^2\sigma_{11} + d_2^2\sigma_{22} + d_3^2\sigma_{33} + 2d_1d_2\sigma_{12} + 2d_1d_3\sigma_{13} + 2d_2d_3\sigma_{23} \\
&= \frac{e^2n_0^2}{\mu^2(en_0+\mu)^2}\cdot\sigma_{11} + \frac{e^4n_0^2(n_0+\mu)^2}{\mu^2\phi_1^2(en_0+\mu)^4}\cdot\sigma_{22} + \frac{e^4n_0^2(n_0+\mu)^2}{\phi_1^2(en_0+\mu)^4}\cdot\sigma_{33} \\
&\quad - 2\frac{e^3n_0^2(n_0+\mu)}{\mu^2\phi_1(en_0+\mu)^3}\cdot\sigma_{12} + 2\frac{e^3n_0^2(n_0+\mu)}{\mu\phi_1(en_0+\mu)^3}\cdot\sigma_{13} - 2\frac{e^4n_0^2(n_0+\mu)^2}{\mu\phi_1^2(en_0+\mu)^4}\cdot\sigma_{23}.
\end{aligned}$$

Now, we can replace σ_{12} and σ_{13} according to (S.7), yielding

$$\begin{aligned}
\sigma^2_{\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp}} = &\left[\frac{e^2n_0^2}{\mu^2(en_0+\mu)^2} - 2\frac{e^3n_0^2(n_0+\mu)}{\mu^2(en_0+\mu)^3}\big(1+\mu(e^{-1}-1)\big) + 2\frac{e^3n_0^2(n_0+\mu)}{\mu(en_0+\mu)^3}(e^{-1}-1)\right]\cdot\sigma_{11} \\
&+ \frac{e^4n_0^2(n_0+\mu)^2}{\mu^2\phi_1^2(en_0+\mu)^4}\cdot\sigma_{22} + \frac{e^4n_0^2(n_0+\mu)^2}{\phi_1^2(en_0+\mu)^4}\cdot\sigma_{33} - 2\frac{e^4n_0^2(n_0+\mu)^2}{\mu\phi_1^2(en_0+\mu)^4}\cdot\sigma_{23}.
\end{aligned}$$

This simplifies to

$$\sigma^2_{\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp}} = -\frac{\sigma_{11}}{\mu^2}\frac{e^2n_0^2\big(en_0-\mu(1-2e)\big)}{(en_0+\mu)^3} + \frac{e^4n_0^2(n_0+\mu)^2}{(en_0+\mu)^4}\frac{\sigma_{22}-2\mu\cdot\sigma_{23}+\mu^2\cdot\sigma_{33}}{\mu^2\phi_1^2}.$$

Recall that when calculating $\sigma^2_{\hat{\mathrm{T}}^{\mathrm{Poi}}_{\exp}}$ under the Poi-null (see p. 7), we already computed

$$\frac{\sigma_{22}-2\mu\,\sigma_{23}+\mu^2\,\sigma_{33}}{\mu^2\,\phi_1^2} \ = \ \frac{\psi(-2)}{\psi(-1)^2}\Big(\frac{1}{\mu}+(1-e^{-1})^2\Big),$$

while $\sigma_{11}=\mu$. Thus,

$$\sigma^2_{\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp}} \ = \ -\frac{1}{\mu}\frac{e^2n_0^2\big(en_0-\mu(1-2e)\big)}{(en_0+\mu)^3} + \frac{e^4n_0^2(n_0+\mu)^2}{(en_0+\mu)^4}\,\frac{\psi(-2)}{\psi(-1)^2}\Big(\frac{1}{\mu}+(1-e^{-1})^2\Big),$$

as stated in part (i) of Theorem 3.

For the bias correction according to (20), we derive the Hessian $\mathbf{H}_\tau(u,v,w)$, and evaluate it as $\mathbf{H} := \mathbf{H}_\tau\big(\mu,\mu\,\phi_1,\phi_1\big)$. We get

$$\mathbf{H}_\tau(u,v,w) \ = \ \begin{pmatrix}
\frac{2en_0v}{u^3(v+en_0w)} & -\frac{e^2n_0^2w}{u^2(v+en_0w)^2} & \frac{e^2n_0^2v}{u^2(v+en_0w)^2} \\
-\frac{e^2n_0^2w}{u^2(v+en_0w)^2} & -\frac{2e^2n_0w(n_0+u)}{u(v+en_0w)^3} & -\frac{e^2n_0(n_0+u)(en_0w-v)}{u(v+en_0w)^3} \\
\frac{e^2n_0^2v}{u^2(v+en_0w)^2} & -\frac{e^2n_0(n_0+u)(en_0w-v)}{u(v+en_0w)^3} & \frac{2e^3n_0^2v(n_0+u)}{u(v+en_0w)^3}
\end{pmatrix},$$

hence

$$\mathbf{H} \;=\; \begin{pmatrix} \frac{2en_0}{\mu^2(en_0+\mu)} & -\frac{e^2n_0^2}{\mu^2\phi_1(en_0+\mu)^2} & \frac{e^2n_0^2}{\mu\phi_1(en_0+\mu)^2} \\ -\frac{e^2n_0^2}{\mu^2\phi_1(en_0+\mu)^2} & -\frac{2e^2n_0(n_0+\mu)}{\mu\phi_1^2(en_0+\mu)^3} & -\frac{e^2n_0(n_0+\mu)(en_0-\mu)}{\mu\phi_1^2(en_0+\mu)^3} \\ \frac{e^2n_0^2}{\mu\phi_1(en_0+\mu)^2} & -\frac{e^2n_0(n_0+\mu)(en_0-\mu)}{\mu\phi_1^2(en_0+\mu)^3} & \frac{2e^3n_0^2(n_0+\mu)}{\phi_1^2(en_0+\mu)^3} \end{pmatrix}.$$

Next, let us compute the bias-corrected mean as

$$\begin{aligned}
\mu_{\hat{\mathrm{T}}_{\exp}^{\mathrm{NB}}} &= \mathrm{T}_{\exp}^{\mathrm{NB}} + \tfrac{1}{T}\Big(\tfrac{1}{2}\big(h_{11}\sigma_{11} + h_{22}\sigma_{22} + h_{33}\sigma_{33}\big) + h_{12}\sigma_{12} + h_{13}\sigma_{13} + h_{23}\sigma_{23}\Big) \\
&= \mathrm{T}_{\exp}^{\mathrm{NB}} + \frac{1}{T}\Bigg[\frac{en_0}{\mu^2(en_0+\mu)}\cdot\sigma_{11} - \frac{e^2n_0(n_0+\mu)}{\mu\phi_1^2(en_0+\mu)^3}\cdot\sigma_{22} + \frac{e^3n_0^2(n_0+\mu)}{\phi_1^2(en_0+\mu)^3}\cdot\sigma_{33} \\
&\qquad - \frac{e^2n_0^2}{\mu^2\phi_1(en_0+\mu)^2}\cdot\sigma_{12} + \frac{e^2n_0^2}{\mu\phi_1(en_0+\mu)^2}\cdot\sigma_{13} - \frac{e^2n_0(n_0+\mu)(en_0-\mu)}{\mu\phi_1^2(en_0+\mu)^3}\cdot\sigma_{23}\Bigg].
\end{aligned}$$

Once again, we replace σ_{12} and σ_{13} according to (S.7), which yields

$$\begin{aligned}
\mu_{\hat{\mathrm{T}}_{\exp}^{\mathrm{NB}}} &= \mathrm{T}_{\exp}^{\mathrm{NB}} + \frac{1}{T}\Bigg[\Bigg(\frac{en_0}{\mu^2(en_0+\mu)} - \frac{e^2n_0^2\big(1+\mu(e^{-1}-1)\big)}{\mu^2(en_0+\mu)^2} + \frac{e^2n_0^2(e^{-1}-1)}{\mu(en_0+\mu)^2}\Bigg)\cdot\sigma_{11} \\
&\qquad - \frac{e^2n_0(n_0+\mu)}{\mu\phi_1^2(en_0+\mu)^3}\cdot\sigma_{22} + \frac{e^3n_0^2(n_0+\mu)}{\phi_1^2(en_0+\mu)^3}\cdot\sigma_{33} - \frac{e^2n_0(n_0+\mu)(en_0-\mu)}{\mu\phi_1^2(en_0+\mu)^3}\cdot\sigma_{23}\Bigg] \\
&= \mathrm{T}_{\exp}^{\mathrm{NB}} + \frac{1}{T}\,\frac{e^2n_0(n_0+\mu)}{(en_0+\mu)^3}\,\frac{n_0e\mu\,\sigma_{33} - (en_0-\mu)\,\sigma_{23} - \sigma_{22}}{\mu\phi_1^2} + \frac{1}{T}\frac{en_0}{\mu(en_0+\mu)^2}\,\sigma_{11}.
\end{aligned}$$

Using (S.7), we compute

$$\begin{aligned}
&n_0e\mu\,\sigma_{33} - (en_0-\mu)\,\sigma_{23} - \sigma_{22} \\
&\quad= n_0e\mu\,e^{-2}\,\psi(-2) - (en_0-\mu)\,e^{-2}\mu\,e^{-1}\psi(-2) - e^{-2}\mu\,(1+e^{-2}\mu)\,\psi(-2) \\
&\qquad - n_0e\mu\,e^{-2}\,\psi(-1)^2 + (en_0-\mu)\,e^{-2}\mu\,\psi(-1)^2 + e^{-2}\mu^2\,\psi(-1)^2 \\
&\quad= e^{-2}\mu\,\psi(-2)\,\Big(n_0e - (en_0-\mu)\,e^{-1} - (1+e^{-2}\mu)\Big) \\
&\quad= e^{-2}\mu\,\psi(-2)\,\big((e-1)\big(n_0+e^{-2}\mu\big) - 1\big).
\end{aligned}$$

Thus, using (S.7), the bias-corrected mean follows as

$$\mu_{\hat{\mathrm{T}}_{\exp}^{\mathrm{NB}}} = \mathrm{T}_{\exp}^{\mathrm{NB}} + \frac{1}{T}\,\frac{\psi(-2)}{\psi(-1)^2}\,\frac{e^2n_0(n_0+\mu)\,\big((e-1)\big(n_0+e^{-2}\mu\big)-1\big)}{(en_0+\mu)^3} + \frac{1}{T}\,\frac{en_0}{(en_0+\mu)^2},$$

which completes the proof of part (i) of Theorem 3.

For part (ii) of Theorem 3, we need to derive the asymptotics of $\hat{\mathrm{T}}_{\exp}^{\mathrm{NB}}$ from (29) under the NB-null. Again, we apply the Delta method to $\tau(\overline{\boldsymbol{Z}})$ with $\tau(u,v,w) = \frac{(n_0+u)v}{u(n_0w+e^{-1}v)}$. Here,

$$\mathrm{T}_{\exp}^{\mathrm{NB}} \;=\; \tau(\boldsymbol{\mu_Z}) \;=\; \tau\Big(\mu,\ \psi'(-1),\ e^{-1}\psi(-1)\Big) \;=\; \frac{e\,(n_0+\mu)\,\psi'(-1)}{\mu\,\mathrm{A}(-1)},$$

where we used the abbreviation $\mathrm{A}(u) := \psi'(u) + n_0\,\psi(u)$. We already know that

$$\operatorname{grad}\boldsymbol{\tau}(u,v,w) \;=\; \frac{en_0}{u(v+en_0w)}\left(-\frac{v}{u},\ \frac{ew(n_0+u)}{v+en_0w},\ -\frac{ev(n_0+u)}{v+en_0w}\right)^{\top}.$$

Evaluating $\mathbf{D} := \operatorname{grad}\boldsymbol{\tau}\big(\boldsymbol{\mu_Z}\big)$, we get

$$\mathbf{D} \;=\; \tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)}\left(-\tfrac{\psi'(-1)}{\mu},\ \tfrac{(n_0+\mu)\,\psi(-1)}{\mathrm{A}(-1)},\ -\tfrac{e\,(n_0+\mu)\,\psi'(-1)}{\mathrm{A}(-1)}\right). \tag{S.13}$$

We calculate the variance of $\sqrt{T}\left(\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp} - \mathrm{T}^{\mathrm{NB}}_{\exp}\right)$ as $\sigma^2_{\hat{\mathrm{T}}^{\mathrm{NB}}_{\exp}} = \mathbf{D\Sigma D}^{\top}$, with $\boldsymbol{\Sigma}$ taken from Equation (S.11), as

$$\begin{aligned}
\sigma^2_{\hat{\mathrm{T}}_f} \;=\;& \left(\tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)}\right)^2\Bigg(\left(\tfrac{\psi'(-1)}{\mu}\right)^2\sigma_{11} - 2\left(\tfrac{\psi'(-1)}{\mu}\right)\left(\tfrac{(n_0+\mu)\,\psi(-1)}{\mathrm{A}(-1)}\right)\sigma_{12}\\
&+\left(\tfrac{(n_0+\mu)\,\psi(-1)}{\mathrm{A}(-1)}\right)^2\sigma_{22} + 2\left(\tfrac{\psi'(-1)}{\mu}\right)\left(\tfrac{e\,(n_0+\mu)\,\psi'(-1)}{\mathrm{A}(-1)}\right)\sigma_{13}\\
&-2\left(\tfrac{(n_0+\mu)\psi(-1)}{\mathrm{A}(-1)}\right)\left(\tfrac{e\,(n_0+\mu)\,\psi'(-1)}{\mathrm{A}(-1)}\right)\sigma_{23} + \left(\tfrac{e\,(n_0+\mu)\,\psi'(-1)}{\mathrm{A}(-1)}\right)^2\sigma_{33}\Bigg)\\
=\;& \left(\tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)}\right)^2\Bigg(\left(\tfrac{\psi'(-1)}{\mu}\right)^2\sigma^2 - 2\left(\tfrac{\psi'(-1)\,(n_0+\mu)\,\psi(-1)}{\mu\,\mathrm{A}(-1)}\right)\big(\psi''(-1)-\mu\psi'(-1)\big)\\
&+\left(\tfrac{(n_0+\mu)\,\psi(-1)}{\mathrm{A}(-1)}\right)^2\big(\psi''(-2)-\psi'(-1)^2\big) + 2\left(\tfrac{(n_0+\mu)\,\psi'(-1)^2}{\mu\,\mathrm{A}(-1)}\right)\big(\psi'(-1)-\mu\psi(-1)\big)\\
&-2\left(\tfrac{(n_0+\mu)^2\psi(-1)\,\psi'(-1)}{\mathrm{A}(-1)^2}\right)\big(\psi'(-2)-\psi'(-1)\psi(-1)\big)\\
&+\left(\tfrac{(n_0+\mu)\,\psi'(-1)}{\mathrm{A}(-1)}\right)^2\big(\psi(-2)-\psi(-1)^2\big)\Bigg)\\
=\;& \left(\tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)}\right)^2\Bigg(\tfrac{\psi'(-1)^2\,\sigma^2}{\mu^2} + \tfrac{2(n_0+\mu)\psi'(-1)}{\mu\,\mathrm{A}(-1)}\big(\psi'(-1)^2-\psi(-1)\psi''(-1)\big)\\
&+\tfrac{(n_0+\mu)^2}{\mathrm{A}(-1)^2}\Big(\psi(-1)^2\psi''(-2) - 2\psi(-1)\psi'(-1)\psi'(-2) + \psi'(-1)^2\psi(-2)\Big)\Bigg).
\end{aligned}$$

To compute the approximate bias correction according to (20), we first need to evaluate $\mathbf{H} := \mathbf{H}_{\tau}\big(\boldsymbol{\mu_Z}\big)$. We already know that

$$\mathbf{H}_{\tau}(u,v,w) \;=\; \frac{e\,n_0}{u(v+en_0w)^2}\begin{pmatrix} \frac{2v(v+en_0w)}{u^2} & -\frac{en_0w}{u} & \frac{en_0v}{u}\\ -\frac{en_0w}{u} & -\frac{2ew(n_0+u)}{v+en_0w} & -\frac{e(n_0+u)(en_0w-v)}{v+en_0w}\\ \frac{en_0v}{u} & -\frac{e(n_0+u)(en_0w-v)}{v+en_0w} & \frac{2e^2n_0v(n_0+u)}{v+en_0w}\end{pmatrix},$$

so

$$\mathbf{H} \;=\; \tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)^2}\begin{pmatrix} \frac{2\psi'(-1)\,\mathrm{A}(-1)}{\mu^2} & -\frac{n_0\psi(-1)}{\mu} & \frac{en_0\psi'(-1)}{\mu} \\ -\frac{n_0\psi(-1)}{\mu} & -\frac{2(n_0+\mu)\psi(-1)}{\mathrm{A}(-1)} & -\frac{e(n_0+\mu)(n_0\psi(-1)-\psi'(-1))}{\mathrm{A}(-1)} \\ \frac{en_0\psi'(-1)}{\mu} & -\frac{e(n_0+\mu)(n_0\psi(-1)-\psi'(-1))}{\mathrm{A}(-1)} & \frac{2e^2n_0(n_0+\mu)\psi'(-1)}{\mathrm{A}(-1)} \end{pmatrix}.$$

This leads to the bias correction

$$\begin{aligned}
&\tfrac{1}{T}\Big(\tfrac{1}{2}\,h_{11}\,\sigma_{11}+\tfrac{1}{2}\,h_{22}\,\sigma_{22}+\tfrac{1}{2}\,h_{33}\,\sigma_{33}+h_{12}\,\sigma_{12}+h_{13}\,\sigma_{13}+h_{23}\,\sigma_{23}\Big)\\
&= \tfrac{1}{T}\,\tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)^2}\Big(\tfrac{\psi'(-1)\,\mathrm{A}(-1)\,\sigma_{11}}{\mu^2}-\tfrac{n_0\psi(-1)\,\sigma_{12}}{\mu}+\tfrac{en_0\psi'(-1)\,\sigma_{13}}{\mu}-\tfrac{(n_0+\mu)\psi(-1)\,\sigma_{22}}{\mathrm{A}(-1)}\\
&\qquad -\tfrac{e(n_0+\mu)\big(n_0\psi(-1)-\psi'(-1)\big)\,\sigma_{23}}{\mathrm{A}(-1)}+\tfrac{e^2n_0(n_0+\mu)\psi'(-1)\,\sigma_{33}}{\mathrm{A}(-1)}\Big)\\
&= \tfrac{1}{T}\,\tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)^2}\Bigg(\tfrac{\psi'(-1)\,\mathrm{A}(-1)\,\sigma_{11}}{\mu^2}+\tfrac{n_0}{\mu}\big(-\psi(-1)\,\sigma_{12}+e\psi'(-1)\,\sigma_{13}\big)\\
&\qquad +\tfrac{(n_0+\mu)}{\mathrm{A}(-1)}\Big(-\psi(-1)\,\sigma_{22}-e\big(n_0\psi(-1)-\psi'(-1)\big)\,\sigma_{23}+e^2n_0\psi'(-1)\,\sigma_{33}\Big)\Bigg)\\
&= \tfrac{1}{T}\,\tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)^2}\Bigg(\tfrac{\psi'(-1)\,\mathrm{A}(-1)\,\sigma^2}{\mu^2}+\tfrac{n_0}{\mu}\Big(-\psi(-1)\big(\psi''(-1)-\mu\,\psi'(-1)\big)\\
&\qquad +\psi'(-1)\big(\psi'(-1)-\mu\,\psi(-1)\big)\Big)+\tfrac{(n_0+\mu)}{\mathrm{A}(-1)}\Big(-\psi(-1)\big(\psi''(-2)-\psi'(-1)^2\big)\\
&\qquad -\big(n_0\psi(-1)-\psi'(-1)\big)\big(\psi'(-2)-\psi'(-1)\,\psi(-1)\big)+n_0\psi'(-1)(\psi(-2)-\psi(-1)^2)\Big)\Bigg)\\
&= \tfrac{1}{T}\,\tfrac{e\,n_0}{\mu\,\mathrm{A}(-1)^2}\Bigg(\tfrac{\psi'(-1)\,\mathrm{A}(-1)\,\sigma^2}{\mu^2}+\tfrac{n_0}{\mu}\big(\psi'(-1)^2-\psi(-1)\psi''(-1)\big)\\
&\qquad +\tfrac{(n_0+\mu)}{\mathrm{A}(-1)}\Big(\psi'(-1)\big(\psi'(-2)+n_0\psi(-2)\big)-\psi(-1)\big(\psi''(-2)+n_0\psi'(-2)\big)\Big)\Bigg).
\end{aligned}$$

This completes the proof of part (ii) of Theorem 3.

S.8 Proof of Theorem 4

While the convergence of the finite-dimensional distributions as well as the tightness follow similarly as in Rueda et al. (1991), it remains to compute the mean function $\mu_{\hat{\mathrm{T}}^{\mathrm{Poi}}_{\exp}}(s)$ and the covariance kernel $\sigma_{\hat{\mathrm{T}}^{\mathrm{Poi}}_{\exp}}(s_1,s_2)$. According to Theorem 1 in Aleksandrov et al. (2022), we have

$$\mu_{\hat{\mathrm{T}}^{\mathrm{Poi}}_{s^x}}(s) \;=\; \frac{1}{T}\,\frac{\mu\,\sigma_{33}-\sigma_{23}}{\mu\,\phi_1^2} \quad \text{with } \sigma_{23}=\mu\,\phi_{12}(0)-\mu\,\phi_1^2 \text{ and } \sigma_{33}=\phi_{11}(0)-\phi_1^2,$$

where $\phi_{ij}(k) := E\big[f_s(X_t{+}i)\,f_s(X_{t+k}{+}j)\big]$ with $f_s(x)=s^x$, and where $\phi_1=E\big[f_s(X_t{+}1)\big]=E[s^{X_t+1}]=s\,E[s^{X_t}]=s\,\mathrm{pgf}(s)=s\,\exp(\mu(s-1))$ in the i. i. d. Poi(μ)-case.

Here, we compute

$$\phi_{11}(0) = E\big[f_s^2(X_t+1)\big] = E[s^{2X_t+2}] = s^2\, E\big[(s^2)^{X_t}\big] = s^2\,\mathrm{pgf}(s^2) = s^2\,\exp\!\big(\mu(s^2-1)\big),$$

and

$$\phi_{12}(0) = E\big[f_s(X_t+1)\, f_s(X_t+2)\big] = E[s^{2X_t+3}] = s^3\,\mathrm{pgf}(s^2) = s^3\,\exp\!\big(\mu(s^2-1)\big).$$

Thus,

$$\begin{aligned}\sigma_{33} \;&=\phi_{11}(0)-\phi_1^2 \;=\; s^2\,\Big(\exp\!\big(\mu(s^2-1)\big) - \big(\exp(\mu(s-1))\big)^2\Big)\\ &= s^2\,\Big(\exp\!\big(\mu(s^2-1)\big) - \exp\!\big(2\mu(s-1)\big)\Big),\end{aligned}$$

and similarly, we have

$$\begin{aligned}\sigma_{23} \;&=\mu\,\big(\phi_{12}(0)-\phi_1^2\big) \;=\; \mu\, s^2\,\Big(s\,\exp\!\big(\mu(s^2-1)\big) - \big(\exp(\mu(s-1))\big)^2\Big)\\ &= \mu\, s^2\,\Big(s\,\exp\!\big(\mu(s^2-1)\big) - \exp\!\big(2\mu(s-1)\big)\Big).\end{aligned}$$

Putting everything together, we get

$$\begin{aligned}\mu_{\hat{\Upsilon}^{\mathrm{Poi}}_{sx}}(s) \;&= \frac{1}{T}\,\frac{\mu\,\sigma_{33}-\sigma_{23}}{\mu\,\phi_1^2}\\ &= \frac{1}{T}\,\frac{\mu\, s^2\,\exp\!\big(\mu(s^2-1)\big) - \mu\, s^3\,\exp\!\big(\mu(s^2-1)\big)}{\mu\,\big(s\exp(\mu(s-1))\big)^2} \;=\; \frac{1}{T}\,\frac{(1-s)\,\exp\!\big(\mu(s^2-1)\big)}{\exp\!\big(2\mu(s-1)\big)}\\ &= \frac{1}{T}\,(1-s)\,\exp\!\big(\mu(s^2-2s+1)\big) \;=\; \frac{1}{T}\,(1-s)\,\exp\!\big(\mu(s-1)^2\big).\end{aligned}$$

Next, for the covariance kernel, we use Theorem 2 in Aleksandrov et al. (2022) together with the functions $f_{s_1}(x) = s_1^x$ and $f_{s_2}(x) = s_2^x$, which leads to

$$\sigma_{\hat{\Upsilon}^{\mathrm{Poi}}_{sx}}(s_1,s_2) \;=\; -\frac{\sigma_{11}}{\mu^2} + \frac{\sigma_{24}-\mu\,(\sigma_{25}+\sigma_{34})+\mu^2\,\sigma_{35}}{\mu^2\,\phi_1\varphi_1}.$$

Here, $\phi_1 = E\big[f_{s_1}(X_t{+}1)\big] = s_1\,\exp\!\big(\mu(s_1{-}1)\big)$ and $\varphi_1 = E\big[f_{s_2}(X_t{+}1)\big] = s_2\,\exp\big(\mu(s_2{-}1)\big)$ like before Furthermore, $\vartheta_{ij}(k) := E\big[f_{s_1}(X_t+i)\, f_{s_2}(X_{t+k}+j)\big]$ is required for expressing the σ_{mn}. We compute

$$\vartheta_{ij}(0) \;=\; E[s_1^{X_t+i}s_2^{X_t+j}] \;=\; s_1^i s_2^j\, E\big[(s_1s_2)^{X_t}\big] \;=\; s_1^i s_2^j\,\exp\!\big(\mu(s_1s_2-1)\big).$$

According to Lemma 2 in Aleksandrov et al. (2022), we have $\sigma_{11} = \mu$,

$$\begin{aligned}\sigma_{24} \;&= \mu\,\vartheta_{11}(0) + \mu^2\,\vartheta_{22}(0) - \mu^2\,\phi_1\varphi_1\\ &= s_1s_2\,\exp\!\big(\mu(s_1s_2-1)\big) + \mu^2\, s_1^2s_2^2\,\exp\!\big(\mu(s_1s_2-1)\big)\\ &\qquad - \mu^2\, s_1s_2\,\exp\!\big(\mu(s_1-1)\big)\,\exp\!\big(\mu(s_2-1)\big)\\ &= (\mu\, s_1s_2 + \mu^2\, s_1^2s_2^2)\,\exp\!\big(\mu(s_1s_2-1)\big) - \mu^2\, s_1s_2\,\exp\!\big(\mu(s_1+s_2-2)\big),\end{aligned}$$

and

$$\begin{aligned}\sigma_{25} &= \mu\,\vartheta_{12}(0) - \mu\,\phi_1\varphi_1 \\ &= \mu\, s_1 s_2^2 \,\exp\bigl(\mu(s_1s_2-1)\bigr) - \mu\, s_1 s_2 \,\exp\bigl(\mu(s_1-1)\bigr)\,\exp\bigl(\mu(s_2-1)\bigr) \\ &= \mu\, s_1 s_2^2 \,\exp\bigl(\mu(s_1s_2-1)\bigr) - \mu\, s_1 s_2 \,\exp\bigl(\mu(s_1+s_2-2)\bigr).\end{aligned}$$

Analogously, we get

$$\sigma_{34} \;=\; \mu\,\vartheta_{21}(0) - \mu\,\phi_1\varphi_1 \;=\; \mu\, s_1^2 s_2 \,\exp\bigl(\mu(s_1s_2-1)\bigr) - \mu\, s_1 s_2 \,\exp\bigl(\mu(s_1+s_2-2)\bigr),$$

and we compute

$$\sigma_{35} \;=\; \vartheta_{11}(0) - \phi_1\varphi_1 \;=\; s_1 s_2 \,\exp\bigl(\mu(s_1s_2-1)\bigr) - s_1 s_2 \,\exp\bigl(\mu(s_1+s_2-2)\bigr).$$

As a consequence,

$$\begin{aligned}&\sigma_{24} - \mu\,(\sigma_{25}+\sigma_{34}) + \mu^2\,\sigma_{35} \\ &= (\mu\, s_1s_2 + \mu^2\, s_1^2 s_2^2)\,\exp\bigl(\mu(s_1s_2-1)\bigr) - \mu^2\, s_1 s_2\,\exp\bigl(\mu(s_1+s_2-2)\bigr) \\ &\quad - \mu\Bigl(\mu\,(s_1s_2^2 + s_1^2s_2)\,\exp\bigl(\mu(s_1s_2-1)\bigr) - 2\,\mu\, s_1s_2\,\exp\bigl(\mu(s_1+s_2-2)\bigr)\Bigr) \\ &\quad + \mu^2\Bigl(s_1s_2\,\exp\bigl(\mu(s_1s_2-1)\bigr) - s_1s_2\,\exp\bigl(\mu(s_1+s_2-2)\bigr)\Bigr) \\ &= \Bigl(\mu\, s_1s_2 + \mu^2\, s_1^2s_2^2 - \mu^2\,(s_1s_2^2+s_1^2s_2) + \mu^2\, s_1s_2\Bigr)\,\exp\bigl(\mu(s_1s_2-1)\bigr) \\ &= \mu^2\, s_1s_2\,\Bigl(\frac{1}{\mu} + s_1s_2 - s_2 - s_1 + 1\Bigr)\,\exp\bigl(\mu(s_1s_2-1)\bigr) \\ &= \mu^2\, s_1s_2\,\Bigl(\frac{1}{\mu} + (s_1-1)(s_2-1)\Bigr)\,\exp\bigl(\mu(s_1s_2-1)\bigr).\end{aligned}$$

Altogether, we have

$$\begin{aligned}\sigma_{\hat{\mathrm{T}}_{sx}^{\mathrm{Poi}}}(s_1,s_2) &= -\,\frac{\sigma_{11}}{\mu^2} + \frac{\sigma_{24} - \mu\,(\sigma_{25}+\sigma_{34}) + \mu^2\,\sigma_{35}}{\mu^2\,\phi_1\varphi_1} \\ &= -\,\frac{\mu}{\mu^2} + \frac{\mu^2\, s_1s_2\,\bigl(\frac{1}{\mu} + (s_1-1)(s_2-1)\bigr)\,\exp\bigl(\mu(s_1s_2-1)\bigr)}{\mu^2\, s_1s_2\,\exp\bigl(\mu(s_1+s_2-2)\bigr)} \\ &= -\,\frac{1}{\mu} + \Bigl(\frac{1}{\mu} + (s_1-1)(s_2-1)\Bigr)\,\exp\bigl(\mu(s_1-1)(s_2-1)\bigr).\end{aligned}$$

This completes the proof of Theorem 4.

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