Retrieval from Mixed Sampling Frequency: Generic Identifiability in the Unit Root VAR

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S-1 Estimation of β and Moments Observed

In Section 2.2, we call the moments $\mathbb{E}\,\tilde{y}_{t+h}\tilde{y}_t'$ in equation (8) also *observed second moments* since they can be consistently estimated from the data as we will argue in this section. Those moments in $\mathbb{E}\,\tilde{y}_{t+h}\tilde{y}_t'$ which involve only N-differences from the data do not require any further discussion. We are left with considering moments that involve β.

In the following, if N is the sampling rate of the slow variables and T is the number of time observations of high-frequency data, we denote by T_s the number of time observations available for the slow/ aggregated coordinates which is approximately T/N and by $\mathbb{I}_s$ the index set of slow/ aggregated time observations, e.g. for $N = 3$ we could have $\mathbb{I}_s = \{3, 6, 9, ..., T\}$. In Miller (2016) and Chambers (2020) consistent estimators $\hat{\beta}_{T_s}$ that converge in probability to β are provided for the mixed frequency case. In both articles, the corresponding estimator $\hat{\beta}_{T_s}$ converges weakly to a product of the inverse of a functional of Brownian motions and a stochastic integral. By applying Lemmata 4.5 and 4.6 of White (2001), we observe that

$$(\widehat{\beta}_T - \beta) = O_p(T_s^{-1}) \ .$$

Next we consider the high-frequency moments $\mathbb{E}\,\beta' y_t$, $\mathbb{E}\,\beta' y_t \Delta y_s'$, and $\mathbb{E}\,\beta' y_t y_s' \beta$, $s, t \in \mathbb{Z}$. The following calculations work in the same way for the mixed frequency counterparts. We can estimate $\mathbb{E}\,\beta' y_t$ by $\frac{1}{T_s}\sum_{t\in\mathbb{I}_s} \widehat{\beta}_{T_s}' y_t$. We impose the standard assumptions that a weak law of large number and a functional central limit theorem can be applied to properly scaled terms. For a "Low-Frequency Invariance Principle" see, e.g., Miller (2016)[Section 2.3]. Let $B(\tau)$ denote some n-dimensional Brownian motion. Define $\widehat{u}_t^{\mathcal{S}} := \widehat{\beta}_{T_s}' y_t$ and $\widehat{u}_t^{\mathcal{F}} := \widehat{\beta}_{T_s} \sum_{j=0}^{N-1} y_{t-j}$. This results in

$$\begin{aligned} T_s^{-1} \sum_{t\in\mathbb{I}_s} \widehat{u}_t^{\mathcal{S}} &= T_s^{-1} \sum_{t\in\mathbb{I}_s} \widehat{\beta}_{T_s}' y_t = T_s^{-1} \sum_{t\in\mathbb{I}_s} \beta' y_t + T_s^{-1} \sum_{t\in\mathbb{I}_s} \left(\widehat{\beta}_{T_s} - \beta\right)' y_t \\ &= \mathbb{E}\,\beta' y_t + o_p(1) + T_s^{1/2} (\widehat{\beta}_{T_s} - \beta)' \underbrace{T_s^{-3/2} \sum_{t\in\mathbb{I}_s} y_t}_{\Rightarrow \int_0^1 B(\tau) d\tau} \\ &= \mathbb{E}\,\beta' y_t + o_p(1) + O_p(T^{-1/2}) O_p(1) \ . \end{aligned} \tag{S-1}$$

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Next, we look at the second moments for the stock case appearing in $\tilde{\gamma}$, i.e. second moments of the form $\mathbb{E}\, u_t^{\mathcal{S}} \Delta_N y_s$. Observe that we have

$$\begin{aligned} T_s^{-1} \sum_{t \in \mathbb{I}_s} \widehat{u}_t^{\mathcal{S}} \Delta y_s' &= T_s^{-1} \sum_{t \in \mathbb{I}_s} \widehat{\beta}_{T_s}' y_t \Delta y_s' \\ &= T_s^{-1} \sum_{t \in \mathbb{I}_s} \beta' y_t \Delta y_s' + \left(\widehat{\beta}_{T_s} - \beta \right)' \underbrace{T_s^{-1} \sum_{t \in \mathbb{I}_s} y_t \Delta y_s'}_{\Rightarrow \int_0^1 B(\tau) dB(\tau)'} \\ &= \mathbb{E}\, u_t^{\mathcal{S}} \Delta y_s' + o_p(1) + O_p(T_s^{-1}) O_p(1) \ . \end{aligned} \tag{S-2}$$

It follows immediately that also

$$T_s^{-1} \sum_{t \in \mathbb{I}_s} \hat{u}_t^{\mathcal{S}} \Delta_N y_s \xrightarrow{P} \mathbb{E}\, u_t^{\mathcal{S}} \Delta_N y_s^{\cdot} \ ,$$

where "$\xrightarrow{P}$" denotes convergence in probability and "$\cdot$" is shorthand for s, f ("fast" and "slow" coordinates). Finally we look the estimation of $\mathbb{E}\, u_t^{\mathcal{S}} u_s^{\mathcal{S}\prime}$ for $t - s \in N\mathbb{Z}$:

$$\begin{aligned} T_s^{-1} \sum_{t \in \mathbb{I}_s} \widehat{u}_t^{\mathcal{S}} (\widehat{u}_s^{\mathcal{S}})' &= T_s^{-1} \sum_{t \in \mathbb{I}_s} \beta' y_t y_s' \beta + T_s^{1/2} (\widehat{\beta}_{T_s} - \beta)' \underbrace{T_s^{-2} \left(\sum_{t \in \mathbb{I}_s} y_t y_s' \right)}_{\Rightarrow \int_0^1 B(\tau) B(\tau)' d\tau} T_s^{1/2} (\widehat{\beta}_{T_s} - \beta) \\ &= \mathbb{E}\, u_t^{\mathcal{S}} (u_s^{\mathcal{S}})' + o_p(1) + O_p(T_s^{-1/2}) O_p(1) O_p(T_s^{-1/2}) \ . \end{aligned} \tag{S-3}$$

We obtain analogous results for moments involving $u_t^{\mathcal{F}}$, by noting that $u_t^{\mathcal{F}} = \sum_{j=0}^{N-1} \beta' y_{t-j}$. Hence, equations (S-1) to (S-3) demonstrate why the above moments can be considered as moments observed although β has to be estimated.

S-2 Example for Non-Identifiability

We consider a VECM where $n = 2$, $r = 1$, and $p = 1$. The cointegrating vector is $\beta = (1, \beta_s)'$, $\alpha = (\alpha_f, \alpha_s)'$, and $\Pi = \alpha\beta'$. That is

$$\Delta y_t = \Pi y_{t-1} + \nu_t, \quad \nu_t \sim WN\left(\Sigma_\nu\right) \text{ , such that } \Sigma_\nu = \begin{pmatrix} \sigma_{ff} & \sigma_{fs} \\ \sigma_{sf} & \sigma_{ss} \end{pmatrix} \text{ , and}$$

$$y_t = \underbrace{(I_n + \Pi)}_{\mathcal{A}_1} y_{t-1} + \nu_t = \begin{pmatrix} \underbrace{1+\alpha_f}_{a_{ff}} & \underbrace{\alpha_f\beta_s}_{a_{fs}} \\ \underbrace{\alpha_s}_{a_{sf}} & \underbrace{1+\alpha_s\beta_s}_{a_{ff}} \end{pmatrix} y_{t-1} + \nu_t \ . \tag{S-4}$$

By Theorem 1 of Anderson et al. (2016a) the model parameters are not identified if and only if (i) $a_{fs} = 0$, (ii) $a_{sf} + \frac{\sigma_{sf}}{\sigma_{ff}}\left(a_{ss} - a_{ff}\right) = 0$, and (iii) $a_{ss} \neq 0$. $a_{fs} = 0$ implies $\alpha_f = 0$ [with $\beta_s = 0$ we do not have a cointegrating relationship]. From $a_{sf} + \frac{\sigma_{sf}}{\sigma_{ff}}\left(a_{ss} - a_{ff}\right) = \alpha_s\left(1 + \frac{\sigma_{sf}}{\sigma_{ff}}\beta_s\right) = 0$, we conclude that α_s must be non-zero, otherwise (y_t) is a white noise process. Hence, we get $0 = \left(1 + \frac{\sigma_{fs}}{\sigma_{ff}}\beta_s\right)$. The third constraint results in $0 = 1 + \alpha_s\beta_s$. Hence, by considering the second moments in levels and the arguments provided in the proof Theorem 1 of Anderson (2016a), we get identifyability on a generic set. Unfortunately these moments are not "observed moments" since they cannot be estimated from data observed as in the stationary case.
By contrast $\mathbb{E}\Delta_N\tilde{y}_t\Delta_N\tilde{y}'_{t-N\ell}$, $\ell \in \mathbb{Z}$, are observed moments, where we consider stock variables only and let $\Delta_N\tilde{y}_t = y_t - y_{t-N}$. In addition, when β is known [or can be consistently estimated as shown in Miller (2016) and Chambers (2020) also the moments $\mathbb{E}\beta\tilde{y}_t\Delta_N\tilde{y}'_{t-\ell}$, $t, t-\ell \in 2\mathbb{Z}$, can be observed. Applying (10) and (11) to the current example yields

$$\begin{pmatrix} \beta' y_t \\ \Delta y_t \\ \Delta y_{t-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \beta'\alpha + 1 & 0_{1\times 2} & 0_{1\times 2} \\ \alpha & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 1} & I_2 & 0_{2\times 2} \end{pmatrix}}_{A \in \mathbb{R}^{r+2n\times r+2n}} \begin{pmatrix} \beta' y_{t-1} \\ \Delta y_{t-1} \\ \Delta y_{t-2} \end{pmatrix} + \begin{pmatrix} \beta' \\ I_n \\ 0 \end{pmatrix} \nu_t \ , \quad t \in \mathbb{Z} \ . \tag{S-5}$$

Next, by direct calculations we get

$$\begin{aligned} \Delta_2 y_t &= \Delta y_t + \Delta y_{t-1} = \Pi y_{t-1} + \nu_t + \Pi y_{t-2} + \nu_{t-1} \\ &= \Pi\left(y_{t-2} + \Delta y_{t-1}\right) + \nu_t + \Pi y_{t-2} + \nu_{t-1} \\ &= 2\alpha\beta' y_{t-2} + \alpha\beta'\alpha\beta' y_{t-2} + \Pi\nu_{t-1} + \nu_{t-1} + \nu_t \\ &= \underbrace{\begin{pmatrix} 2\alpha_f + \alpha_f^2 + \alpha_f\beta_s\alpha_s \\ 2\alpha_s + \alpha_f\alpha_s + \alpha_2^2\beta_s \end{pmatrix}}_{=:M_{\alpha,\beta}} \beta y_{t-2} + \Pi\nu_{t-1} + \nu_{t-1} + \nu_t \ . \end{aligned} \tag{S-6}$$

In addition, $\beta' y_t = \beta'\left(y_{t-2} + \Delta_2 y_t\right)$ and $\Delta y^f_{t-1} := y_{1t-1} = \alpha_f\beta y_{t-2} + (1,0)'\nu_{t-1}$. This results in the state-space system

$$\begin{aligned} \underbrace{\begin{pmatrix} \beta' y_t \\ \Delta_2 y_t \\ \Delta y^f_{t-1} \end{pmatrix}}_{x_t} &= \underbrace{\begin{pmatrix} 1 + \beta' M_{\alpha,\beta} & 0_{1\times 2} & 0_{1\times 2} \\ M_{\alpha,\beta} & 0_{2\times 2} & 0_{2\times 2} \\ \alpha_f & 0_{1\times 2} & 0_{1\times 2} \end{pmatrix}}_{\tilde{A} \in \mathbb{R}^{r+n+1\times r+n+1}} \begin{pmatrix} \beta' y_{t-2} \\ \Delta_2 y_{t-2} \\ \Delta y_{t-3} \end{pmatrix} + \underbrace{\begin{pmatrix} \beta' I_2 & \beta'(I_2 + \Pi) \\ I_2 & I_2 + \Pi \\ 0 & (1,0)' \end{pmatrix}}_{\tilde{B}} \begin{pmatrix} \nu_t \\ \nu_{t-1} \end{pmatrix} \\ \tilde{y}_t &= I_{\tilde{n}} x_t \quad , \quad t \in 2\mathbb{Z} \ . \end{aligned} \tag{S-7}$$

Finally, we investigate whether there are (different) parameter vectors such that the autocovariance function of $(\tilde{y}_t)_{t\in 2\mathbb{Z}}$, i.e. $\tilde{\gamma}$, is the same. (Recall that $\theta^I := \left(\alpha_f^I, \alpha_s^I, 1, \beta_s, \sigma_{ff}^I, \sigma_{fs}^I, \sigma_{ss}^I\right)$ and

$\theta^{II} := \left(\alpha_f^{II}, \alpha_s^{II}, 1, \beta_s, \sigma_{ff}^{II}, \sigma_{fs}^{II}, \sigma_{ss}^{II}\right)$ as defined in Section 2.2.) This is the case if $\tilde{A}$, and $\tilde{B}\Sigma\tilde{B}'$ are equal though $\theta^I \neq \theta^{II}$. Consider $\tilde{A}$, for $\alpha_f = 0$, from the definition of $M[\alpha, \beta]$ in equation (S-6), we can choose some $\alpha_s^I \neq \alpha_s^{II}$ solving the quadratic equation $2\alpha_s + \beta_s\alpha_s^2 - [M_{\alpha,\beta}]_{2,1} = 0$, in which case $\tilde{A}$ remains the same.
Next we consider $\tilde{B}\Sigma\tilde{B}'$. To get equality with some θ^I, θ^{II}, the covariances of the lagged fast variable have to be equal, this demands for $\sigma_{ff}^I = \sigma_{ff}^{II}$. Second, to get equality of the covariances of the cointegrating terms it is sufficient to look at the covariances of $\Delta_N y_t$. Direct calculations show that (with $\alpha_f = 0$, $\sigma_{ff}^I = \sigma_{ff}^{II}$)

$$\begin{aligned}&\left[\tilde{B}\Sigma\tilde{B}'\right]_{(2:3,2:3)}\\&= \begin{pmatrix} \sigma_{ff} & \alpha_s\sigma_{ff} + (1+\alpha_s\beta_s)\sigma_{sf} \\ \alpha_s\sigma_{ff} + (1+\alpha_s\beta_s)\sigma_{sf} & \alpha_s\left(\alpha_s\sigma_{ff} + (1+\alpha_s\beta_s)\sigma_{sf}\right) + \left(\alpha_s\sigma_{fs} + (1+\alpha_s\beta_s)\sigma_{ss}\right)(1+\alpha_s\beta_s) \end{pmatrix} .\end{aligned} \tag{S-8}$$

This yields $\sigma_{sf}^{II} = \sigma_{fs}^{II} = \frac{\alpha_s^I\sigma_{ff} - \alpha_s^{II} + (1+\alpha_s^I\beta_s)\sigma_{sf}^I}{(1+\alpha_s^{II}\beta_s)}$, where $(1+\alpha_s^{II}\beta_s) \neq 0$. Finally, choose σ_{ss}^{II} such that the (2,2)-elements of (S-8) for parameters θ^I and θ^{II} become equal, this requires $(1+\alpha_s^I\beta_s) \neq 0$. Hence, we have obtained a pair θ^I, θ^{II}, $\theta^I \neq \theta^{II}$, where $\tilde{A}$ and $\tilde{B}\Sigma\tilde{B}'$ are the same. This implies by Lyapunov equations that the model parameters cannot be identified from $\tilde{\gamma}$. In this example we observed the following: If $\alpha_f = 0$, $\left(y_t^f\right)_{t\in\mathbb{Z}}$ is a random walk not affected by $(y_t^s)_{t\in\mathbb{Z}}$. Therefore, the high frequency variable y_t^f does not provide information about the parameter α_s. By contrast, if $\alpha_f \neq 0$ the fast variable y_t^f provides sufficient information to identify α_s. To see this, if $\alpha_f \neq 0$, α_f follows from the (4,1)-element of $\tilde{\mathcal{A}}$, while α_s can be uniquely retrieved from the first coordinate of $M_{\alpha,\beta}$, that is from the equality $2\alpha_f + \alpha_f^2 + \alpha_f\beta_s\alpha_s = [M_{\alpha,\beta}]_{(1,1)}$.

S-3 Deterministic Terms

This section investigates the VECM

$$\Delta y_t = \mu_0 + \mu_1 t + \Pi y_{t-1} + \sum_{j=1}^{p-1} \Phi_j \Delta y_{t-j} + \nu_t \ , \qquad \nu_t \sim WN(\Sigma_\nu) \ , \tag{S-9}$$

containing the deterministic terms μ_0 and $\mu_1 t$. The five cases following from (S-9), namely "$H_2(r)$, $H_1(r)$, $H_1^*(r)$, $H(r)$, and $H^*(r)$", are obtained and defined in Johansen (1995)[page 81]. Recall that the cointegrating vectors β can be identified from mixed frequency data (see Chambers, 2020). For high frequency data $(\Delta y_t)_{t\in\mathbb{Z}}$ and $(\beta' y_t)_{t\in\mathbb{Z}}$ we can compute the expectations $\mathbb{E}\,\Delta y_t$ and $\mathbb{E}\,\beta' y_t$, while for mixed frequency case we get $\mathbb{E}\,\beta' y_t$ and $\mathbb{E}\,\Delta_N y_t = \mathbb{E}\,(y_t - y_{t-N})$ for the stock case and $\mathbb{E}\,\beta' w_t = \mathbb{E}\sum_{j=0}^{N-1}\beta' y_{t-j}$ and $\mathbb{E}\,\Delta_N^\Sigma y_t = \mathbb{E}\sum_{j=0}^{N-1}\Delta_N y_{t-j}$ for the flow case, respectively. To identify the deterministic terms in (S-9) we can proceed as follows:

1. Remove deterministic trends from $(\beta' y_t)_{t\in N\mathbb{Z}}$ and $(\Delta_N y_t)_{t\in N\mathbb{Z}}$ in the stock case or $(\beta' w_t)_{t\in N\mathbb{Z}}$ and $\left(\Delta_N^\Sigma y_t\right)_{t\in N\mathbb{Z}}$ for the flow case [given β], such that $\mathbb{E}\,\beta' y_t = 0$ and $\mathbb{E}\,\Delta_N y_t = 0$, $\mathbb{E}\,\beta' w_t = 0$, and $\mathbb{E}\,\Delta_N^\Sigma y_t = 0$.

2. Apply Theorem 4 to obtain the parameters θ.

3. Given the parameters θ this section shows that the parameters μ_0 and μ_1 can be identified from moments following from data observed.

This results in:

Theorem S 1
Under the assumptions of Theorem 4 the parameters μ_0 and μ_1 can be identified generically from observable first and second moments.

Proof. By considering the Granger-Representation-Theorem for the solution on $\mathbb{Z}$ in Bauer and Wagner (2012)[equation (26)], (see also Theorem 4.3 in Johansen, 1995), we obtain (see also Hansen and Johansen, 1998, Exercise 4.5)

$$\mathbb{E}\,\Delta y_t = C(\mu_0 + \mu_1 t) + M_c\mu_1 \ , \tag{S-10}$$

$$\text{with } C := \beta_\perp \left(\alpha'_\perp \left(I_n - \sum_{j=1}^{p-1} \Phi_j \right) \beta_\perp \right)^{-1} \alpha'_\perp \ , \tag{S-11}$$

and M_c is the limit of the stable part of the impulse responses in the particular solution defined in Bauer and Wagner (2012)[Equation (8); this is $k_\bullet(1)$ in their equation (26)] Let $\bar{\alpha}' = (\alpha'\alpha)^{-1}\alpha'$, then

$$\begin{aligned}\mathbb{E}\,\beta' y_t = \bar{\alpha}' \Bigg[& C(\mu_0 + \mu_1 t) + M_c\mu_1 - \mu_0 - \mu_1 t \ , \\ & - \sum_{j=1}^{p-1} \Phi_j \big(C(\mu_0 + \mu_1(t-j)) + M_c\mu_1 \big) \Bigg] \ .\end{aligned} \tag{S-12}$$

For $N = 1$, both moments in equations (S-10) and (S-12) are observable (or consistently estimable from observed data) for the stock as well as for the flow case. In addition, for $N > 1$ we get

$$\begin{aligned}\mathbb{E}\,\beta' y_{t-\ell} = \bar{\alpha}' \Bigg[& C(\mu_0 + \mu_1(t-\ell)) + M_c\mu_1 - \mu_0 - \mu_1(t-\ell) \\ & - \sum_{j=1}^{p-1} \Phi_j \big(C(\mu_0 + \mu_1(t-j)) + M_c\mu_1 \big) \Bigg] \text{ for the stock case, and}\end{aligned} \tag{S-13}$$

$$\mathbb{E}\,\beta' w_t = \mathbb{E} \sum_{\ell=0}^{N-1} \beta' y_{t-\ell} \ , \quad \text{for the flow case} \ . \tag{S-14}$$

$$\begin{aligned}\mathbb{E}\,\Delta_N y_t &= \sum_{j=0}^{N-1} \left(C(\mu_0 + \mu_1(t-j)) + M_c\mu_1 \right) \\ &= NC\mu_0 + C\mu_1 \left(\sum_{j=0}^{N-1} (t-j) \right) + NCM_c\mu_1 \ , \quad \text{for the stock case, and}\end{aligned} \tag{S-15}$$

$$\begin{aligned}\mathbb{E}\,\Delta_N^\Sigma y_t &= \sum_{\ell=0}^{N-1} \mathbb{E}\,\Delta_N y_{t-\ell} = \sum_{\ell=0}^{N-1} \sum_{j=\ell}^{\ell+N-1} \left(C(\mu_0 + \mu_1(t-j)) + M_c\mu_1 \right) \\ &= N^2 C\mu_0 + C\mu_1 \left(\sum_{\ell=0}^{N-1} \sum_{j=\ell}^{\ell+N-1} (t-j) \right) + N^2 M_c\mu_1 \ , \quad \text{for the flow case} \ .\end{aligned} \tag{S-16}$$

As already stated in Section S-3, we first remove the deterministic trends from $\mathbb{E}\,\beta' y_t$ and $\mathbb{E}\,\Delta_N y_t$ for the stock case or $\mathbb{E}\,\beta' w_t = \mathbb{E} \sum_{j=0}^{N-1} \beta' y_{t-j}$ and $\mathbb{E}\,\Delta_N^\Sigma y_t = \mathbb{E} \sum_{j=0}^{N-1} \Delta_N y_{t-j}$ for the flow case. β follows from mixed frequency data as demonstrated in Chamber (2020). Then the parameters of the model without deterministic terms are generically identified as shown in the

main text. For the five cases arsing from model (S-9) we get:
Case $H_2(r)$**:** ($\mu_0 = \mu_1 = 0$) No deterministic terms. This case was shown above.
Case $H_1(r)$**:** ($\mu_1 = 0$ and $\mu_0 \neq 0$.) We have a linear trend in $\mathbb{E}y_t$, and a constant in $\mathbb{E}\Delta y_t$, and a constant in $\mathbb{E}\beta' y_t$. Let the matrix S_C select $n-r$ basis rows of C (e.g., $S_C = \alpha'_\perp$ can be used). In this case it holds that

$$\mathbb{E}\begin{pmatrix} \beta' y_t \\ S_C \Delta_N y_t \end{pmatrix} = \begin{pmatrix} \bar\alpha'((I_n - \sum_{j=1}^{p-1}\Phi_j)C - I_n) \\ N S_C C, \end{pmatrix} \mu_0 \ , \quad \text{for the stock case, and}$$

$$\mathbb{E}\begin{pmatrix} \beta' w_t \\ S_C \Delta_N^\Sigma y_t \end{pmatrix} = \begin{pmatrix} N\bar\alpha'((I_n - \sum_{j=1}^{p-1}\Phi_j)C - I_n) \\ N^2 S_C C \end{pmatrix} \mu_0 \ , \quad \text{for the flow case} \ ,$$

where the matrix on the LHS before μ_0 is of rank n since the first rowblock has rank r and C has rank $n-r$ and both are mutually orthogonal as $\alpha'_\perp \alpha = 0$ and therefore (with $\alpha'_\perp$ the last term of C)

$$\alpha'_\perp\big((I_n - \sum_{j=1}^{p-1}\Phi_j)C - I_n\big)'\bar\alpha = 0 \ .$$

From this we can compute μ_0.
Case $H_1^*(r)$**:** ($\mu_1 = 0$, $\mu_0 \neq 0$ and $\alpha'_\perp\mu_0 = 0$) no linear trend but constant in $\mathbb{E}\, y_t$, constant in $\mathbb{E}\beta' y_{t-1}$, and $\mathbb{E}\Delta y_t = 0$. This results in r parameters in the cointegration equation. In this case we write $\mu_t = \alpha\rho_0$, where $\rho_0 \in \mathbb{R}^r$. Note that $C\mu_t = C\alpha\rho_0 = 0$, which results in

$$\begin{aligned}\mathbb{E}\begin{pmatrix} \beta' y_t \\ \Delta_N y_t \end{pmatrix} &= \begin{pmatrix} \bar\alpha'((I_n - \sum_{j=1}^{p-1}\Phi_j)C - I_n) \\ NC \end{pmatrix} \alpha\rho_0 \\ &= \begin{pmatrix} -\bar\alpha'\alpha\rho_0 \\ N0 \end{pmatrix} = \begin{pmatrix} -\rho_0 \\ 0 \end{pmatrix} \ , \quad \text{for the stock case.}\end{aligned}$$

For the flow case we get

$$\begin{aligned}\mathbb{E}\begin{pmatrix} \beta' w_t \\ \Delta_N^\Sigma y_t \end{pmatrix} &= \begin{pmatrix} N\bar\alpha'((I_n - \sum_{j=1}^{p-1}\Phi_j)C - I_n) \\ N^2 C \end{pmatrix} \alpha\rho_0 \\ &= \begin{pmatrix} -N\bar\alpha'\alpha\rho_0 \\ N^2 0 \end{pmatrix} = \begin{pmatrix} -N\rho_0 \\ 0 \end{pmatrix} \ .\end{aligned}$$

This allows to uniquely retrieve the unkown parameter ρ_0 from $\mathbb{E}\,\beta' y_{t-1}$.
Case $H(r)$**:** ($\mu_0 \in \mathbb{R}^n$ and $\mu_1 \neq 0$) quadratic trend in $\mathbb{E}y_t$, linear trend in $\mathbb{E}\beta' y_t$, linear trend in $\mathbb{E}\Delta y_t$. For this the stock case we get the system of linear equations

$$\mathbb{E}\begin{pmatrix} \beta' y_t \\ S_C \Delta_N^\Sigma y_t \\ \beta' y_{t-N} \\ S_C \Delta_N^\Sigma y_{t-N} \end{pmatrix} = M_\mu \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix} \ ,$$

where

$$\begin{aligned} M_\mu^{\mathcal{S}} &:= \left(\begin{array}{c|c} \bar\alpha'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j) & \bar\alpha'\left[Ct + M_c - tI_n - \sum_{j=1}^{p-1}\Phi_j\,(C(t-j) - M_c)\right] \\ N S_C C & S_C C\left(\sum_{\ell=0}^{N-1}(t-\ell) + NM_c\right) \\ \hline \bar\alpha'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j) & \bar\alpha'\left[C(t-N) + M_c - (t-N)I_n - \sum_{j=1}^{p-1}\Phi_j\,(C(t-N-j) - M_c)\right] \\ N S_C C & S_C C\left(\sum_{\ell=0}^{N-1}(t-N-\ell) + NM_c\right) \end{array}\right) \\ &= \begin{pmatrix} M_{\mu 11}^{\mathcal{S}} & M_{\mu 12}^{\mathcal{S}} \\ M_{\mu 21}^{\mathcal{S}} & M_{\mu 22}^{\mathcal{S}} \end{pmatrix} \ .\end{aligned}$$

For the flow case we have

$$\mathbb{E}\begin{pmatrix} \beta' w_t \\ S_C \Delta_N^{\Sigma} y_t \\ \beta' w_{t-N} \\ S_C \Delta_N^{\Sigma} y_{t-N} \end{pmatrix} = M_\mu \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix} ,$$

where

$$M_\mu^{\mathcal{F}} := \left(\begin{array}{c|c} N\bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j) & \bar{\alpha}'\left[C\sum_{\ell=0}^{N-1}(t-\ell) + NM_c - \left(\sum_{\ell=0}^{N-1}(t-\ell)\right) I_n - \sum_{j=1}^{p-1}\Phi_j\left(C\sum_{\ell=0}^{N-1}(t-\ell) - NM_c\right)\right] \\ N^2 S_C C & S_C\left(\left(\sum_{\ell=0}^{N-1}\sum_{j=0}^{N-1}(t-j-\ell)\right)C + N^2 M_c\right) \\ \hline N\bar{\alpha}'\left(C - I_n - \sum_{j=1}^{p-1} C\Phi_j\right) & \bar{\alpha}'\left[C\sum_{\ell=0}^{N-1}(t-N-\ell) + NM_c - \sum_{\ell=0}^{N-1}(t-\ell-N)I_n - \sum_{j=1}^{p-1}\Phi_j\left(C\left(\sum_{\ell=0}^{N-1}(t-N-j-\ell)\right) - NM_c\right)\right] \\ N^2 S_C C & S_C\left(\left(\sum_{\ell=0}^{N-1}\sum_{j=0}^{N-1}(t-N-\ell-j)\right)C + N^2 M_c\right) \end{array}\right)$$

$$= \begin{pmatrix} M^{\mathcal{F}}_{\mu 11} & M^{\mathcal{F}}_{\mu 12} \\ M^{\mathcal{F}}_{\mu 21} & M^{\mathcal{F}}_{\mu 22} \end{pmatrix} .$$

The matrix S_C selects $n-r$ basis rows of C (e.g., $S_C = \alpha'_\perp$ can be used) and $M^{\cdot}_{\mu 11} = M^{\cdot}_{\mu 21}$. The matrix $M^{\cdot}_{\mu 11}$ is of full column rank n by the above calculations where we set $\mu_1 = 0$. The determinant of a blocked matrix is given by

$$\det M^{\cdot}_\mu = \det M^{\cdot}_{\mu 11} \det\left(M^{\cdot}_{\mu 22} - \underbrace{M^{\cdot}_{\mu 21} M^{\cdot -1}_{\mu 11}}_{I_n} M^{\cdot}_{\mu 12}\right) = \det M^{\cdot}_{\mu 11} \det\left(M^{\cdot}_{\mu 22} - M^{\cdot}_{\mu 12}\right) .$$

Note that $M^{\mathcal{S}}_{\mu 22} - M^{\mathcal{S}}_{\mu 12} = \begin{pmatrix} N\bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j) \\ N^2 S_C C \end{pmatrix} = N M^{\mathcal{S}}_{\mu 11}$ for the stock case, while for the flow case $M^{\mathcal{F}}_{\mu 22} - M^{\mathcal{F}}_{\mu 12} = \begin{pmatrix} N^2\bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j) \\ N^3 S_C C \end{pmatrix} = N M^{\mathcal{F}}_{\mu 11}$. Therefore, $\det M^{\cdot}_\mu \neq 0$. This allows to uniquely solve for μ_0 and μ_1.

Case $H^*(r)$: ($\mu_0 \in \mathbb{R}^n$, $\mu_1 \neq 0$, with $\alpha'_\perp \mu_1 = 0$) linear trend in $\mathbb{E}y_t$, linear trend in $\mathbb{E}\beta' y_t$, and constant in $\mathbb{E}\Delta y_t$. μ_0 contains n free parameters, while $\mu_t = \mu_0 + \alpha\rho_1 t$, where $\rho_1 \in \mathbb{R}^r$. Hence, $C\mu_t = C\mu_0 + C\alpha\rho_1 t = C\mu_0$. This results in

$$\begin{aligned} \mathbb{E}\begin{pmatrix} \beta' y_t \\ S_C \Delta_N y_t \\ \beta' y_{t-N} \end{pmatrix} &= \left[M_\mu^{\mathcal{S}}\right]_{1:r+n,1:2(r+m)} \begin{pmatrix} \mu_0 \\ \alpha\rho_1 \end{pmatrix} \\ &= \underbrace{\left[M_\mu^{\mathcal{S}}\right]_{1:r+n,1:2(r+n)} \begin{pmatrix} I_n & 0_{n\times r} \\ 0_{n\times n} & \alpha \end{pmatrix}}_{M^{\mathcal{S}}_{\mu, H_1^*(r)}} \begin{pmatrix} \mu_0 \\ \rho_1 \end{pmatrix} . \end{aligned}$$

Since $M_\mu^{\mathcal{F}}$ has full rank $2n$, the matrix $M^{\mathcal{F}}_{\mu H_1^*(r)}$ has rank $r+n$. This allows to uniquely obtain μ_0 and ρ_1. For the flow case this works equivalently. ■

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