

Supplement to “Data-based bandwidth selection for kernel smoothing of empirical distribution functions”

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Abstract

This supplement contains the proofs of the results in the paper “Data-based bandwidth selection for kernel smoothing of empirical distribution functions”.

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A. Proofs

A.1. Proof of (2)

Let $X_1, \dots, X_n$ be real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, we have

$$\begin{aligned}\widehat{F}_{n,h}(x) &= \int K_h(x-y) \widehat{F}_n(y) \lambda(dy) = \int K_h(x-y) \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[X_i, \infty)} \lambda(dy) \\ &= \frac{1}{n} \sum_{i=1}^n \int_{[X_i, \infty)} K_h(x-y) \lambda(dy) = \frac{1}{n} \sum_{i=1}^n \int_{(-\infty, x-X_i]} K_h(z) \lambda(dz) \\ &= \frac{1}{n} \sum_{i=1}^n \overline{K}_h(x-X_i) = \int \overline{K}_h(x-y) d\widehat{F}_n(y).\end{aligned}$$

Hence, (2) indeed holds true. By the way, the same calculation with K replaced by K^* gives

$$\widehat{F}_{n,h}^*(x) = \int \overline{(K^*)_h}(x-y) d\widehat{F}_n(y) = \int \overline{K}_h(x-y)^2 d\widehat{F}_n(y) = \frac{1}{n} \sum_{i=1}^n \overline{K}_h(x-X_i)^2 \quad (\text{A.1})$$

for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, where the second = follows from $\overline{(K^*)_h}(z) = \overline{K^*}(z/h) = \overline{K}(z/h)^2 = \overline{K}_h(x)^2$ (recall that $\overline{\kappa}_h(\cdot) = \overline{\kappa}(\cdot/h)$ and $\overline{\kappa^*}(\cdot) = \overline{\kappa}(\cdot)^2$ for any kernel κ).

A.2. Continuity and boundedness of $\overline{\kappa}$

Let κ be any kernel. In Section 2 we mentioned that the function $\overline{\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\overline{\kappa}(x) := \int_{(-\infty, x]} \kappa d\lambda$ is continuous and bounded. We here verify that this is indeed true. In fact, we show that $\overline{\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ is even absolutely continuous (which will be needed later on in Section A.3), so that $\overline{\kappa}$ is in particular continuous. The boundedness then results from the continuity of $\overline{\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ since $\lim_{x \rightarrow -\infty} \overline{\kappa}(x) = 0$ and $\lim_{x \rightarrow \infty} \overline{\kappa}(x) = 1$. The latter follow from $\overline{\kappa}(x) = \int_{(-\infty, x]} \kappa d\lambda$ (for all $x \in \mathbb{R}$) and the λ -integrability of κ with $\int_{\mathbb{R}} \kappa d\lambda = 1$.

By Definition 7.1.7 in [5] (see also [4, VII.4.18]) a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *absolutely continuous* (on $\mathbb{R}$) if for every compact interval $[a, b]$ in $\mathbb{R}$ the restriction of f to $[a, b]$ is absolutely continuous, where a function $\psi : [a, b] \rightarrow \mathbb{R}$ is said to be absolutely continuous if for every $\varepsilon \in \mathbb{R}_{++}$ one can find a $\delta \in \mathbb{R}_{++}$ such that for each $k \in \mathbb{N}$ and $a \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \dots \leq \alpha_k < \beta_k \leq b$ with $\sum_{j=1}^k (\beta_j - \alpha_j) < \delta$ it holds that $\sum_{j=1}^k |\psi(\beta_j) - \psi(\alpha_j)| < \varepsilon$. The above mentioned absolute continuity of $\overline{\kappa}$ follows from the absolute continuity of its restrictions to compact intervals, where the latter follows from [4, VII.4.14.a] and the fact that the restriction of $\overline{\kappa}$ to a compact interval $[a, b]$ differs from the function $\kappa^{a,b} : [a, b] \rightarrow \mathbb{R}$ defined by $\kappa^{a,b}(x) := \int_{[a, x]} \kappa d\lambda$ only by the constant $c_{a,b} := \int_{(-\infty, a)} \kappa d\lambda$ (i.e., more precisely, $\kappa = \kappa^{a,b} + c_{a,b}$ on $[a, b]$).

A.3. Proof of statements on κ^*

Let κ be a kernel and set $\kappa^*(x) := 2\kappa(x)\bar{\kappa}(x)$, $x \in \mathbb{R}$. Note that $\kappa^* : \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable and satisfies $|\kappa^*(x)| \leq 2\|\bar{\kappa}\|_\infty|\kappa(x)|$ for all $x \in \mathbb{R}$. Since $\|\bar{\kappa}\|_\infty < \infty$ (see Section A.2) and κ is λ -integrable, the function $\kappa^* : \mathbb{R} \rightarrow \mathbb{R}$ is λ -integrable. In Section 2 we asserted that κ^* is again a kernel and satisfies $\overline{\kappa^*}(\cdot) = \bar{\kappa}(\cdot)^2$. We will now show that these assertions are indeed true.

Step 1. Since we have seen in Section A.2 that $\bar{\kappa}$ is absolutely continuous, we know from Theorem 7.20 in [5] that $\bar{\kappa}$ is λ -a.e. differentiable (see also [4, VII.4.12.b]), and by [4, VII.4.14.a] we can find a λ -null set $N \in \mathcal{B}(\mathbb{R})$ such that $\bar{\kappa}' = \kappa$ on N^c . In particular, we obtain that $\bar{\kappa}^2$ is differentiable at every point $x \in N^c$ with $(\bar{\kappa}^2)'(x) = 2\kappa(x)\bar{\kappa}(x) = \kappa^*(x)$. For all $x \in N$ we set $(\bar{\kappa}^2)'(x) := 0$. Since $\bar{\kappa}^2$ is absolutely continuous (recall that $\bar{\kappa}$ is absolutely continuous), we obtain by from Theorem 7.20 in [5] that $\bar{\kappa}^2(x) - \bar{\kappa}^2(a) = \int_{[a,x]} (\bar{\kappa}^2)' d\lambda = \int_{[a,x]} \kappa^* d\lambda$ for all $a, x \in \mathbb{R}$ with $a \leq x$ (see also [4, VII.4.14.b]). Since $\lim_{a \rightarrow -\infty} \bar{\kappa}(a) = 0$ (by the λ -integrability of κ) and $\lim_{a \rightarrow -\infty} \int_{[a,x]} \kappa^* d\lambda = \int_{(-\infty,x]} \kappa^* d\lambda$ (by the dominated convergence theorem and the λ -integrability of κ^*), we arrive at $\bar{\kappa}^2(x) = \int_{(-\infty,x]} \kappa^* d\lambda = \overline{\kappa^*}(x)$ for all $x \in \mathbb{R}$.

Step 2. Since both κ and κ^* are λ -integrable and $\lim_{x \rightarrow \infty} \bar{\kappa}(x) = \lim_{x \rightarrow \infty} \int_{(-\infty,x]} \kappa d\lambda = \int \kappa d\lambda = 1$ (by the dominated convergence theorem and the assumption on κ), we have that $\int \kappa^* d\lambda = \lim_{x \rightarrow \infty} \int_{(-\infty,x]} \kappa^* d\lambda = \lim_{x \rightarrow \infty} \overline{\kappa^*}(x) = \lim_{x \rightarrow \infty} \bar{\kappa}^2(x) = 1$, where the second last = relies on Step 1. Therefore, κ^* is a kernel.

A.4. Proof of (4)

If we choose $\kappa := K_h$, then the elaborations in Section A.2 show that $\overline{K_h}$ is absolutely continuous. Moreover, F is absolutely continuous by assumption. From Theorem 7.20 in [5] it follows that $\overline{K_h}$ and F are λ -a.e. differentiable (see also [4, VII.4.12.b]) and by [4, VII.4.14.a] we can find a λ -null set $N \in \mathcal{B}(\mathbb{R})$ such that $\overline{K_h}' = K_h$ on N^c . Without loss of generality we can assume that F is differentiable on N^c . On N , we set $\overline{K_h}' := 0$ and $F' := 0$. With the help of [4, VII.4.16] we therefore obtain

$$\begin{aligned} \int_{[a,b]} K_h(x-y)F(y) \lambda(dy) &= \int_{[a,b]} \overline{K_h}'(x-y)F(y) \lambda(dy) \\ &= \left[-\overline{K_h}(x-\cdot)F(y) \right]_a^b - \int_{[a,b]} (-\overline{K_h}(x-y))F'(y) \lambda(dy) \\ &= \left[-\overline{K_h}(x-\cdot)F(y) \right]_a^b + \int_{[a,b]} \overline{K_h}(x-y) dF(y). \end{aligned}$$

Letting $[a, b] \nearrow \mathbb{R}$, we can conclude that $\int K_h(x-y)F(y) \lambda(dy) = \int \overline{K_h}(x-y) dF(y)$. Hence, the first equation in (4) indeed holds true. The second equation in (4) can be shown in the same way where K has to be replaced by K^* .

A.5. Proofs of Lemmas 2.4–2.6

Note that F is λ -a.e. differentiable, since we assumed that it is absolutely continuous. The derivative F' can be extended to all of $\mathbb{R}$ by setting $F'(x) := 0$ for all points $x \in \mathbb{R}$ at which F is not differentiable. Theorem 7.20 in [5] shows that $F(b) - F(a) = \int_{[a,b]} F' d\lambda$ for all $a, b \in \mathbb{R}$ with $a < b$ (see also [4, VII.4.1.14.b]). This can be used to show that

$$F(x - uh) - F(x) = -uh \int_{[0,1]} F'(x - uht) \lambda(dt) \quad (\text{A.2})$$

for all $x, u \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$. Indeed: $F(x - uh) - F(x) = -\text{sgn}(u) \int_{I(x, x-uh)} F'(y) \lambda(dy) = -\text{sgn}(u) \int_{I(0, -uh)} F'(x + y) \lambda(dy) = -\text{sgn}(u) \int_{I(0, uh)} F'(x - z) \lambda(dz) = -\text{sgn}(u)^2 \int_{I(0,1)} F'(x - uht) uh \lambda(dt) = -uh \int_{[0,1]} F'(x - uht) \lambda(dt)$ for all $x, u \in \mathbb{R}$, $h \in \mathbb{R}_{++}$, where, for any $a, b \in \mathbb{R}$, $I(a, b)$ is used to denote the closed interval in $\mathbb{R}$ with endpoints a and b .

Proof of Lemma 2.4. With the help of the transformation theorem, (A.2) and Fubini's theorem we obtain that, for every $h \in \mathbb{R}_{++}$,

$$\begin{aligned} \int |F_h(x) - F(x)| \lambda(dx) &= \int \left| \int K_h(x - y) (F(y) - F(x)) \lambda(dy) \right| \lambda(dx) \\ &= \int \left| \int K(u) (F(x - hu) - F(x)) \lambda(du) \right| \lambda(dx) \\ &= \int \left| \int K(u) uh \int_{[0,1]} F'(x - uht) \lambda(dt) \lambda(du) \right| \lambda(dx) \\ &\leq h \int |u| |K(u)| \int_{[0,1]} \int F'(x - uht) \lambda(dx) \lambda(dt) \lambda(du) \\ &= h \int |u| |K(u)| \int_{[0,1]} \int F'(y) \lambda(dy) \lambda(dt) \lambda(du) = h \int |u| |K(u)| \lambda(du) = K_{|1|} h. \end{aligned}$$

Proof of Lemma 2.5. With the help of the transformation theorem, (A.2) and Fubini's theorem we obtain that, for every $h \in \mathbb{R}_{++}$,

$$\begin{aligned} \sup_{x \in \mathbb{R}} |F_h(x) - F(x)| &= \sup_{x \in \mathbb{R}} \left| \int K_h(x - y) (F(y) - F(x)) \lambda(dy) \right| \\ &= \sup_{x \in \mathbb{R}} \left| \int K(u) (F(x - hu) - F(x)) \lambda(du) \right| \\ &= \sup_{x \in \mathbb{R}} \left| \int K(u) uh \int_{[0,1]} F'(x - uht) \lambda(dt) \lambda(du) \right| \\ &\leq h \sup_{x \in \mathbb{R}} \int |u| |K(u)| \int_{[0,1]} |F'(x - uht)| \lambda(dt) \lambda(du) \\ &= h \|F'\|_\infty \int |u| |K(u)| \lambda(du) = \|F'\|_\infty K_{|1|} h. \end{aligned}$$

Proof of Lemma 2.6. With the help of the transformation theorem, (A.2) and Fubini's theorem we obtain that, for every $h \in (0, 1]$,

$$\begin{aligned}
\sup_{x \in \mathbb{R}} |F_h(x) - F(x)| \phi(x) &= \sup_{x \in \mathbb{R}} \left| \int K_h(x-y) (F(y) - F(x)) \lambda(dy) \right| \phi(x) \\
&= \sup_{x \in \mathbb{R}} \left| \int K(u) (F(x-hu) - F(x)) \lambda(du) \right| \phi(x) \\
&= \sup_{x \in \mathbb{R}} \left| \int K(u) u h \int_{[0,1]} F'(x-uh t) \lambda(dt) \lambda(du) \right| \phi(x) \\
&\leq h \sup_{x \in \mathbb{R}} \int |u| |K(u)| \int_{[0,1]} F'(x-uh t) \lambda(dt) \lambda(du) \phi(x) \\
&\leq h \sup_{x \in \mathbb{R}} \int |u| |K(u)| \int_{[0,1]} F'(x-uh t) \phi(x-uh t) \frac{\phi(x)}{\phi(x-uh t)} \lambda(dt) \lambda(du) \\
&\leq h \|F' \phi\|_\infty \sup_{x \in \mathbb{R}} \int |u| |K(u)| \int_{[0,1]} \frac{\phi(x)}{\phi(x-uh t)} \lambda(dt) \lambda(du) \\
&\leq h \|F' \phi\|_\infty L_\phi \int |u| |K(u)| \lambda(du) \leq h \|F' \phi\|_\infty L_\phi \int |u| |K(u)| \lambda(du) \\
&= \|F' \phi\|_\infty L_\phi K_{[1]} h,
\end{aligned}$$

where we have used the assumption that $K = 0$ outside $[-1, 1]$.

A.6. Proof of Theorem 3.1

By (2) and (4) we have

$$\mathbb{E}[\widehat{F}_{n,h}(x)] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\overline{K}_h(x - X_i)] = \mathbb{E}[\overline{K}_h(x - X_1)] = \int \overline{K}_h(x - y) dF(y) = F_h(x)$$

as well as

$$\begin{aligned}
\text{Var}[\widehat{F}_{n,h}(x)] &= \frac{1}{n^2} \text{Var}\left[\sum_{i=1}^n \overline{K}_h(x - X_i)\right] = \frac{1}{n} \text{Var}[\overline{K}_h(x - X_1)] \\
&= \frac{1}{n} (\mathbb{E}[\overline{K}_h(x - X_1)^2] - \mathbb{E}[\overline{K}_h(x - X_1)]^2) = \frac{1}{n} (\mathbb{E}[\overline{K}_h(x - X_1)^2] - F_h(x)^2) \\
&= \frac{1}{n} \left(\int \overline{K}_h(x - y)^2 dF(y) - F_h(x)^2 \right) = \frac{1}{n} \int \overline{K}_h(x - y)^2 dF(y) - \frac{1}{n} F_h(x)^2.
\end{aligned}$$

In view of $\text{MSE}^F(\widehat{F}_{n,h}(x), F(x)) = \text{Var}[\widehat{F}_{n,h}(x)] + (\mathbb{E}[\widehat{F}_{n,h}(x)] - F(x))^2$, this gives the claim.

A.7. Taylor expansion of F_h

Lemma A.1 *Let $m \in \mathbb{N}$. Let K be a kernel of order m for which we have $K_{|m+1|} < \infty$. Assume that F is $m+1$ times continuously differentiable and that $\int |F^{(m+1)}| d\lambda < \infty$*

and $\|F^{(m+1)}\|_\infty < \infty$. Then there exists a continuous function $R_{K,m} : \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ such that

$$F_h(x) = F(x) + (-h)^m \frac{K^{(m)}}{m!} F^{(m)}(x) + h^{m+1} R_{K,m}(x, h) \quad (\text{A.3})$$

for all $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$, and $\sup_{h \in \mathbb{R}_{++}} \int |R_{K,m}(x, h)| \lambda(dx)$ and $\sup_{h \in \mathbb{R}_{++}} \|R_{K,m}(\cdot, h)\|_\infty$ are finite.

Proof For every $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$, we obtain by using the Taylor expansion of F at x that

$$\begin{aligned} F_h(x) &= \int K_h(x-y) F(y) \lambda(dy) = \int K(u) F(x-hu) \lambda(du) \\ &= \int K(u) \left(F(x) + \sum_{j=1}^m \frac{(-hu)^j}{j!} F^{(j)}(x) + R_m(x, u, h) \right) \lambda(du) \\ &= F(x) + \sum_{j=1}^m \frac{(-h)^j}{j!} F^{(j)}(x) \int u^j K(u) \lambda(du) + \int R_m(x, u, h) K(u) \lambda(du) \\ &= F(x) + \sum_{j=1}^m \frac{(-h)^j}{j!} F^{(j)}(x) K_{(j)} + \int R_m(x, u, h) K(u) \lambda(du) \\ &= F(x) + \frac{(-h)^m}{m!} F^{(m)}(x) K_{(m)} + \int R_m(x, u, h) K(u) \lambda(du), \end{aligned} \quad (\text{A.4})$$

where $R_m(x, u, h)$ is the remainder. The last step follows, since K is of order m . The integral form of the remainder is $R_m(x, u, h) = -\frac{\text{sgn}(u)}{m!} \int_{I(x, x-uh)} (x-hu-y)^m F^{(m+1)}(y) \lambda(dy)$, where $I(a, b)$ is used to denote the closed interval in $\mathbb{R}$ with endpoints a and b . Thus, for every $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$,

$$\begin{aligned} R_m(x, u, h) &= -\frac{\text{sgn}(u)}{m!} \int_{I(x, x-uh)} (x-hu-y)^m F^{(m+1)}(y) \lambda(dy) \\ &= -\frac{\text{sgn}(u)}{m!} \int_{I(0, uh)} (z-uh)^m F^{(m+1)}(x-z) \lambda(dz) \\ &= -\frac{\text{sgn}(u)^2}{m!} \int_{I(0,1)} (uht-uh)^m F^{(m+1)}(x-uht) uh \lambda(dt) \\ &= \frac{(-uh)^{m+1}}{m!} \int_{[0,1]} (1-t)^m F^{(m+1)}(x-uht) \lambda(dt). \end{aligned}$$

Together with (A.4), this implies

$$F_h(x) = F(x) + \frac{(-h)^m}{m!} F^{(m)}(x) K_{(m)} + h^{m+1} R_{K,m}(h, x)$$

for every $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$, where the function $R_{K,m} : \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ is defined by $R_{K,m}(x, h) := \frac{(-1)^{m+1}}{m!} \iint_{[0,1]} (1-t)^m F^{(m+1)}(x-uht) u^{m+1} K(u) \lambda(dt) \lambda(du)$. This function is

continuous (this is ensured by the continuity lemma) and satisfies $\sup_{h \in \mathbb{R}_{++}} \int |R_{K,m}(x, h)| \lambda(dx) < \infty$. Indeed: For every $h \in \mathbb{R}_{++}$ we have

$$\begin{aligned}
& \int |R_{K,m}(x, h)| \lambda(dx) \\
&= \int \left| \int \frac{(-1)^{m+1}}{m!} \int_{[0,1]} (1-t)^m F^{(m+1)}(x - uht) \lambda(dt) u^{m+1} K(u) \lambda(du) \right| \lambda(dx) \\
&\leq \int \frac{1}{m!} \int_{[0,1]} (1-t)^m \int |F^{(m+1)}(x - uht)| \lambda(dx) \lambda(dt) |u|^{m+1} |K(u)| \lambda(du) \\
&= \frac{1}{m!} \int_{[0,1]} (1-t)^m \lambda(dt) \int |u|^{m+1} |K(u)| \lambda(du) \int |F^{(m+1)}| d\lambda \\
&= \frac{1}{(m+1)!} \int |u|^{m+1} |K(u)| \lambda(du) \int |F^{(m+1)}| d\lambda = \frac{K_{|m+1|}}{(m+1)!} \int |F^{(m+1)}| d\lambda.
\end{aligned}$$

The latter expression is finite (since we assumed that $K_{|m+1|}$ and $\int |F^{(m+1)}| d\lambda$ are finite) and independent of $h \in \mathbb{R}_{++}$.

The function $R_{K,m} : \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ also satisfies $\sup_{h \in \mathbb{R}_{++}} \|R_{K,m}(\cdot, h)\| < \infty$. Indeed: For every $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$ we have

$$\begin{aligned}
& |R_{K,m}(x, h)| \\
&= \left| \int \frac{(-1)^{m+1}}{m!} \int_{[0,1]} (1-t)^m F^{(m+1)}(x - uht) \lambda(dt) u^{m+1} K(u) \lambda(du) \right| \\
&\leq \int \frac{1}{m!} \int_{[0,1]} (1-t)^m |F^{(m+1)}(x - uht)| \lambda(dt) |u|^{m+1} |K(u)| \lambda(du) \\
&= \frac{1}{m!} \int_{[0,1]} (1-t)^m \lambda(dt) \int |u|^{m+1} |K(u)| \lambda(du) \|F^{(m+1)}\|_{\infty} \\
&= \frac{1}{(m+1)!} \int |u|^{m+1} |K(u)| \lambda(du) \|F^{(m+1)}\|_{\infty} = \frac{K_{|m+1|}}{(m+1)!} \|F^{(m+1)}\|_{\infty}.
\end{aligned}$$

The latter expression is finite by assumption and independent of $h \in \mathbb{R}_{++}$. $\square$

A.8. Taylor expansion of MSE and MISE of $\widehat{F}_{n,h}$

Lemma A.2 *Let $m \in \mathbb{N}$. Let K be a kernel of order m for which we have $K_{|m+1|} < \infty$. Assume that F is $(m+1)$ -times continuously differentiable with $\int |F^{(m+1)}| d\lambda < \infty$ and $\|F^{(m+1)}\|_{\infty} < \infty$. Then there exists a function $R_{K,m} : \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ such that*

$$\begin{aligned}
& \text{MSE}^F(\widehat{F}_{n,h}(x), F(x)) \\
&= \frac{1}{n} F(x)(1 - F(x)) + \frac{1}{n} (F_h^*(x) - F(x)) - \frac{2(-h)^m K_{(m)}}{n m!} F^{(m)}(x) F(x) \\
&\quad + \frac{n-1}{n} h^{2m} \frac{K_{(m)}^2}{(m!)^2} F^{(m)}(x)^2 - \frac{2}{n} h^{m+1} R_{K,m}(x, h) F(x)
\end{aligned} \tag{A.5}$$

$$+ 2 \frac{n-1}{n} (-1)^m h^{2m+1} \frac{K_{(m)}}{m!} R_{K,m}(x, h) F^{(m)}(x) + \frac{n-1}{n} h^{2m+2} R_{K,m}(x, h)^2$$

for all $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$, and $\sup_{h \in \mathbb{R}_{++}} \int |R_{K,m}(x, h)| \lambda(dx)$ and $\sup_{h \in \mathbb{R}_{++}} \|R_{K,m}(\cdot, h)\|_\infty$ are finite.

Proof According to Lemma A.1, we can find a continuous function $R_{K,m} : \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ such that $\sup_{h \in \mathbb{R}_{++}} \int |R_{K,m}(x, h)| < \infty$ and $\sup_{h \in \mathbb{R}_{++}} \|R_{K,m}(\cdot, h)\|_\infty < \infty$ as well as

$$\begin{aligned} \frac{n-1}{n} (F_h(x) - F(x))^2 &= \frac{n-1}{n} \left((-h)^m \frac{K_{(m)}}{m!} F^{(m)}(x) + h^{m+1} R_{K,m}(x, h) \right)^2 \\ &= \frac{n-1}{n} h^{2m} \frac{K_{(m)}^2}{(m!)^2} F^{(m)}(x)^2 + 2 \frac{n-1}{n} (-1)^m h^{2m+1} \frac{K_{(m)}}{m!} R_{K,m}(x, h) F^{(m)}(x) \\ &\quad + \frac{n-1}{n} h^{2m+2} R_{K,m}(x, h)^2 \end{aligned}$$

and

$$\begin{aligned} \frac{2}{n} (F_h(x) - F(x)) F(x) &= \frac{2}{n} \left((-h)^m \frac{K_{(m)}}{m!} F^{(m)}(x) + h^{m+1} R_{K,m}(x, h) \right) F(x) \\ &= \frac{2(-h)^m}{n} \frac{K_{(m)}}{m!} F^{(m)}(x) F(x) + \frac{2}{n} h^{m+1} R_{K,m}(x, h) F(x) \end{aligned}$$

for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$. Thus, in view of (7),

$$\begin{aligned} \text{MSE}^F(\widehat{F}_{n,h}(x), F(x)) &= \frac{1}{n} F(x)(1 - F(x)) + \frac{1}{n} (F_h^*(x) - F(x)) \\ &\quad - \frac{2}{n} (F_h(x) - F(x)) F(x) + \frac{n-1}{n} (F_h(x) - F(x))^2 \\ &= \frac{1}{n} F(x)(1 - F(x)) + \frac{1}{n} (F_h^*(x) - F(x)) - \frac{2(-h)^m}{n} \frac{K_{(m)}}{m!} F^{(m)}(x) F(x) \\ &\quad + \frac{n-1}{n} h^{2m} \frac{K_{(m)}^2}{(m!)^2} F^{(m)}(x)^2 - \frac{2}{n} h^{m+1} R_{K,m}(x, h) F(x) \\ &\quad + 2 \frac{n-1}{n} (-1)^m h^{2m+1} \frac{K_{(m)}}{m!} R_{K,m}(x, h) F^{(m)}(x) + \frac{n-1}{n} h^{2m+2} R_{K,m}(x, h)^2 \end{aligned}$$

for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$. □

Lemma A.3 *In the setting of Lemma A.2, assume in addition that the integrals $\int F(1-F) d\lambda$ and $\int |F^{(m)}| d\lambda$ are finite. Then we have*

$$\begin{aligned} \text{MISE}^F(\widehat{F}_{n,h}, F) &= \frac{1}{n} \int F(1-F) d\lambda - \frac{h}{n} K_{(1)}^* - \frac{2(-h)^m}{n} \frac{K_{(m)}}{m!} \int F^{(m)} F d\lambda \end{aligned} \tag{A.6}$$

$$\begin{aligned}
& + \frac{n-1}{n} h^{2m} \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda - \frac{2}{n} h^{m+1} \int R_{K,m}(\cdot, h) F d\lambda \\
& + 2 \frac{n-1}{n} (-1)^m h^{2m+1} \frac{K_{(m)}}{m!} \int R_{K,m}(\cdot, h) F^{(m)} d\lambda + \frac{n-1}{n} h^{2m+2} \int R_{K,m}(\cdot, h)^2 d\lambda
\end{aligned}$$

for all $h \in \mathbb{R}_{++}$, where $R_{K,m}$ is as in Lemma A.2.

Proof By integrating both sides of Equation (A.5) we obtain that

$$\begin{aligned}
& \text{MISE}^F(\hat{F}_{n,h}, F) \tag{A.7} \\
& = \frac{1}{n} \int F(1-F) d\lambda + \frac{1}{n} \int (F_h^* - F) d\lambda - \frac{2(-h)^m K_{(m)}}{n m!} \int F^{(m)} F d\lambda \\
& + \frac{n-1}{n} h^{2m} \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda - \frac{2}{n} h^{m+1} \int R_{K,m}(\cdot, h) F d\lambda \\
& + 2 \frac{n-1}{n} (-1)^m h^{2m+1} \frac{K_{(m)}}{m!} \int R_{K,m}(\cdot, h) F^{(m)} d\lambda + \frac{n-1}{n} h^{2m+2} \int R_{K,m}(\cdot, h)^2 d\lambda
\end{aligned}$$

for every $h \in \mathbb{R}_{++}$, where $R_{K,m}$ is as in Lemma A.2. Note at this point that $F^{(m)}$ is differentiable with bounded derivative, so $\int |F^{(m)}| d\lambda < \infty$ implies both $\int (F^{(m)})^2 d\lambda < \infty$ and $\|F^{(m)}\|_\infty < \infty$. Since $R_{K,m}$ is continuous, we also have that $\sup_{h \in \mathbb{R}_{++}} \int |R_{K,m}(\cdot, h)| d\lambda < \infty$ implies $\sup_{h \in \mathbb{R}_{++}} \int R_{K,m}(\cdot, h)^2 d\lambda < \infty$. The right-hand side of (A.7) differs from the right-hand side of (A.6) only by $\frac{1}{n} \int (F_h^* - F) d\lambda - (-\frac{h}{n} K_{(1)}^*)$. It is therefore sufficient to show that $\frac{1}{n} \int (F_h^* - F) d\lambda = -\frac{h}{n} K_{(1)}^*$ for all $h \in \mathbb{R}_{++}$.

Note that F is λ -a.e. differentiable, since we assumed that it is absolutely continuous. The derivative F' can be extended to all of $\mathbb{R}$ by setting $F'(x) := 0$ for all points $x \in \mathbb{R}$ at which F is not differentiable. In Section A.5 we have seen that this implies (A.2) for all $x, u \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$. Then it follows from the transformation theorem, (A.2) and Fubini's theorem that the integral $\int (F_h^*(x) - F(x)) \lambda(dx)$ is well-defined with

$$\begin{aligned}
& \frac{1}{n} \int (F_h^*(x) - F(x)) \lambda(dx) = \frac{1}{n} \iint (K^*)_h(x-y) (F(y) - F(x)) \lambda(dy) \lambda(dx) \\
& = \frac{1}{n} \iint K^*(u) (F(x-uh) - F(x)) \lambda(du) \lambda(dx) \\
& = -\frac{h}{n} \iint u K^*(u) \int_{[0,1]} F'(x-uh t) \lambda(dt) \lambda(du) \lambda(dx) \\
& = -\frac{h}{n} \iint \int_{[0,1]} u K^*(u) F'(x-uh t) \lambda(dt) \lambda(du) \lambda(dx) \\
& = -\frac{h}{n} \int \int_{[0,1]} \int u K^*(u) F'(x-uh t) \lambda(dx) \lambda(dt) \lambda(du) \\
& = -\frac{h}{n} \int u K^*(u) \int_{[0,1]} \int F'(y) \lambda(dy) \lambda(dt) \lambda(du) \\
& = -\frac{h}{n} \int u K^*(u) \lambda(du) = -\frac{h}{n} K_{(1)}^* \tag{A.8}
\end{aligned}$$

for every $h \in \mathbb{R}_{++}$. For the well-definedness and the application of Fubini's theorem, we used that the first absolute moment of K^* is finite when the first absolute moment of K is finite. $\square$

A.9. Proof of Lemma 3.2

The proof of Lemma 3.2 was already given in [6]; see the proof of Theorem 1 there. As the proof in the presentation there was quite brief, a few more details are given here.

For the first claim in Lemma 3.2 we have to show that the function $M_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is well-defined and possesses a minimum point. For this it suffices to show that $M_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is well-defined and continuous (see Step 1) and that $\liminf_{h \rightarrow \infty} M_n^F(h) = \infty$ (see Step 2). For the second claim in Lemma 3.2 we have to show that $h = 0$ is *not* a minimum point if $K_{(1)}^* > K_{(1)}$. For this it suffices to show that M_n^F is right-sided differentiable at 0 with right-sided derivative $(M_n^F)'(0) = \frac{1}{n}(K_{(1)} - K_{(1)}^*)$ (see Step 3) since $K_{(1)}^* > K_{(1)}$ implies $\frac{1}{n}(K_{(1)} - K_{(1)}^*) < 0$.

Step 1. First, we have $M_n^F(h) = \text{MISE}^F(\hat{F}_{n,h}, F) = \int \text{MSE}^F(\hat{F}_{n,h}(x), F(x)) \lambda(dx)$. Since $\text{MSE}^F(\hat{F}_{n,h}(x), F(x)) \in \mathbb{R}_+$ for every $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$, M_n^F takes values only in $\overline{\mathbb{R}}_+$. Second, by (7), we have for any $h \in \mathbb{R}_{++}$ that

$$\begin{aligned} M_n^F(h) - M_n^F(0) &= \frac{1}{n} \int (F_h^* - F) d\lambda - \frac{2}{n} \int (F_h - F) F d\lambda \\ &\quad + \frac{n-1}{n} \int (F_h - F)^2 d\lambda =: S_1(h) + S_2(h) + S_3(h). \end{aligned} \quad (\text{A.9})$$

Since $M_n^F(0)$ is well-defined and finite (note that $\int |x| dF(x) < \infty$ implies $\int F(1 - F) d\lambda < \infty$), it follows that the function $M_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is well-defined and continuous if we can show that, for every $j = 1, 2, 3$, the summand $S_j(\cdot)$, regarded as a real-valued function on $\mathbb{R}_{++}$, is well-defined and continuous and satisfies $\lim_{h \rightarrow 0} S_j(h) = 0$. For the proof of these conditions we will use that the assumptions of Lemma 3.2 imply Equations (A.2) and (A.8).

For the function S_1 , it follows from (A.8) that it is well-defined with $S_1(h) = \frac{1}{n} \int (F_h^* - F) d\lambda = -\frac{h}{n} K_{(1)}^*$ for every $h \in \mathbb{R}_{++}$. Thus, it immediately follows that $S_1 : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is continuous with $\lim_{h \rightarrow 0} S_1(h) = 0$.

For the function S_2 , it follows from the transformation theorem, (A.2) and Fubini's theorem that it is well-defined with

$$\begin{aligned} S_2(h) &= -\frac{2}{n} \int (F_h(x) - F(x)) F(x) \lambda(dx) \\ &= -\frac{2}{n} \iint K_h(x - y) (F(y) - F(x)) F(x) \lambda(dy) \lambda(dx) \\ &= -\frac{2}{n} \iint K(u) (F(x - uh) - F(x)) F(x) \lambda(du) \lambda(dx) \end{aligned}$$

$$\begin{aligned}
&= \frac{2h}{n} \iint \int_{[0,1]} uK(u)F'(x - uht)F(x) \lambda(dt) \lambda(du) \lambda(dx) \\
&= \frac{2h}{n} \iint \int_{[0,1]} uK(u)F'(y)F(y + uht) \lambda(dt) \lambda(dy) \lambda(du) \\
&= \frac{2h}{n} \iint \int_{[0,1]} uK(u)F(y + uht) \lambda(dt) \lambda(du) F'(y) \lambda(dy) \\
&= \frac{2h}{n} \iint \int_{[0,1]} uK(u)F(y + uht) \lambda(dt) \lambda(du) dF(y) \tag{A.10}
\end{aligned}$$

for every $h \in \mathbb{R}_{++}$. For the well-definedness note that the latter triple integral is absolutely bounded uniformly in $h \in \mathbb{R}_{++}$ since $|uK(u)F(y + uht)| \leq |u||K(u)|$ for all u, y, h, t (recall that K is assumed to have a finite first absolute moment). This also implies that $\lim_{h \rightarrow 0} S_2(h) = 0$. Moreover, by the continuity lemma (in the form of Lemma 16.1 in [1]), we can derive from (A.10) that $S_2 : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is continuous; take into account that F is (absolutely) continuous.

Finally, for the function S_3 , we obtain again by using the transformation theorem, (A.2) and Fubini's theorem that it is well-defined with

$$\begin{aligned}
S_3(h) &= \frac{n-1}{n} \int (F_h(x) - F(x))(F_h(x) - F(x)) \lambda(dx) \\
&= \frac{n-1}{n} \int \left(\int K_h(x-y)F(y) \lambda(dy) - F(x) \right) \left(\int K_h(x-z)F(z) \lambda(dz) - F(x) \right) \lambda(dx) \\
&= \frac{n-1}{n} \int \int K_h(x-y)(F(y) - F(x)) \lambda(dy) \int K_h(x-z)(F(z) - F(x)) \lambda(dz) \lambda(dx) \\
&= \frac{n-1}{n} \iiint K(u)K(v)(F(x-hu) - F(x))(F(x-hv) - F(x)) \lambda(du) \lambda(dv) \lambda(dx) \\
&= -h \frac{n-1}{n} \iiint uK(u)K(v) \int_{[0,1]} F'(x - uht) \lambda(dt) \\
&\quad (F(x - hv) - F(x)) \lambda(du) \lambda(dv) \lambda(dx) \\
&= -h \frac{n-1}{n} \iiint \int_{[0,1]} uK(u)K(v)F'(y) \\
&\quad (F(y + uht - hv) - F(y + uht)) \lambda(dt) \lambda(du) \lambda(dv) \lambda(dy) \\
&= -h \frac{n-1}{n} \iiint \int_{[0,1]} uK(u)K(v) \\
&\quad (F(y + uht - hv) - F(y + uht)) \lambda(dt) \lambda(du) \lambda(dv) dF(y) \tag{A.11}
\end{aligned}$$

for every $h \in \mathbb{R}_{++}$. For the well-definedness note that the latter quadruple integral is absolutely bounded uniformly in $h \in \mathbb{R}_{++}$ since $|uK(u)K(v)(F(y + uht - hv) - F(y + uht))| \leq 2|u||K(u)||K(v)|$ for all u, v, y, h, t (recall that K is assumed to have a finite first absolute moment). This implies that $\lim_{h \rightarrow 0} S_3(h) = 0$. Moreover, by the continuity

lemma (in the form of Lemma 16.1 in [1]), we can derive from (A.11) that $S_3 : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is continuous; take into account that F is (absolutely) continuous.

Step 2. To show that $\liminf_{h \rightarrow \infty} M_n^F(h) = \infty$, we first note that

$$\begin{aligned} M_n^F(h) &= \int \left(\frac{1}{n} \int \overline{K}_h(x-y)^2 dF(y) - \frac{1}{n} F_h(x)^2 + (F_h(x) - F(x))^2 \right) \lambda(dx) \\ &= \frac{1}{n} \int \left(\int \overline{K}_h(x-y)^2 dF(y) - \left(\int \overline{K}_h(x-y) dF(y) \right)^2 \right) \lambda(dx) \\ &\quad + \int (F_h(x) - F(x))^2 \lambda(dx) \geq \int (F_h(x) - F(x))^2 \lambda(dx) \end{aligned} \quad (\text{A.12})$$

for every $h \in \mathbb{R}_{++}$. Thus, in view of Fatou's lemma, we have $\liminf_{h \rightarrow \infty} M_n^F(h) \geq \liminf_{h \rightarrow \infty} \int (F_h(x) - F(x))^2 \lambda(dx) \geq \int \liminf_{h \rightarrow \infty} (F_h(x) - F(x))^2 \lambda(dx) \geq \int (\overline{K}(0) - F(x))^2 \lambda(dx) = \infty$. The second last step can be seen as follows. We have $|F_h(x) - \overline{K}(0)| = \left| \int (\overline{K}_h(x-y) - \overline{K}(0)) dF(y) \right| \leq \int |\overline{K}((x-y)/h) - \overline{K}(0)| dF(y)$, and $\overline{K}$ is continuous (see Section A.2) and $\lim_{h \rightarrow \infty} (x-y)/h = 0$ holds true for all $x, y \in \mathbb{R}$. Thus, $\lim_{h \rightarrow \infty} F_h(x) = \overline{K}(0)$.

Step 3. It remains to show that M_n^F is right-sided differentiable at 0 with right-sided derivative $(M_n^F)'(0) = \frac{1}{n}(K_{(1)} - K_{(1)}^*)$. In view of (A.9), it suffices to show that $\lim_{h \rightarrow 0} S_1(h)/h = -K_{(1)}^*/n$, $\lim_{h \rightarrow 0} S_2(h)/h = K_{(1)}/n$ and $\lim_{h \rightarrow 0} S_3(h)/h = 0$. The convergence $\lim_{h \rightarrow 0} S_1(h) = -K_{(1)}^*/n$ directly follows from (A.8). By using (A.10), we get

$$\begin{aligned} \lim_{h \rightarrow 0} S_2(h)/h &= \lim_{h \rightarrow 0} \frac{2}{n} \iint \int_{[0,1]} uK(u)F(y+uht) \lambda(dt) \lambda(du) dF(y) \\ &= \frac{2}{n} \iint \int_{[0,1]} uK(u) \lim_{h \rightarrow 0} F(y+uht) \lambda(dt) \lambda(du) dF(y) \\ &= \frac{1}{n} \int uK(u) \lambda(du) \int 2F(y)F'(y) \lambda(dy) = \frac{K_{(1)}}{n}, \end{aligned}$$

where the second $=$ is ensured by the dominated convergence theorem (with majorant $|uK(u)|$), the third $=$ relies on the continuity of F and the last $=$ is true since $\int 2F(y)F'(y) \lambda(dy) = \int (F^2)'(y) \lambda(dy) = 1$ (note here that $(F^2)'(\cdot)$ is a Lebesgue density of the distribution function $\tilde{F} := F^2$). Finally, by using (A.11), we also get

$$\begin{aligned} \lim_{h \rightarrow 0} S_3(h)/h &= \lim_{h \rightarrow 0} \frac{n-1}{n} \iiint \int_{[0,1]} uK(u)K(v)(F(y+uht-hv) - F(y+uht)) \\ &\quad \lambda(dt) \lambda(du) \lambda(dv) dF(y) \\ &= \frac{n-1}{n} \iiint \int_{[0,1]} uK(u)K(v) \lim_{h \rightarrow 0} (F(y+uht-hv) - F(y+uht)) \\ &\quad \lambda(dt) \lambda(du) \lambda(dv) dF(y) \\ &= \frac{n-1}{n} \iiint \int_{[0,1]} uK(u)K(v) 0 \lambda(dt) \lambda(du) \lambda(dv) dF(y) = 0, \end{aligned}$$

where the second $=$ is ensured by the dominated convergence theorem (with majorant $|uK(u)K(v)|$) and the third $=$ relies on the continuity of F .

A.10. Proof of Remark 3.3

Below we will show that

$$\kappa_{(1)} = \int_{(0,\infty)} (1 - \bar{\kappa}) d\lambda - \int_{(-\infty,0)} \bar{\kappa} d\lambda \quad (\text{A.13})$$

for any kernel κ with $\kappa_{|1|} < \infty$. Since $K_{|1|} < \infty$ by assumption (and therefore also $K_{|1|}^* < \infty$), we can apply (A.13) to $\kappa := K$ and to $\kappa := K^*$ (where K^* is derived from K as κ^* was derived from κ in Section 2). So together with $\overline{K^*}(\cdot) = \overline{K}(\cdot)^2$ (see Section A.3) we get

$$\begin{aligned} K_{(1)}^* - K_{(1)} &= \int_{(0,\infty)} (1 - \overline{K^2}) d\lambda - \int_{(0,\infty)} (1 - \overline{K}) d\lambda - \int_{(-\infty,0)} \overline{K^2} d\lambda + \int_{(-\infty,0)} \overline{K} d\lambda \\ &= \int_{(0,\infty)} ((1 - \overline{K^2}) - (1 - \overline{K})) d\lambda - \int_{(-\infty,0)} (\overline{K^2} - \overline{K}) d\lambda \\ &= \int_{(0,\infty)} (\overline{K} - \overline{K^2}) d\lambda + \int_{(-\infty,0)} (\overline{K} - \overline{K^2}) d\lambda = \int (\overline{K} - \overline{K^2}) d\lambda. \end{aligned}$$

This proves the first claim in Remark 3.3. If K is nonnegative, then $\overline{K}$ is nondecreasing (with $\lim_{x \rightarrow -\infty} \overline{K}(x) = 0$ and $\lim_{x \rightarrow \infty} \overline{K}(x) = 1$) and therefore takes values only in $[0, 1]$. Thus, $\overline{K^2} \leq \overline{K}$, and therefore the latter integral is nonnegative. This proves the second claim in Remark 3.3.

It remains to show (A.13). Let κ be any kernel with $\kappa_{|1|} < \infty$. Recall that $\bar{\kappa}$ is absolutely continuous (see Section A.2). By Theorem 7.20 in [5] it follows that $\bar{\kappa}$ is λ -a.e. differentiable (see also [4, VII.4.12.b]) and by [4, VII.4.14.a] we can find a λ -null set $N \in \mathcal{B}(\mathbb{R})$ such that $\bar{\kappa}' = \kappa$ on N^c . On N , we set $\bar{\kappa}' := 0$, and we note that $(\bar{\kappa} - 1)' = \kappa$ on N^c .

Let $a \in (-\infty, 0)$ and $b \in \mathbb{R}_{++}$ be arbitrary. With the help of [4, VII.4.16] we obtain

$$\begin{aligned} \int_{[a,b]} u\kappa(u) \lambda(du) &= \int_{[a,0]} u\bar{\kappa}'(u) \lambda(du) + \int_{[0,b]} u\bar{\kappa}'(u) \lambda(du) \\ &= [u\bar{\kappa}(u)]_a^0 - \int_{[a,0]} \bar{\kappa}(u) \lambda(du) + [u(\bar{\kappa}(u) - 1)]_0^b - \int_{[0,b]} (\bar{\kappa}(u) - 1) \lambda(du) \\ &= [u\bar{\kappa}(u)]_a^0 + [u(\bar{\kappa}(u) - 1)]_0^b + \int_{[0,b]} (1 - \bar{\kappa}(u)) \lambda(du) - \int_{[a,0]} \bar{\kappa}(u) \lambda(du) \\ &= -a\bar{\kappa}(a) - b(1 - \bar{\kappa}(b)) + \int_{[0,b]} (1 - \bar{\kappa}(u)) \lambda(du) - \int_{[a,0]} \bar{\kappa}(u) \lambda(du). \end{aligned}$$

Thus, using the dominated convergence theorem twice (for this we need the assumption $\kappa_{|1|} < \infty$), we get $\kappa_{(1)} = \lim_{n \rightarrow \infty} \int_{[-n,n]} u\kappa(u) \lambda(du) = \lim_{n \rightarrow \infty} (n\bar{\kappa}(-n) - n(1 - \bar{\kappa}(n)) +$

$\int_{[0,n]}(1 - \bar{\kappa}) d\lambda - \int_{[-n,0]} \bar{\kappa} d\lambda = \int_{(0,\infty)}(1 - \bar{\kappa}) d\lambda - \int_{(-\infty,0)} \bar{\kappa} d\lambda$, i.e. (A.13). Note at this point that $|n\bar{\kappa}(-n)| = |n \int_{(-\infty,-n]} \kappa d\lambda| \leq \int_{(-\infty,-n]} n|\kappa| d\lambda \leq \int_{(-\infty,-n]} |u||\kappa(u)| \lambda(du) = \int \mathbb{1}_{(-\infty,-n]}(u)|u||\kappa(u)| \lambda(du)$ for all $n \in \mathbb{N}$ and that the latter integral converges to 0 as $n \rightarrow \infty$ (by the dominated convergence theorem and $K_{[1]} < \infty$). Moreover, we have that $|n(1 - \bar{\kappa}(n))| = |n \int_{(n,\infty)} \kappa d\lambda| \leq \int_{(n,\infty)} n|\kappa| d\lambda \leq \int_{(n,\infty)} |u||\kappa(u)| \lambda(du) = \int \mathbb{1}_{(n,\infty)}(u)|u||\kappa(u)| \lambda(du)$ for all $n \in \mathbb{N}$ and the latter integral converges to 0 as $n \rightarrow \infty$ (by the dominated convergence theorem and $K_{[1]} < \infty$).

A.11. Proof of Lemma 3.4

In a more general framework, Lemma 3.4 has already been proved in [3]; see the proof of Theorem 2 there. Since we will need most of the arguments later for the proofs of Theorem 3.9 (see Section A.13) and Theorem 4.2 (see Section A.16), we will carry out the proof here in our special case. We need the following lemma:

Lemma A.4 *For every $h \in \mathbb{R}_{++}$, the function F_h is continuous and of bounded variation, and the signed measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ induced by F_h possesses the λ -density $f_h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_h(x) := \int K_h(x - y) dF(y)$.*

Proof Let $h \in \mathbb{R}_{++}$ be arbitrary but fixed.

Step 1. We first show that F_h is continuous. Note that K_h is continuous (choose $\kappa := K_h$ in Section A.2) and that $|\overline{K_h}(x)| = |\overline{K}(x/h)| \leq \|\overline{K}\|_\infty$ for all $x \in \mathbb{R}$. Since $\|\overline{K}\|_\infty < \infty$ (choose $\kappa := K$ in Section A.2), it follows from the continuity lemma (in the form of Lemma 16.1 in [1]) that the map $x \mapsto F_h(x) = \int_{\mathbb{R}} \overline{K_h}(x - y) dF(y)$ is continuous.

Step 2. We next show that F_h is of bounded variation. First note that $\overline{K_h}$ is of bounded variation. Indeed: For every $a, b \in \mathbb{R}$, $r \in \mathbb{N}$ and $x_0, \dots, x_r \in [a, b]$ with $a \leq x_0 < x_1 < \dots < x_r \leq b$ we have that

$$\sum_{i=1}^r |\overline{K_h}(x_i) - \overline{K_h}(x_{i-1})| = \sum_{i=1}^r \left| \int_{(x_{i-1}, x_i]} K_h d\lambda \right| \leq \sum_{i=1}^r \int_{(x_{i-1}, x_i]} |K_h| d\lambda \leq \int_{\mathbb{R}} |K_h| d\lambda,$$

and the latter upper bound is finite and independent of the specific choice of a, b , r and $x_0, \dots, x_r$. Consequently, for every $a, b \in \mathbb{R}$, $r \in \mathbb{N}$ and $x_0, \dots, x_r \in [a, b]$ with $a \leq x_0 < x_1 < \dots < x_r \leq b$, we have by (4) that

$$\begin{aligned} \sum_{i=1}^r |F_h(x_i) - F_h(x_{i-1})| &= \sum_{i=1}^r \left| \int_{\mathbb{R}} \overline{K_h}(x_i - y) dF(y) - \int_{\mathbb{R}} \overline{K_h}(x_{i-1} - y) dF(y) \right| \\ &\leq \int_{\mathbb{R}} \sum_{i=1}^r |\overline{K_h}(x_i - y) - \overline{K_h}(x_{i-1} - y)| dF(y) \leq \int_{\mathbb{R}} V(\overline{K_h}) dF(y) = V(\overline{K_h}) < \infty, \end{aligned}$$

where $V(\overline{K_h})$ is the total variation of $\overline{K_h}$. Thus, F_h is of bounded variation.

Step 3. In view of Steps 1 and 2, F_h is a measure generating function of a (finite) signed measure. By (4) and Fubini's theorem we obtain

$$\begin{aligned}
F_h(x) &= \int_{\mathbb{R}} \overline{K_h}(x-y) dF(y) = \int_{\mathbb{R}} \int_{(-\infty, x-y]} K_h(z) \lambda(dz) dF(y) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{(-\infty, x]}(u) K_h(u-y) \lambda(du) dF(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{(-\infty, x]}(u) K_h(u-y) dF(y) \lambda(du) \\
&= \int_{(-\infty, x]} \int_{\mathbb{R}} K_h(u-y) dF(y) \lambda(du) = \int_{(-\infty, x]} f_h(u) \lambda(du)
\end{aligned}$$

for all $x \in \mathbb{R}$, and therefore f_h is a λ -density of the Borel signed measure induced by F_h . $\square$

We will now prove that $\lim_{n \rightarrow \infty} h_{n, \text{opt}}^F = 0$. Note that the sequence $(h_{n, \text{opt}}^F)_{n \in \mathbb{N}}$ is well-defined in $\mathbb{R}_+^{\mathbb{N}}$ by Lemma 3.2.

From (A.12) we know that $M_n^F(h) \geq g^F(h) \geq 0$ for all $h \in \mathbb{R}_+$ and $n \in \mathbb{N}$, where the function $g^F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by $g^F(h) := \int (F_h - F)^2 d\lambda$, $h \in \mathbb{R}_{++}$, and $g^F(0) := 0$. Since $\lim_{n \rightarrow \infty} M_n^F(0) = \lim_{n \rightarrow \infty} \frac{1}{n} \int F(1-F) d\lambda = 0$ (note that $\int |x| dF(x) < \infty$ implies $\int F(1-F) d\lambda < \infty$), we necessarily have $\lim_{n \rightarrow \infty} M_n^F(h_{n, \text{opt}}^F) = 0$ (recall that $h_{n, \text{opt}}^F$ is a minimiser of $M_n^F(\cdot)$). Thus, in particular, $\lim_{n \rightarrow \infty} g^F(h_{n, \text{opt}}^F) = 0$.

Below we will show that $g^F(h) > 0$ for all $h \in \mathbb{R}_{++}$. Moreover, we know that $g^F(\cdot)$ is continuous (see Step 1 in Section A.9; take into account that $g^F(h) = \frac{n}{n-1} S_3(h)$ for all $h \in \mathbb{R}_{++}$) and that $\lim_{h \rightarrow \infty} g^F(h) = \infty$ (see Step 2 in Section A.9, specifically the lines after (A.12)). Thus, $\lim_{n \rightarrow \infty} g^F(h_{n, \text{opt}}^F) = 0$ implies that $\lim_{n \rightarrow \infty} h_{n, \text{opt}}^F = 0$.

It remains to show that $g^F(h) (= \int (F_h - F)^2 d\lambda)$ is strictly positive for every $h \in \mathbb{R}_{++}$. Let $h \in \mathbb{R}_{++}$ be arbitrary but fixed. Since F is (absolutely) continuous by assumption and F_h is continuous by Lemma A.4, it suffices to show that F_h and F are *not* identical. If F_h is not a probability distribution function, then the claim follows directly. Hence, we can assume that F_h is a probability distribution function. In this case it suffices to show that the characteristic functions φ_F and φ_{F_h} of (the distributions corresponding to) F_h and F , respectively, are *not* identical. In view of Lemma A.4 (according to which $f_h(\cdot) := \int K_h(\cdot - y) dF(y)$ defines a λ -density of dF_h) and Fubini's theorem, we obtain for every $t \in \mathbb{R}$ that

$$\begin{aligned}
\varphi_{F_h}(t) &= \int e^{itx} dF_h(x) = \int e^{itx} f_h(x) \lambda(dx) = \int e^{itx} \int K_h(x-y) dF(y) \lambda(dx) \\
&= \iint e^{itx} K_h(x-y) dF(y) \lambda(dx) = \iint e^{it(z+y)} K_h(z) \lambda(dz) dF(y) \\
&= \int e^{itz} K_h(z) \lambda(dz) \int e^{ity} dF(y) = \int e^{ithx} K(x) \lambda(dx) \int e^{ity} dF(y) \\
&= \int e^{ithx} K(x) \lambda(dx) \varphi_F(t).
\end{aligned}$$

Thus, for every $t \in \mathbb{R}$, we have $\varphi_F(t) = \varphi_{F_h}(t)$ if and only if $\int e^{ithx} K(x) \lambda(dx) = 1$. If we assume by way of contradiction that $\varphi_F(t) = \varphi_{F_h}(t)$ for all $t \in \mathbb{R}$, then we obtain $\int e^{i\xi x} K(x) \lambda(dx) = 1$ for all $\xi \in \mathbb{R}$. Since the mapping $\xi \mapsto \int e^{i\xi x} K(x) \lambda(dx)$ is, up to a constant factor, nothing but the Fourier transform of the $L^1(d\lambda)$ -function K , and the Fourier transform of any $L^1(d\lambda)$ -function is a continuous function vanishing at infinity, we indeed obtain a contradiction (take into account that the function that is $\equiv 1$ does not vanish at infinity).

A.12. Proof of Theorem 3.5

From Lemma 3.4 we know that $\lim_{n \rightarrow \infty} h_{n,\text{opt}}^F = 0$. We are now going to prove that $\lim_{n \rightarrow \infty} n^{1/(2m-1)} h_{n,\text{opt}}^F = c$ with c as in (8). We will use $\mathbb{R}_{+,0}^{\mathbb{N}}$ to denote the set of all sequences $(h_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ with $\lim_{n \rightarrow \infty} h_n = 0$. In particular, $(h_{n,\text{opt}}^F)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$. We will also use the notation

$$\underline{c}((h_n)_{n \in \mathbb{N}}) := \liminf_{n \rightarrow \infty} n^{1/(2m-1)} h_n \quad \text{and} \quad \bar{c}((h_n)_{n \in \mathbb{N}}) := \limsup_{n \rightarrow \infty} n^{1/(2m-1)} h_n$$

for every sequence $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$.

For every $n \in \mathbb{N}$, $h_{n,\text{opt}}^F$ minimises $M_n^F(h) = \text{MISE}^F(\hat{F}_{n,h}, F)$ over $h \in \mathbb{R}_+$ and therefore it also minimises

$$N_n^F(h) := n^{2m/(2m-1)} (M_n^F(h) - M_n^F(0)) = n^{1/(2m-1)} \left(n M_n^F(h) - \int F(1-F) d\lambda \right)$$

over all $h \in \mathbb{R}_+$. This implies that the sequence $(h_{n,\text{opt}}^F)_{n \in \mathbb{N}}$ is a minimiser of the function $\overline{N}^F : \mathbb{R}_{+,0}^{\mathbb{N}} \rightarrow \overline{\mathbb{R}}$ defined by

$$\overline{N}^F((h_n)_{n \in \mathbb{N}}) := \limsup_{n \rightarrow \infty} N_n^F(h_n).$$

Below we will show that

- (a) for every $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ with $\underline{c}((h_n)_{n \in \mathbb{N}}) = \bar{c}((h_n)_{n \in \mathbb{N}}) < \infty$, the expression $\overline{N}^F((h_n)_{n \in \mathbb{N}})$ depends on $(h_n)_{n \in \mathbb{N}}$ only through $\bar{c}((h_n)_{n \in \mathbb{N}})$,
- (b) $\underline{c}((h_{n,\text{opt}}^F)_{n \in \mathbb{N}}) = \bar{c}((h_{n,\text{opt}}^F)_{n \in \mathbb{N}}) < \infty$.

In view of (a), we can uniquely define a function $\overline{\nu}^F : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}$ by

$$\overline{\nu}^F(c) := \overline{N}^F((h_{c;n})_{n \in \mathbb{N}})$$

for an arbitrary $(h_{c;n})_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ with $\underline{c}((h_{c;n})_{n \in \mathbb{N}}) = \bar{c}((h_{c;n})_{n \in \mathbb{N}}) = c$. Below we will also show that

- (c) c defined in (8) is the unique minimum point of $\overline{\nu}^F : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}$.

Since $(h_{n,\text{opt}}^F)_{n \in \mathbb{N}}$ is a minimiser of the function $\overline{N}^F : \mathbb{R}_{+,0}^{\mathbb{N}} \rightarrow \overline{\mathbb{R}}$ (see above), assertions (a) and (b) imply that $\overline{c}((h_{n,\text{opt}}^F)_{n \in \mathbb{N}}) (= \underline{c}((h_{n,\text{opt}}^F)_{n \in \mathbb{N}}))$ is a minimiser of the function $\overline{\nu}^F : \mathbb{R}_+ \rightarrow \mathbb{R}$. Along with assertion (c), we can infer that $\underline{c}((h_{n,\text{opt}}^F)_{n \in \mathbb{N}}) = \overline{c}((h_{n,\text{opt}}^F)_{n \in \mathbb{N}}) = c$. Thus, $\lim_{n \rightarrow \infty} n^{1/(2m-1)} h_n = c$.

Proof of (a): For every $h \in \mathbb{R}_+$, we define

$$\begin{aligned} L_1^F(h) &:= K_{(1)}^* - (-h)^{m-1} \frac{K_{(m)}}{m!} \int 2F^{(m)} F d\lambda + 2h^m \int R_{K,m}(\cdot, h) F d\lambda, \\ L_2^F(h) &:= \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \\ &\quad + (-1)^m 2h \frac{K_{(m)}}{m!} \int R_{K,m}(\cdot, h) F^{(m)} d\lambda + h^2 \int R_{K,m}(\cdot, h)^2 d\lambda \end{aligned}$$

(which will appear in (A.16)–(A.18) below). Since $\sup_{h \in \mathbb{R}_{++}} \int |R_{K,m}(x, h)| \lambda(dx) < \infty$ and $\sup_{h \in \mathbb{R}_{++}} \|R_{K,m}(\cdot, h)\|_{\infty} < \infty$ (see Lemma A.3), $\|F^{(m)}\|_{\infty} < \infty$ (see the proof of Lemma A.3 in Section A.8) and $K_{(1)}^*$ is well-defined in $\mathbb{R}$ (since $K_{|1|} < \infty$), we have

$$\lim_{h \rightarrow 0} L_1^F(h) = K_{(1)}^* - K_{(1)} \in \mathbb{R}_{++}, \quad (\text{A.14})$$

$$\lim_{h \rightarrow 0} L_2^F(h) = \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \in \mathbb{R}_{++}. \quad (\text{A.15})$$

The term $-K_{(1)}$ in (A.14) is required for the case $m = 1$ and is equal to 0 when $m \geq 2$. For the case $m = 1$ note that F^2 is the distribution function of the distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with λ -density $(F^2)' = 2F'F$, which implies $\int 2F'F d\lambda = \int 1 dF^2 = 1$.

By Lemma A.3 we obtain

$$\begin{aligned} &n(M_n^F(h) - M_n^F(0)) \\ &= -hK_{(1)}^* - 2(-h)^m \frac{K_{(m)}}{m!} \int F^{(m)} F d\lambda + (n-1)h^{2m} \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \\ &\quad - 2h^{m+1} \int R_{K,m}(\cdot, h) F d\lambda + 2(n-1)(-1)^m h^{2m+1} \frac{K_{(m)}}{m!} \int R_{K,m}(\cdot, h) F^{(m)} d\lambda \\ &\quad + (n-1)h^{2m+2} \int R_{K,m}(\cdot, h)^2 d\lambda \\ &= -h \left(K_{(1)}^* - (-h)^{m-1} \frac{K_{(m)}}{m!} \int 2F^{(m)} F d\lambda + 2h^m \int R_{K,m}(\cdot, h) F d\lambda \right) \\ &\quad + (n-1)h^{2m} \left(\frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda + (-1)^m 2h \frac{K_{(m)}}{m!} \int R_{K,m}(\cdot, h) F^{(m)} d\lambda \right. \\ &\quad \left. + h^2 \int R_{K,m}(\cdot, h)^2 d\lambda \right) \\ &= -hL_1^F(h) + (n-1)h^{2m}L_2^F(h) = h(-L_1^F(h) + (n-1)h^{2m-1}L_2^F(h)) \end{aligned} \quad (\text{A.16})$$

for all $h \in \mathbb{R}_{++}$ and $n \in \mathbb{N}$. Since also $n(M_n^F(0) - M_n^F(0)) = 0 = 0 \cdot (-L_1^F(0) + (n-1)h^{2m-1}L_2^F(0))$, we therefore have

$$\begin{aligned} N_n^F(h) &= n^{2m/(2m-1)}(M_n^F(h) - M_n^F(0)) = n^{1/(2m-1)}n(M_n^F(h) - M_n^F(0)) \\ &= n^{1/(2m-1)}h(-L_1^F(h) + (n-1)h^{2m-1}L_2^F(h)) \end{aligned} \quad (\text{A.17})$$

for all $h \in \mathbb{R}_+$ and $n \in \mathbb{N}$. From (A.17) and (A.14)–(A.15) we obtain

$$\begin{aligned} \overline{N}^F((h_n)_{n \in \mathbb{N}}) &= \limsup_{n \rightarrow \infty} N_n^F(h_n) \\ &= \limsup_{n \rightarrow \infty} n^{1/(2m-1)}h_n(-L_1^F(h_n) + (n-1)h_n^{2m-1}L_2^F(h_n)) \\ &= \left(\lim_{n \rightarrow \infty} n^{1/(2m-1)}h_n \right) \left(-\lim_{n \rightarrow \infty} L_1^F(h_n) + \left(\lim_{n \rightarrow \infty} (n-1)h_n^{2m-1} \right) \lim_{n \rightarrow \infty} L_2^F(h_n) \right) \\ &= \bar{c}((h_n)_{n \in \mathbb{N}}) \left(-(K_{(1)}^* - K_{(1)}) + \bar{c}((h_n)_{n \in \mathbb{N}})^{2m-1} \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \right) \end{aligned} \quad (\text{A.18})$$

for any sequence $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ with $\underline{c}((h_n)_{n \in \mathbb{N}}) = \bar{c}((h_n)_{n \in \mathbb{N}}) < \infty$. Clearly, this expression depends on $(h_n)_{n \in \mathbb{N}}$ only through $\bar{c}((h_n)_{n \in \mathbb{N}})$. By the way, (A.18) shows that the function $\bar{\nu}^F : \mathbb{R}_+ \rightarrow \mathbb{R}$ admits the representation

$$\bar{\nu}^F(c) = c \left(-(K_{(1)}^* - K_{(1)}) + c^{2m-1} \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \right). \quad (\text{A.19})$$

Proof of (c): Since $\bar{\nu}^F$ is a polynomial of the form $c \mapsto -ac + bc^{2m}$ with $a, b \in \mathbb{R}_{++}$ (see (A.19)), it can be seen easily that $\bar{\nu}^F$ possesses exactly one minimum point and that this minimum point is given by c introduced in (8).

Proof of (b): Recall that $(h_{n,\text{opt}}^F)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ and that $h_{n,\text{opt}}^F$ minimises $N_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}$ for every $n \in \mathbb{N}$. Thus, it suffices to show that for any sequence $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$, which does *not* satisfy $\underline{c}((h_n)_{n \in \mathbb{N}}) = \bar{c}((h_n)_{n \in \mathbb{N}}) < \infty$, we can *not* have that h_n is a minimiser of $N_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}$ for every $n \in \mathbb{N}$.

Case 1. Let $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ be such that $\bar{c}((h_n)_{n \in \mathbb{N}}) (= \limsup_{n \rightarrow \infty} n^{1/(2m-1)}h_n) = \infty$. In this case, we have from (A.14)–(A.15) that $\limsup_{n \rightarrow \infty} (-L_1^F(h_n) + (n-1)h_n^{2m-1}L_2^F(h_n)) = \infty$. Thus, in view of (A.17), $\overline{N}^F((h_n)_{n \in \mathbb{N}}) = \limsup_{n \rightarrow \infty} n^{1/(2m-1)}h_n(-L_1^F(h_n) + (n-1)h_n^{2m-1}L_2^F(h_n)) = \infty$. So $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ can *not* be a minimiser of $\overline{N}^F : \mathbb{R}_{+,0}^{\mathbb{N}} \rightarrow \mathbb{R}$ (since, for instance, $\overline{N}^F((0)_{n \in \mathbb{N}}) = 0$). In particular, we can *not* have that h_n is a minimiser of $N_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}$ for every $n \in \mathbb{N}$.

Case 2. Let $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ be such that $\underline{c}((h_n)_{n \in \mathbb{N}}) < \bar{c}((h_n)_{n \in \mathbb{N}}) < \infty$. By way of contradiction, assume that h_n is a minimiser of $N_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}$ for every $n \in \mathbb{N}$. In this case we have that for every subsequence $(h_{n_j})_{j \in \mathbb{N}}$ of $(h_n)_{n \in \mathbb{N}}$ and every $j \in \mathbb{N}$ the term h_{n_j} minimises $N_{n_j}^F : \mathbb{R}_+ \rightarrow \mathbb{R}$. In particular, $(h_{n_j})_{j \in \mathbb{N}}$ minimises $\limsup_{j \rightarrow \infty} N_{n_j}^F(h_j)$ over all $(h_j)_{j \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$. We can choose two subsequences $(h_{\underline{n}_j})_{j \in \mathbb{N}}$ and $(h_{\bar{n}_j})_{j \in \mathbb{N}}$ of $(h_n)_{n \in \mathbb{N}}$

such that $\lim_{j \rightarrow \infty} \underline{n}_j^{1/(2m-1)} h_{\underline{n}_j} = \underline{c}((h_n)_{n \in \mathbb{N}})$ and $\lim_{j \rightarrow \infty} \bar{n}_j^{1/(2m-1)} h_{\bar{n}_j} = \bar{c}((h_n)_{n \in \mathbb{N}})$, respectively. If we set

$$\begin{aligned} \underline{c}_*((h_{\underline{n}_j})_{j \in \mathbb{N}}) &:= \liminf_{j \rightarrow \infty} \underline{n}_j^{1/(2m-1)} h_{\underline{n}_j}, & \underline{c}^*((h_{\underline{n}_j})_{j \in \mathbb{N}}) &:= \limsup_{j \rightarrow \infty} \underline{n}_j^{1/(2m-1)} h_{\underline{n}_j}, \\ \bar{c}_*((h_{\bar{n}_j})_{j \in \mathbb{N}}) &:= \liminf_{j \rightarrow \infty} \bar{n}_j^{1/(2m-1)} h_{\bar{n}_j}, & \bar{c}^*((h_{\bar{n}_j})_{j \in \mathbb{N}}) &:= \limsup_{j \rightarrow \infty} \bar{n}_j^{1/(2m-1)} h_{\bar{n}_j}, \end{aligned}$$

then we have in particular $\underline{c}((h_n)_{n \in \mathbb{N}}) = \underline{c}_*((h_{\underline{n}_j})_{j \in \mathbb{N}}) = \underline{c}^*((h_{\underline{n}_j})_{j \in \mathbb{N}})$ and $\bar{c}((h_n)_{n \in \mathbb{N}}) = \bar{c}_*((h_{\bar{n}_j})_{j \in \mathbb{N}}) = \bar{c}^*((h_{\bar{n}_j})_{j \in \mathbb{N}})$, and therefore (analogously to (A.18)) we can use (A.17) and (A.14)–(A.15) to get

$$\begin{aligned} &\limsup_{j \rightarrow \infty} N_{\underline{n}_j}^F(h_{\underline{n}_j}) \\ &= \underline{c}^*((h_{\underline{n}_j})_{j \in \mathbb{N}}) \left(- (K_{(1)}^* - K_{(1)}) + \underline{c}^*((h_{\underline{n}_j})_{j \in \mathbb{N}})^{2m-1} \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \right), \\ &\limsup_{j \rightarrow \infty} N_{\bar{n}_j}^F(h_{\bar{n}_j}) \\ &= \bar{c}^*((h_{\bar{n}_j})_{j \in \mathbb{N}}) \left(- (K_{(1)}^* - K_{(1)}) + \bar{c}^*((h_{\bar{n}_j})_{j \in \mathbb{N}})^{2m-1} \frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \right). \end{aligned}$$

Since $(h_{\underline{n}_j})_{j \in \mathbb{N}}$ and $(h_{\bar{n}_j})_{j \in \mathbb{N}}$ are minimisers of $\limsup_{j \rightarrow \infty} N_{\underline{n}_j}^F(\mathfrak{h}_j)$ and $\limsup_{j \rightarrow \infty} N_{\bar{n}_j}^F(\mathfrak{h}_j)$ respectively over all sequences $(\mathfrak{h}_j)_{j \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$, they are of course also minimisers of these limes superiors over all sequences $(\mathfrak{h}_j)_{j \in \mathbb{N}} \in \mathbb{R}_{+,0}^{\mathbb{N}}$ with $\liminf_{j \rightarrow \infty} \underline{n}_j^{1/(2m-1)} \mathfrak{h}_j = \limsup_{j \rightarrow \infty} \underline{n}_j^{1/(2m-1)} \mathfrak{h}_j$ and $\liminf_{j \rightarrow \infty} \bar{n}_j^{1/(2m-1)} \mathfrak{h}_j = \limsup_{j \rightarrow \infty} \bar{n}_j^{1/(2m-1)} \mathfrak{h}_j$ respectively. So we can easily conclude that both $\underline{c}^*((h_{\underline{n}_j})_{j \in \mathbb{N}})$ and $\bar{c}^*((h_{\bar{n}_j})_{j \in \mathbb{N}})$ are minimisers of the function $\bar{\nu}^F$ (recall (A.19)). But we already know from (c) that the only minimiser of $\bar{\nu}^F$ is c . This contradicts $\underline{c}^*((h_{\underline{n}_j})_{j \in \mathbb{N}}) = \underline{c}((h_n)_{n \in \mathbb{N}}) < \bar{c}((h_n)_{n \in \mathbb{N}}) = \bar{c}^*((h_{\bar{n}_j})_{j \in \mathbb{N}})$.

A.13. Proof of Theorem 3.9

Let $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ be arbitrary but fixed. Whenever $\lim_{n \rightarrow \infty} h_n = 0$, we can conclude from (A.16) and (A.14)–(A.15) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} n M_n^F(h_n) &= \limsup_{n \rightarrow \infty} \left(n M_n^F(0) + h_n (-L_1^F(h_n) + (n-1) h_n^{2m-1} L_2^F(h_n)) \right) \\ &= n M_n^F(0) - \lim_{n \rightarrow \infty} h_n L_1^F(h_n) + \left(\limsup_{n \rightarrow \infty} (n-1) h_n^{2m} \right) \left(\lim_{n \rightarrow \infty} L_2^F(h_n) \right) \\ &= c^F - 0 + \left(\limsup_{n \rightarrow \infty} n h_n^{2m} \right) \left(\frac{K_{(m)}^2}{(m!)^2} \int (F^{(m)})^2 d\lambda \right), \end{aligned} \tag{A.20}$$

where $c^F := \int F(1-F) d\lambda = n M_n^F(0)$ is contained in $\mathbb{R}_{++}$. For the second = we used that $\limsup_{n \rightarrow \infty} (c + a_n + \ell_n b_n) = c + \lim_{n \rightarrow \infty} a_n + (\limsup_{n \rightarrow \infty} \ell_n)(\lim_{n \rightarrow \infty} b_n)$ for any

$c \in \mathbb{R}$, $(\ell_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ and convergent sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with limits $a \in \mathbb{R}$ and $b \in \mathbb{R}_{++}$, respectively.

Let us first assume that $\limsup_{n \rightarrow \infty} n^{1/(2m)} h_n \in \mathbb{R}_+$ (in particular $\lim_{n \rightarrow \infty} h_n = 0$). In this case, we obtain from (A.20) that $\limsup_{n \rightarrow \infty} n M_n^F(h_n) \in \mathbb{R}_{++}$.

Conversely, assume that $\limsup_{n \rightarrow \infty} n M_n^F(h_n) \in \mathbb{R}_{++}$. Below we will show that, in this case, $\lim_{n \rightarrow \infty} h_n = 0$. Thus, we can again conclude from (A.20) that $\limsup_{n \rightarrow \infty} n h_n^{2m} \in \mathbb{R}_+$, since $(K_{(m)}^2/(m!)^2) \int (F^{(m)})^2 d\lambda \in \mathbb{R}_{++}$. In fact, we even have $\limsup_{n \rightarrow \infty} n h_n^{2m} \in \mathbb{R}_+$, since $h_n \in \mathbb{R}_+$ for all $n \in \mathbb{N}$. In particular, $\limsup_{n \rightarrow \infty} n^{1/(2m)} h_n \in \mathbb{R}_+$.

It remains to show that $\lim_{n \rightarrow \infty} h_n = 0$. From Section A.11 we know that $M_n^F(h) \geq g^F(h) \geq 0$ for all $h \in \mathbb{R}_+$ and $n \in \mathbb{N}$, where the function $g^F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by $g^F(h) := \int (F_h - F)^2 d\lambda$, $h \in \mathbb{R}_{++}$, and $g^F(0) := 0$. Hence, $M_n^F(h_n)$ can converge to 0 as $n \rightarrow \infty$ only if $g(h_n)$ converges to 0 as $n \rightarrow \infty$. The function g^F is continuous on $\mathbb{R}_+$, strictly positive on $\mathbb{R}_{++}$ (see Section A.11) and satisfies $\lim_{h \rightarrow \infty} g^F(h) = \infty$ (see Step 2 in the proof of Lemma 3.2 in Section A.9). Hence, $\lim_{n \rightarrow \infty} M_n^F(h_n) = 0$ can hold only if $\lim_{n \rightarrow \infty} h_n = 0$. Since $\limsup_{n \rightarrow \infty} n M_n^F(h_n) \in \mathbb{R}_{++}$ implies $\lim_{n \rightarrow \infty} M_n^F(h_n) = 0$, we can conclude that indeed $\lim_{n \rightarrow \infty} h_n = 0$.

A.14. Proof of Lemma 4.1

Let $n \in \mathbb{N}$ be arbitrary but fixed. For the first claim in Lemma 4.1 we have to show that, for every $\mathbf{x}_n \in \mathbb{R}^n$, the function $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is well-defined and continuous (see Step 1) and that $\liminf_{h \rightarrow \infty} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) = \infty$ (see Step 2). For the second claim in Lemma 4.1, we have to show that $h = 0$ is not a minimum point of $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; \cdot)$ for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$, when the kernel K also satisfies $K_{(1)}^* > K_{(1)} + \frac{n}{2n-1} \int |u| K(u) \lambda(du)$. For this it suffices to show that $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; \cdot)$ is right-sided differentiable at 0 with right-sided derivative $\widehat{M}'_{n,\text{Emp}}(\mathbf{x}_n; 0) = \frac{2n-1}{n^2} (K_{(1)} - K_{(1)}^*) + \frac{1}{n} \int |u| K(u) \lambda(du)$ for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$ (see Step 3), since $K_{(1)}^* > K_{(1)} + \frac{n}{2n-1} \int |u| K(u) \lambda(du)$ implies that $\frac{2n-1}{n^2} (K_{(1)} - K_{(1)}^*) + \frac{1}{n} \int |u| K(u) \lambda(du) < 0$.

Step 1. Let $\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^n$ be arbitrary but fixed. In view of (A.1) and $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0) = \frac{1}{n} \int \widehat{F}_n(\mathbf{x}_n; \cdot) (1 - \widehat{F}_n(\mathbf{x}_n; \cdot)) d\lambda$, a calculation similar to (7) gives

$$\begin{aligned} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) - \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0) &= \frac{1}{n} \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \widehat{F}_n(\mathbf{x}_n; x) \right) \lambda(dx) \\ &\quad - \frac{2}{n} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x)) \widehat{F}_n(\mathbf{x}_n; x) \lambda(dx) \\ &\quad + \frac{n-1}{n} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \\ &=: S_1(\mathbf{x}_n; h) + S_2(\mathbf{x}_n; h) + S_3(\mathbf{x}_n; h) \end{aligned} \tag{A.21}$$

for all $h \in \mathbb{R}_{++}$. Since $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0)$ is well-defined (note that $\widehat{F}_n(\mathbf{x}_n; \cdot) (1 - \widehat{F}_n(\mathbf{x}_n; \cdot))$ is bounded and has bounded support) and independent of h , it is proved that the function

$\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; \cdot)$ is well-defined and continuous if we can show that, for every $j = 1, 2, 3$, the summand $S_j(\mathbf{x}_n; \cdot)$, regarded as a real-valued function on $\mathbb{R}_{++}$, is well-defined and continuous and satisfies $\lim_{h \rightarrow 0} S_j(\mathbf{x}_n; h) = 0$. To show that the latter conditions are fulfilled, we first note that $\widehat{F}_n(\mathbf{x}_n; \cdot - uh) - \widehat{F}_n(\mathbf{x}_n; \cdot)$ is bounded with bounded support, for every $(u, h) \in \mathbb{R} \times \mathbb{R}_+$. Thus, for any function $G : \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$, for which $G(\cdot, h)$ is bounded for every $h \in \mathbb{R}_{++}$, we have that $(\widehat{F}_n(\mathbf{x}_n; \cdot - uh) - \widehat{F}_n(\mathbf{x}_n; \cdot))G(\cdot, h)$ is also bounded with bounded support and is therefore λ -integrable with

$$\begin{aligned}
& \int (\widehat{F}_n(\mathbf{x}_n; x - uh) - \widehat{F}_n(\mathbf{x}_n; x)) G(x, h) \lambda(dx) \\
&= \int \frac{1}{n} \sum_{i=1}^n (\mathbb{1}_{[x_i, \infty)}(x - uh) - \mathbb{1}_{[x_i, \infty)}(x)) G(x, h) \lambda(dx) \\
&= \frac{1}{n} \sum_{i=1}^n \int (\mathbb{1}_{[x_i, \infty)}(x - uh) - \mathbb{1}_{[x_i, \infty)}(x)) G(x, h) \lambda(dx) \\
&= \frac{1}{n} \sum_{i=1}^n \int (\mathbb{1}_{[x_i + uh, \infty)}(x) - \mathbb{1}_{[x_i, \infty)}(x)) G(x, h) \lambda(dx) \\
&= \frac{1}{n} \sum_{i=1}^n (-\text{sgn}(u)) \int_{I(x_i, x_i + uh)} G(x, h) \lambda(dx) \\
&= -\frac{1}{n} \sum_{i=1}^n \text{sgn}(u)^2 \int_{I(0,1)} G(x_i + uht, h) uh \lambda(dt) = -\frac{uh}{n} \sum_{i=1}^n \int_{[0,1]} G(x_i + uht, h) \lambda(dt)
\end{aligned} \tag{A.22}$$

for every $(u, h) \in \mathbb{R} \times \mathbb{R}_+$, where $I(a, b)$ is the closed interval in $\mathbb{R}$ with end points a and b (for arbitrary $a, b \in \mathbb{R}$). Note that (A.22) will also be used in the proof of Lemma 4.3 in Section A.18.

For the function $S_1(\mathbf{x}_n, \cdot)$, it follows from (A.1), the transformation theorem, Fubini's theorem and (A.22) (with specifically $G \equiv 1$) that it is well-defined with

$$\begin{aligned}
S_1(\mathbf{x}_n; h) &= \frac{1}{n} \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K_h}(x - x_i)^2 - \widehat{F}_n(\mathbf{x}_n; x) \right) \lambda(dx) \\
&= \frac{1}{n} \int \left(\widehat{F}_{n,h}^*(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x) \right) \lambda(dx) \\
&= \frac{1}{n} \int \left(\int (K^*)_h(x - y) \widehat{F}_n(\mathbf{x}_n; y) \lambda(dy) - \widehat{F}_n(\mathbf{x}_n; x) \right) \lambda(dx) \\
&= \frac{1}{n} \iint K^*(u) (\widehat{F}_n(\mathbf{x}_n; x - uh) - \widehat{F}_n(\mathbf{x}_n; x)) \lambda(du) \lambda(dx) \\
&= \frac{1}{n} \int K^*(u) \int (\widehat{F}_n(\mathbf{x}_n; x - uh) - \widehat{F}_n(\mathbf{x}_n; x)) \lambda(dx) \lambda(du) \\
&= -\frac{h}{n^2} \sum_{i=1}^n \int u K^*(u) \int_{[0,1]} 1 \lambda(dt) \lambda(du) = -\frac{K_{(1)}^*}{n} h
\end{aligned} \tag{A.23}$$

for all $h \in \mathbb{R}_{++}$. For the well-definedness and the application of Fubini's theorem, we used that the first absolute moment of K^* is finite when the first absolute moment of K is finite. It follows immediately from (A.23) that $S_1(\mathbf{x}_n; \cdot)$ is continuous with $\lim_{h \rightarrow 0} S_1(\mathbf{x}_n; h) = 0$.

For the function $S_2(\mathbf{x}_n, \cdot)$, it follows from the transformation theorem, Fubini's theorem and (A.22) (with specifically $G(x, h) := \widehat{F}_n(\mathbf{x}_n; x)$, $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$) that it is well-defined with

$$\begin{aligned}
S_2(\mathbf{x}_n; h) &= -\frac{2}{n} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x)) \widehat{F}_n(\mathbf{x}_n; x) \lambda(dx) \\
&= -\frac{2}{n} \int K(u) \int (\widehat{F}_n(\mathbf{x}_n; x - uh) - \widehat{F}_n(\mathbf{x}_n; x)) \widehat{F}_n(\mathbf{x}_n; x) \lambda(du) \lambda(dx) \\
&= -\frac{2}{n} \int K(u) \int (\widehat{F}_n(\mathbf{x}_n; x - uh) - \widehat{F}_n(\mathbf{x}_n; x)) \widehat{F}_n(\mathbf{x}_n; x) \lambda(dx) \lambda(du) \\
&= \frac{2h}{n^2} \sum_{i=1}^n \iint_{[0,1]} uK(u) \widehat{F}_n(\mathbf{x}_n; x_i + uht) \lambda(dt) \lambda(du) \tag{A.24}
\end{aligned}$$

for all $h \in \mathbb{R}_{++}$. For the well-definedness and the application of Fubini's theorem we used that $|uK(u) \widehat{F}_n(\mathbf{x}_n; x_i + uht)| \leq |uK(u)|$ for all $i = 1, \dots, n$, $u \in \mathbb{R}$, $h \in \mathbb{R}_{++}$, $t \in [0, 1]$, and $\int |uK(u)| \lambda(du) = K_{|1|} < \infty$ by assumption. From this we can also infer that $\lim_{h \rightarrow 0} S_2(\mathbf{x}_n; h) = 0$. Moreover, by the continuity lemma (in the form of Lemma 16.1 in [1]), we can infer from (A.24) that $S_2(\mathbf{x}_n, \cdot) : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is continuous. Note at this point that, for every $i = 1, \dots, n$ and $h_0 \in \mathbb{R}_{++}$, the function $\mathbb{R}_{++} \rightarrow [0, 1]$, $h \mapsto \widehat{F}_n(\mathbf{x}_n; x_i + uht)$ is continuous at h_0 for $\lambda^{\otimes 2}$ -a.e. $(u, t) \in \mathbb{R} \times [0, 1]$ (more precisely, for every $(u, t) \in \mathbb{R} \times [0, 1]$ with $uth_0 \notin \{c \in \mathbb{R} \setminus \{0\} : \exists j \in \{1, \dots, n\} \text{ with } c = x_j - x_i\}$).

For the function $S_3(\mathbf{x}_n; \cdot)$, it follows from the transformation theorem, Fubini's theorem and (A.22) (with specifically $G(x, h) := (\widehat{F}_n(\mathbf{x}_n; x - vh) - \widehat{F}_n(\mathbf{x}_n; x))$, $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$) that it is well-defined with

$$\begin{aligned}
S_3(\mathbf{x}_n; h) &= \frac{n-1}{n} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \\
&= \frac{n-1}{n^2} \sum_{i=1}^n \int \int (\widehat{F}_n(\mathbf{x}_n; x - uh) - \widehat{F}_n(\mathbf{x}_n; x)) K(u) \lambda(du) \\
&\quad \int (\widehat{F}_n(\mathbf{x}_n; x - vh) - \widehat{F}_n(\mathbf{x}_n; x)) K(v) \lambda(dv) \lambda(dx) \\
&= \frac{n-1}{n^2} \sum_{i=1}^n \iint K(u) K(v) \\
&\quad \int (\widehat{F}_n(\mathbf{x}_n; x - uh) - \widehat{F}_n(\mathbf{x}_n; x)) (\widehat{F}_n(\mathbf{x}_n; x - vh) - \widehat{F}_n(\mathbf{x}_n; x)) \lambda(dx) \lambda(du) \lambda(dv) \\
&= -h \frac{n-1}{n^2} \sum_{i=1}^n \iint uK(u) K(v)
\end{aligned}$$

$$\int_{[0,1]} (\widehat{F}_n(\mathbf{x}_n; x_i + uht - vh) - \widehat{F}_n(\mathbf{x}_n; x_i + uht)) \lambda(dt) \lambda(du) \lambda(dv) \quad (\text{A.25})$$

for all $h \in \mathbb{R}_{++}$. For the well-definedness and the application of Fubini's theorem we used that $|uK(u)K(v)(\widehat{F}_n(\mathbf{x}_n; x_i + uht - vh) - \widehat{F}_n(\mathbf{x}_n; x_i + uht))| \leq |uK(u)K(v)|$ for all $i = 1, \dots, n$, $u, v \in \mathbb{R}$, $h \in \mathbb{R}$ and $t \in [0, 1]$, and $\iint |uK(u)K(v)| \lambda(du) \lambda(dv) = \int K_{|1|} |K(v)| \lambda(dv) = K_{|1|} \int |K(v)| \lambda(dv) < \infty$ by the assumption $K_{|1|} < \infty$ and the λ -integrability of K . From this we can also infer that $\lim_{h \rightarrow 0} S_3(\mathbf{x}_n; h) = 0$. Moreover, by the continuity lemma (in the form of Lemma 16.1 in [1]), we can infer from (A.25) that $S_3(\mathbf{x}_n, \cdot) : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is continuous. Note at this point that, for every $i = 1, \dots, n$ and $h_0 \in \mathbb{R}_{++}$, the function $\mathbb{R}_{++} \rightarrow [-1, 1]$, $h \mapsto (\widehat{F}_n(\mathbf{x}_n; x_i + uht - vh) - \widehat{F}_n(\mathbf{x}_n; x_i + uht))$ is continuous at h_0 for $\lambda^{\otimes 3}$ -a.e. $(u, v, t) \in \mathbb{R}^2 \times [0, 1]$ (more precisely, for every $(u, v, t) \in \mathbb{R}^2 \times [0, 1]$ with $(ut - v)h_0, u h_0 \notin \{c \in \mathbb{R} \setminus \{0\} : \exists j \in \{1, \dots, n\} \text{ with } c = x_j - x_i\}$).

Step 2. Let $\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^n$ be arbitrary but fixed. To show that $\liminf_{h \rightarrow \infty} \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; h) = \infty$, we first note that

$$\begin{aligned} & \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; h) \\ &= \int \left(\frac{1}{n^2} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \frac{1}{n} \widehat{F}_{n,h}(\mathbf{x}_n; x)^2 + (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \right) \lambda(dx) \\ &= \frac{1}{n} \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \left(\frac{1}{n} \sum_{i=1}^n \overline{K}_h(x - x_i) \right)^2 \right) \lambda(dx) \\ &\quad + \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \geq \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \end{aligned}$$

for all $h \in \mathbb{R}_{++}$. Thus, in view of Fatou's lemma, we have $\liminf_{h \rightarrow \infty} \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; h) \geq \liminf_{h \rightarrow \infty} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \geq \int \liminf_{h \rightarrow \infty} (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \geq \int (\overline{K}(0) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) = \infty$. The second last step can be seen as follows. We have $|\widehat{F}_{n,h}(\mathbf{x}_n; x) - \overline{K}(0)| = |\frac{1}{n} \sum_{i=1}^n (\overline{K}_h(x - x_i) - \overline{K}(0))| \leq \frac{1}{n} \sum_{i=1}^n |(\overline{K}((x - x_i)/h) - \overline{K}(0))|$, and $\overline{K}$ is continuous (see Section A.2) and $\lim_{h \rightarrow \infty} (x - x_i)/h = 0$ holds true for all $x \in \mathbb{R}$. Thus, $\lim_{h \rightarrow \infty} \widehat{F}_{n,h}(\mathbf{x}_n; x) = \overline{K}(0)$.

Step 3. It remains to show that, for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$, the function $\widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; \cdot)$ is right-sided differentiable at 0 with right-sided derivative $\widehat{M}'_{n, \text{Emp}}(\mathbf{x}_n; 0) = \frac{2n-1}{n^2} (K_{(1)} - K_{(1)}^*) + \frac{1}{n} \int |u| K(u) \lambda(du)$. In view of (A.21), it suffices to show that $\lim_{h \rightarrow 0} S_1(\mathbf{x}_n; h)/h = -\frac{1}{n} K_{(1)}^*$, $\lim_{h \rightarrow 0} S_2(\mathbf{x}_n; h)/h = \frac{1}{n} K_{(1)} + \frac{1}{n^2} \int |u| K(u) \lambda(du)$ and $\lim_{h \rightarrow 0} S_3(\mathbf{x}_n; h)/h = \frac{n-1}{n^2} (K_{(1)} - K_{(1)}^* + \int |u| K(u) \lambda(du))$ for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$.

The convergence $\lim_{h \rightarrow 0} S_1(\mathbf{x}_n; h)/h = -\frac{1}{n} K_{(1)}^*$ directly follows from (A.23).

To show the stated convergence of $S_2(\mathbf{x}_n; h)/h$, we first note that, since $\widehat{F}_n(\mathbf{x}_n; \cdot)$ is càdlàg, we have

$$\lim_{h \rightarrow 0} \widehat{F}_n(\mathbf{x}_n; x_i + uht) = \begin{cases} \widehat{F}_n(\mathbf{x}_n; x_i) & , \quad ut \geq 0 \\ \widehat{F}_n(\mathbf{x}_n; x_i) - c_i(\mathbf{x}_n) & , \quad ut < 0 \end{cases}$$

for every $i = 1, \dots, n$, where $c_i(\mathbf{x}_n) := \#\{j \in \{1, \dots, n\} : x_j = x_i\}/n$. Since $c_i(\mathbf{x}_n) = 1/n$ for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$ (recall that F is assumed to be continuous), we obtain by applying the dominated convergence theorem (with majorant $|uK(u)|$) to the very right-hand side in (A.24) that

$$\begin{aligned}
& \lim_{h \rightarrow 0} S_2(\mathbf{x}_n; h)/h \\
&= \frac{2}{n^2} \sum_{i=1}^n \iint_{[0,1]} uK(u) \lim_{h \rightarrow 0} \widehat{F}_n(\mathbf{x}_n; x_i + uht) \lambda(dt) \lambda(du) \\
&= \frac{2}{n^2} \sum_{i=1}^n \left(\int_{\mathbb{R}_{++}} \int_{[0,1]} uK(u) \widehat{F}_n(\mathbf{x}_n; x_i) \lambda(dt) \lambda(du) \right. \\
&\quad \left. + \int_{\mathbb{R}_-} \int_{[0,1]} uK(u) \left(\widehat{F}_n(\mathbf{x}_n; x_i) - \frac{1}{n} \right) \lambda(dt) \lambda(du) \right) \\
&= \frac{1}{n^2} \left(2 \sum_{i=1}^n \widehat{F}_n(\mathbf{x}_n; x_i) \int_{\mathbb{R}_{++}} uK(u) \lambda(du) + 2 \sum_{i=1}^n \left(\widehat{F}_n(\mathbf{x}_n; x_i) - \frac{1}{n} \right) \int_{\mathbb{R}_-} uK(u) \lambda(du) \right) \\
&= \frac{1}{n^2} \left((n+1) \int_{\mathbb{R}_{++}} uK(u) \lambda(du) + (n-1) \int_{\mathbb{R}_-} uK(u) \lambda(du) \right) \\
&= \frac{1}{n} K_{(1)} + \frac{1}{n^2} \left(\int_{\mathbb{R}_{++}} uK(u) \lambda(du) - \int_{\mathbb{R}_-} uK(u) \lambda(du) \right) = \frac{1}{n} K_{(1)} + \frac{1}{n^2} \int |u|K(u) \lambda(du)
\end{aligned}$$

for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$, where we used that $\sum_{i=1}^n \widehat{F}_n(\mathbf{x}_n; x_i) = \sum_{j=1}^n \frac{j}{n} = \frac{n+1}{2}$ for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$.

To show the stated convergence of $S_3(\mathbf{x}_n; h)/h$, we first note that, since $\widehat{F}_n(\mathbf{x}_n; \cdot)$ is càdlàg, we have

$$\lim_{h \rightarrow 0} (\widehat{F}_n(\mathbf{x}_n; x_i + uht - vh) - \widehat{F}_n(\mathbf{x}_n; x_i + uht)) = \begin{cases} c_i(\mathbf{x}_n) & , v \leq ut < 0 \\ -c_i(\mathbf{x}_n) & , 0 \leq ut < v \\ 0 & , \text{ else} \end{cases}$$

for every $i = 1, \dots, n$, where $c_i(\mathbf{x}_n) := \#\{j \in \{1, \dots, n\} : x_j = x_i\}/n$. Since $c_i(\mathbf{x}_n) = 1/n$ for $i = 1, \dots, n$, for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$ (recall that F is assumed to be continuous), we obtain by applying the dominated convergence theorem (with majorant $2|uK(u)K(v)|$) to the very-right-hand side in (A.25) that

$$\begin{aligned}
& \lim_{h \rightarrow 0} S_3(\mathbf{x}_n; h)/h \\
&= -\frac{n-1}{n^2} \sum_{i=1}^n \iint uK(u)K(v) \\
&\quad \int_{[0,1]} \lim_{h \rightarrow 0} (\widehat{F}_n(\mathbf{x}_n; x_i + uht - vh) - \widehat{F}_n(\mathbf{x}_n; x_i + uht)) \lambda(dt) \lambda(du) \lambda(dv)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{n-1}{n^2} \sum_{i=1}^n \iint uK(u)K(v) \left(\int_{[0,1]} \frac{1}{n} \mathbb{1}_{(-\infty,0)}(ut) \mathbb{1}_{[v,\infty)}(ut) \lambda(dt) \right. \\
&\quad \left. + \int_{[0,1]} \left(-\frac{1}{n} \mathbb{1}_{[0,\infty)}(ut) \right) \mathbb{1}_{(-\infty,v)}(ut) \lambda(dt) \right) \lambda(du) \lambda(dv) \\
&= -\frac{n-1}{n^2} \frac{n}{n} \left(\iint uK(u)K(v) \mathbb{1}_{(-\infty,0)}(u) \int_{[0,1]} \mathbb{1}_{[v,\infty)}(ut) \lambda(dt) \lambda(du) \lambda(dv) \right. \\
&\quad \left. + \iint uK(u)K(v) \mathbb{1}_{(0,\infty)}(u) \int_{[0,1]} \left(-\mathbb{1}_{(-\infty,v)}(ut) \right) \lambda(dt) \lambda(du) \lambda(dv) \right) \\
&= \frac{n-1}{n^2} \left(\int_{\mathbb{R}_+} \int_{[u,\infty)} uK(u)K(v) \lambda(dv) \lambda(du) - \int_{\mathbb{R}_-} \int_{(-\infty,u]} uK(u)K(v) \lambda(dv) \lambda(du) \right. \\
&\quad \left. + \int_{\mathbb{R}_+} \int_{[v,\infty)} K(u)vK(v) \lambda(du) \lambda(dv) - \int_{\mathbb{R}_-} \int_{(-\infty,v]} K(u)vK(v) \lambda(du) \lambda(dv) \right)
\end{aligned}$$

for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x} \in \mathbb{R}^n$, where the last = follows from

$$u \mathbb{1}_{(-\infty,0)}(u) \int_{[0,1]} \mathbb{1}_{[v,\infty)}(ut) \lambda(dt) = \begin{cases} u & , 0 > u \geq v \\ v & , 0 > v > u \\ 0 & , v \geq 0 \text{ or } u \geq 0 \end{cases}$$

and

$$u \mathbb{1}_{(0,\infty)}(u) \int_{[0,1]} \left(-\mathbb{1}_{(-\infty,v)}(ut) \right) \lambda(dt) = \begin{cases} -u & , v \geq u > 0 \\ -v & , u > v > 0 \\ 0 & , v \leq 0 \text{ or } u \leq 0 \end{cases}$$

for every $u, v \in \mathbb{R}$. Since $\overline{K}(x) = \int_{(-\infty,x]} K(y) \lambda(dy) = \int_{(-\infty,x)} K(y) \lambda(dy)$ for all $x \in \mathbb{R}$, we can continue with

$$\begin{aligned}
&= \frac{n-1}{n^2} \left(\int_{\mathbb{R}_+} uK(u)(1 - \overline{K}(u)) \lambda(du) - \int_{\mathbb{R}_-} uK(u)\overline{K}(u) \lambda(du) \right. \\
&\quad \left. + \int_{\mathbb{R}_+} (1 - \overline{K}(v))vK(v) \lambda(dv) - \int_{\mathbb{R}_-} \overline{K}(v)vK(v) \lambda(dv) \right) \\
&= \frac{n-1}{n^2} \left(2 \int_{\mathbb{R}_+} uK(u) \lambda(du) - \int u(2K(u)\overline{K}(u)) \lambda(du) \right) \\
&= \frac{n-1}{n^2} \left(2 \int_{\mathbb{R}_+} uK(u) \lambda(du) - K_{(1)}^* \right) = \frac{n-1}{n^2} \left(K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du) \right),
\end{aligned} \tag{A.26}$$

where we used for the last step that $2 \int_{\mathbb{R}_+} uK(u) \lambda(du) = K_{(1)} + \int_{\mathbb{R}_+} uK(u) \lambda(du) - \int_{\mathbb{R}_-} uK(u) \lambda(du) = K_{(1)} + \int |u|K(u) \lambda(du)$.

A.15. On the condition in the second part of Lemma 4.1

To verify $K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du) \geq 0$, it is sufficient to show that there exists a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ for which $\lim_{h \rightarrow \infty} g(h)/h = K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du)$ and $\lim_{h \rightarrow \infty} g(h) = \infty$. Such a function g can be defined by $g(h) := \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx)$ for arbitrary $n \geq 2$ and $\mathbf{x}_n \in \mathbb{R}^n$. Indeed: It was shown in Step 2 in Section A.14 that $\lim_{h \rightarrow \infty} g(h) = \lim_{h \rightarrow \infty} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) = \infty$, and it will be shown in Step 2 in Section A.18 (see the chain of equations from (A.35) onwards) that $\lim_{h \rightarrow \infty} g(h)/h = \lim_{h \rightarrow \infty} \frac{1}{h} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) = K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du)$. Take into account that the expression $S_{\mathbb{C},3}(\mathbf{x}_n; h)$ that appears in (A.35) coincides with $g(h)$ for every $h \in \mathbb{R}_{++}$.

A.16. Proof of Theorem 4.2

As before, we use $\mathbf{x}_n$ to denote the projection of $\mathbf{x} \in \mathbb{R}^N$ to the first n coordinates.

Lemma A.5 *Assume that $\int |x| dF(x) < \infty$. Then, for $\mathbb{P}_{X_1}^{\otimes N}$ -a.e. $\mathbf{x} \in \mathbb{R}^N$ we have: $\lim_{n \rightarrow \infty} |\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h_n) - M_n^F(h_n)| = 0$ for all $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^N$.*

Proof For every $\mathbf{x} \in \mathbb{R}^N$, $n \in \mathbb{N}$ and $h_n \in \mathbb{R}_{++}$, we have

$$\begin{aligned}
& |\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h_n) - M_n^F(h_n)| \\
&= \left| \int \left(\frac{1}{n^2} \sum_{i=1}^n \overline{K_{h_n}}(x - x_i)^2 - \frac{1}{n} \widehat{F}_{n,h_n}(\mathbf{x}_n; x)^2 + (\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \right) \lambda(dx) \right. \\
&\quad \left. - \int \left(\frac{1}{n} \int \overline{K_{h_n}}(x - y)^2 dF(y) - \frac{1}{n} F_{h_n}(x)^2 + (F_{h_n}(x) - F(x))^2 \right) \lambda(dx) \right| \\
&= \left| \int \left((\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 - (F_{h_n}(x) - F(x))^2 \right) \lambda(dx) \right. \\
&\quad \left. + \frac{1}{n} \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K_{h_n}}(x - x_i)^2 - \int \overline{K_{h_n}}(x - y)^2 dF(y) \right. \right. \\
&\quad \left. \left. - \widehat{F}_{n,h_n}(\mathbf{x}_n; x)^2 + F_{h_n}(x)^2 \right) \lambda(dx) \right| \\
&\leq \int |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - F_{h_n}(x)| \cdot |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) + F_{h_n}(x) - 2F(x)| \lambda(dx) \\
&\quad + \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \cdot |\widehat{F}_n(\mathbf{x}_n; x) + F(x) - 2\widehat{F}_{n,h_n}(\mathbf{x}_n; x)| \lambda(dx) \\
&\quad + \frac{1}{n} \left| \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K_{h_n}}(x - x_i)^2 - \int \overline{K_{h_n}}(x - y)^2 dF(y) \right) \lambda(dx) \right| \\
&\quad + \frac{1}{n} \int |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - F_{h_n}(x)| \cdot |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) + F_{h_n}(x)| \lambda(dx). \\
&\leq (\|\widehat{F}_{n,h_n}(\mathbf{x}_n; \cdot)\|_\infty + \|F_{h_n}\|_\infty + 2\|F\|_\infty) \int |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - F_{h_n}(x)| \lambda(dx)
\end{aligned}$$

$$\begin{aligned}
& + (\|\widehat{F}_n(\mathbf{x}_n; \cdot)\|_\infty + \|F\|_\infty + 2\|\widehat{F}_{n,h_n}(\mathbf{x}_n; \cdot)\|_\infty) \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \lambda(dx) \\
& + \frac{1}{n} \left| \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K_{h_n}}(x - x_i)^2 - \int \overline{K_{h_n}}(x - y)^2 dF(y) \right) \lambda(dx) \right| \\
& + (\|\widehat{F}_{n,h_n}(\mathbf{x}_n; \cdot)\|_\infty + \|F_{h_n}\|_\infty) \frac{1}{n} \int |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - F_{h_n}(x)| \lambda(dx).
\end{aligned}$$

Moreover, for every $\mathbf{x} \in \mathbb{R}^N$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
|\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0) - M_n^F(0)| & \leq \frac{1}{n} \int |(\widehat{F}_n(\mathbf{x}_n; x) - F(x)) - (\widehat{F}_n(\mathbf{x}_n; x)^2 - F(x)^2)| \lambda(dx) \\
& = \frac{1}{n} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \cdot |1 - (\widehat{F}_n(\mathbf{x}_n; x) + F(x))| \lambda(dx) \\
& \leq \frac{1}{n} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \lambda(dx).
\end{aligned}$$

Thus, it suffices to show that there exists a $\mathbb{P}_{X_1}^{\otimes N}$ -null set $N \in \mathcal{B}(\mathbb{R})^{\otimes N}$ such that for every $\mathbf{x} \in \mathbb{R}^N \setminus N$ the following assertions hold for any $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{++}^N$:

- (a) $\sup_{n \in \mathbb{N}} \|\widehat{F}_{n,h_n}(\mathbf{x}_n; \cdot)\|_\infty \leq \|\overline{K}\|_\infty$,
- (b) $\sup_{n \in \mathbb{N}} \|F_{h_n}\|_\infty \leq \|\overline{K}\|_\infty$,
- (c) $\sup_{n \in \mathbb{N}} \|\widehat{F}_n(\mathbf{x}_n; \cdot)\|_\infty = 1$,
- (d) $\lim_{n \rightarrow \infty} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \lambda(dx) = 0$,
- (e) $\lim_{n \rightarrow \infty} \int |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - F_{h_n}(x)| \lambda(dx) = 0$,
- (f) $\sup_{n \in \mathbb{N}} \left| \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K_{h_n}}(x - x_i)^2 - \int \overline{K_{h_n}}(x - y)^2 dF(y) \right) \lambda(dx) \right| < \infty$.

Recall at this point that $\overline{K}$ is bounded (which is known from Section A.2), i.e. that $\|\overline{K}\|_\infty < \infty$ holds true.

- (a): $\|\widehat{F}_{n,h_n}(\mathbf{x}_n; \cdot)\|_\infty \leq \|\overline{K_{h_n}}\|_\infty = \|\overline{K}\|_\infty$ for all $\mathbf{x} \in \mathbb{R}^N$, $n \in \mathbb{N}$ and $h_n \in \mathbb{R}_{++}$.
- (b): $\|F_{h_n}\|_\infty \leq \int \|\overline{K_{h_n}}(\cdot - y)\|_\infty dF(y) \leq \|\overline{K}\|_\infty$ for all $\mathbf{x} \in \mathbb{R}^N$, $n \in \mathbb{N}$ and $h_n \in \mathbb{R}_{++}$.
- (c): This equation is obvious, since $\widehat{F}_n(\mathbf{x}_n; \cdot)$ is a distribution function for all $\mathbf{x} \in \mathbb{R}^N$ and $n \in \mathbb{N}$.

(d): $\int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \lambda(dx)$ is the L^1 -Wasserstein distance between the probability measures $\widehat{\mu}_n(\mathbf{x}_n; \cdot)$ and μ associated with the distribution functions $\widehat{F}_n(\mathbf{x}_n; \cdot)$ and F , respectively. It is well known (see e.g. [2, Lemma 8.3]) that this distance converges to 0 as $n \rightarrow \infty$ if and only if $\widehat{\mu}_n(\mathbf{x}_n; \cdot)$ converges weakly to μ and $\lim_{n \rightarrow \infty} \int |x| \widehat{\mu}_n(\mathbf{x}_n; dx) = \int |x| \mu(dx)$. But the latter two conditions hold true for $\mathbb{P}_{X_1}^{\otimes N}$ -a.e. $\mathbf{x} \in \mathbb{R}^N$ by the classical Glivenko–Cantelli theorem and Kolmogorov’s strong law of large numbers. That is, there exists a $\mathbb{P}_{X_1}^{\otimes N}$ -null set $N \in \mathcal{B}(\mathbb{R})^{\otimes N}$ such that $\lim_{n \rightarrow \infty} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \lambda(dx) = 0$ for all $\mathbf{x} \in \mathbb{R}^N \setminus N$.

(e): For every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $h_n \in \mathbb{R}_{++}$, we have

$$\begin{aligned}
& \int |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - F_{h_n}(x)| \lambda(dx) \\
&= \int \left| \int K_{h_n}(x-y) (\widehat{F}_n(\mathbf{x}_n; y) - F(y)) \lambda(dy) \right| \lambda(dx) \\
&\leq \int |\widehat{F}_n(\mathbf{x}_n; y) - F(y)| \int |K_{h_n}(x-y)| \lambda(dx) \lambda(dy) \\
&= \int |\widehat{F}_n(\mathbf{x}_n; y) - F(y)| \int |K(z)| \lambda(dz) \lambda(dy) = L \int |\widehat{F}_n(\mathbf{x}_n; y) - F(y)| \lambda(dy),
\end{aligned}$$

where $L := \int |K(z)| \lambda(dz)$ is finite (since K is a kernel). By (d) the latter integral converges to 0 as $n \rightarrow \infty$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$. Thus, $\lim_{n \rightarrow \infty} \int |\widehat{F}_{n,h_n}(\mathbf{x}_n; x) - F_{h_n}(x)| \lambda(dx) = 0$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$.

(f): On the one hand, we have $|\int (\frac{1}{n} \sum_{i=1}^n \overline{K_{h_n}}(x-x_i)^2 - \int \overline{K_{h_n}}(x-y)^2 dF(y)) \lambda(dx)| = |\int (\widehat{F}_{n,h_n}^*(x) - F_h^*(x)) \lambda(dx)| \leq \int |\widehat{F}_{n,h_n}^*(x) - F_h^*(x)| \lambda(dx)$, where the first step is ensured by (A.1) and (4). On the other hand, as in the proof (e) (with K replaced by K^*), we obtain that $\lim_{n \rightarrow \infty} \int |\widehat{F}_{n,h_n}^*(x) - F_h^*(x)| \lambda(dx) = 0$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$. In particular, $\sup_{n \in \mathbb{N}} |\int (\frac{1}{n} \sum_{i=1}^n \overline{K_{h_n}}(x-x_i)^2 - \int \overline{K_{h_n}}(x-y)^2 dF(y)) \lambda(dx)| < \infty$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$. $\square$

We now show that $\lim_{n \rightarrow \infty} \widehat{h}_n(\mathbf{x}_n) = 0$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$.

Step 1. There exists a $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -null set $N \in \mathcal{B}(\mathbb{R})^{\otimes \mathbb{N}}$ such that for every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ we have $\lim_{n \rightarrow \infty} M_n^F(\widehat{h}_n(\mathbf{x}_n)) = 0$. Indeed: By Lemma A.5 there exists a $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -null set $N \in \mathcal{B}(\mathbb{R})^{\otimes \mathbb{N}}$ such that for every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ we have $\lim_{n \rightarrow \infty} |\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h_n) - M_n^F(h_n)| = 0$. Since $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) = \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) - M_n^F(h) + M_n^F(h)$ for all $h \in \mathbb{R}_+$, we can conclude that, for every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, $\lim_{n \rightarrow \infty} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h_n) = 0$ if and only if $\lim_{n \rightarrow \infty} M_n^F(h_n) = 0$. It easily follows that $\lim_{n \rightarrow \infty} M_n^F(\widehat{h}_n(\mathbf{x}_n)) = 0$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$. Indeed: Let $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ be arbitrary but fixed. From what has just been said we can infer that $\lim_{n \rightarrow \infty} M_n^F(\widehat{h}_n(\mathbf{x}_n)) = 0$ if and only if $\lim_{n \rightarrow \infty} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; \widehat{h}_n(\mathbf{x}_n)) = 0$, and the latter convergence holds true if $\lim_{n \rightarrow \infty} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0) = 0$ (since $\widehat{h}_n(\mathbf{x}_n)$ minimises $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h)$ in $h \in \mathbb{R}_+$, in particular $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; \widehat{h}_n(\mathbf{x}_n)) \leq \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0)$ for every $n \in \mathbb{N}$). Now, $\lim_{n \rightarrow \infty} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0) = 0$ holds true if and only if $\lim_{n \rightarrow \infty} M_n^F(0) = 0$ (this is the above mentioned equivalence with specifically $h_n := 0$, $n \in \mathbb{N}$), and $\lim_{n \rightarrow \infty} M_n^F(0) = \lim_{n \rightarrow \infty} \frac{1}{n} \int F(1-F) d\lambda = 0$ holds true since $\int |x| dF(x) < \infty$ implies $\int F(1-F) d\lambda < \infty$.

Step 2. In view of Step 1, the claim follows if we can show that for every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ we have that $\lim_{n \rightarrow \infty} M_n^F(\widehat{h}_n(\mathbf{x}_n)) = 0$ implies $\lim_{n \rightarrow \infty} \widehat{h}_n(\mathbf{x}_n) = 0$. In the rest of the proof we will show that the latter implication is true. Let $\mathbf{x}_n \in \mathbb{R}^{\mathbb{N}} \setminus N$ be arbitrary but fixed, and assume that $\lim_{n \rightarrow \infty} M_n^F(\widehat{h}_n(\mathbf{x}_n)) = 0$. From (A.12) we know that $M_n^F(h) \geq g^F(h) \geq 0$ for all $h \in \mathbb{R}_+$ and $n \in \mathbb{N}$, where the function $g^F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by $g^F(h) := \int (F_h - F)^2 d\lambda$, $h \in \mathbb{R}_{++}$, and $g^F(0) := 0$. Thus, $\lim_{n \rightarrow \infty} g^F(\widehat{h}_n(\mathbf{x}_n)) =$

0. Since $g^F(h) > 0$ for all $h \in \mathbb{R}_{++}$ (see the last paragraph in Section A.11), g^F is continuous (see Step 1 in Section A.9; note that $g^F(h) = \frac{n}{n-1}S_3(h)$ for all $h \in \mathbb{R}_+$) and $\lim_{h \rightarrow \infty} g^F(h) = \infty$ (see Step 2 in Section A.9, specifically the lines after (A.12)), it follows that $\lim_{n \rightarrow \infty} \widehat{h}_n(\mathbf{x}_n) = 0$.

A.17. An unbiased estimator of $B_n^F(h)$ introduced in (11)

Let $n \geq 2$ be arbitrary but fixed and assume that $K_{|1|} < \infty$ and $\int |x| dF(x) < \infty$. For notational simplicity we will write $\widehat{F}_{n,h}$ and $\widehat{F}_n$ instead of $\widehat{F}_{n,h}(\mathbf{X}_n; \cdot)$ and $\widehat{F}_n(\mathbf{X}_n; \cdot)$, respectively. Recall that we defined $B_n^F(h) := \mathbb{E}[\widehat{M}_{n,\text{Emp}}(\mathbf{X}_n; h)] - M_n^F(h)$, $h \in \mathbb{R}_+$.

First of all note that $B_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}$ is well-defined since the function $M_n^F : \mathbb{R}_+ \rightarrow \mathbb{R}$ is well-defined (see Lemma 3.2) and the function $\mathbb{E}[\widehat{M}_{n,\text{Emp}}(\mathbf{X}_n; \cdot)] : \mathbb{R}_+ \rightarrow \mathbb{R}$ is well-defined. To see the latter assertion, note that $\mathbb{E}[\widehat{M}_{n,\text{Emp}}(\mathbf{X}_n; h)] = \mathbb{E}[\widehat{M}_{n,\text{Emp}}(\mathbf{X}_n; h) - \widehat{M}_{n,\text{Emp}}(\mathbf{X}_n; 0)] + \mathbb{E}[\widehat{M}_{n,\text{Emp}}(\mathbf{X}_n; 0)]$, where $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) - \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0)$ is bounded uniformly in $\mathbf{x}_n \in \mathbb{R}^n$, for every $h \in \mathbb{R}_{++}$, and $\mathbb{E}[\widehat{M}_{n,\text{Emp}}(\mathbf{X}_n; 0)] = \frac{n-1}{n^2} \int F(1-F) \lambda(dx)$ (see (A.28) below). The uniform boundedness can be seen as follows. From (A.21) we have $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) - \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0) = S_1(\mathbf{x}_n; h) + S_2(\mathbf{x}_n; h) + S_3(\mathbf{x}_n; h)$ (for $S_1(\mathbf{x}_n; h)$, $S_2(\mathbf{x}_n; h)$ and $S_3(\mathbf{x}_n; h)$ as defined in (A.21)) for every $\mathbf{x}_n \in \mathbb{R}^n$ and $h \in \mathbb{R}_{++}$. It can be inferred from (A.23), (A.24) and (A.25), respectively, that, for every $h \in \mathbb{R}_{++}$, $S_1(\mathbf{x}_n; h)$ is finite and does not depend on $\mathbf{x}_n$, $|S_2(\mathbf{x}_n; h)|$ is bounded by $(2h/n)K_{|1|}$ uniformly in $\mathbf{x}_n \in \mathbb{R}^n$ and $|S_3(\mathbf{x}_n; h)|$ is bounded by $hK_{|1|} \int |K| d\lambda$ uniformly in $\mathbf{x}_n \in \mathbb{R}^n$.

Next we note that

$$B_n^F(0) = \int \mathbb{E} \left[\frac{1}{n} \widehat{F}_n(x) (1 - \widehat{F}_n(x)) - \frac{1}{n} F(x) (1 - F(x)) \right] \lambda(dx) = \int B_{n,0}^F(x) \lambda(dx)$$

and

$$\begin{aligned} B_n^F(h) &= \int \left(\mathbb{E} \left[\frac{1}{n^2} \sum_{i=1}^n \overline{K}_h(x - X_i)^2 - \frac{1}{n} \widehat{F}_{n,h}(x)^2 - \frac{1}{n} \int \overline{K}_h(x - y)^2 dF(y) + \frac{1}{n} F_h(x)^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[(\widehat{F}_{n,h}(x) - \widehat{F}_n(x))^2 - (F_h(x) - F(x))^2 \right] \right) \lambda(dx) \\ &= \int (B_{n,1}^F(x, h) + B_{n,2}^F(x, h)) \lambda(dx) \end{aligned} \quad (\text{A.27})$$

for all $h \in \mathbb{R}_{++}$, where $B_{n,0}^F(x) := \mathbb{E}[\frac{1}{n} \widehat{F}_n(x)(1 - \widehat{F}_n(x)) - \frac{1}{n} F(x)(1 - F(x))]$, $B_{n,1}^F(x, h) := \mathbb{E}[\frac{1}{n^2} \sum_{i=1}^n \overline{K}_h(x - X_i)^2 - \frac{1}{n} \widehat{F}_{n,h}(x)^2 - \frac{1}{n} \int \overline{K}_h(x - y)^2 dF(y) + \frac{1}{n} F_h(x)^2]$ and $B_{n,2}^F(x, h) := \mathbb{E}[(\widehat{F}_{n,h}(x) - \widehat{F}_n(x))^2 - (F_h(x) - F(x))^2]$ for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$.

We intend to define an unbiased estimator $\widehat{B}_n(h)$ of $B_n^F(h)$ for every $h \in \mathbb{R}_+$. In view of the above representations, it suffices to find, for every $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, unbiased and $\lambda(dx)$ -integrable estimators $\widehat{B}_{n,0}(\cdot; x)$, $\widehat{B}_{n,1}(\cdot; x, h)$, $\widehat{B}_{n,2}(\cdot; x, h)$ for $B_{n,0}^F(x)$, $B_{n,1}^F(x, h)$, $B_{n,2}^F(x, h)$, respectively. Such estimators will be constructed in Steps 1 to 3 below. The resulting estimator $\widehat{B}_n(h)$ for $B_n^F(h)$ can be found in (A.29)–(A.30) below.

Step 1. For every $x \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{n}\widehat{F}_n(x)(1 - \widehat{F}_n(x))\right] &= \frac{1}{n}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \mathbb{1}_{[0,\infty)}(x - X_i)\left(1 - \frac{1}{n}\sum_{j=1}^n \mathbb{1}_{[0,\infty)}(x - X_j)\right)\right] \\
&= \frac{1}{n}\left(\mathbb{E}[\mathbb{1}_{[0,\infty)}(x - X_1)] - \frac{1}{n}\sum_{j=1}^n \mathbb{E}[\mathbb{1}_{[0,\infty)}(x - X_1)\mathbb{1}_{[0,\infty)}(x - X_j)]\right) \\
&= \frac{1}{n}\left(\mathbb{E}[\mathbb{1}_{[0,\infty)}(x - X_1)] - \frac{1}{n}\mathbb{E}[\mathbb{1}_{[0,\infty)}(x - X_1)] - \frac{n-1}{n}\mathbb{E}[\mathbb{1}_{[0,\infty)}(x - X_1)]^2\right) \\
&= \frac{n-1}{n^2}\mathbb{E}[\mathbb{1}_{[0,\infty)}(x - X_1)] - \frac{n-1}{n^2}\mathbb{E}[\mathbb{1}_{[0,\infty)}(x - X_1)]^2 \\
&= \frac{n-1}{n^2}F(x) - \frac{n-1}{n^2}F(x)^2 = \frac{n-1}{n^2}F(x)(1 - F(x)) \tag{A.28}
\end{aligned}$$

and therefore

$$\begin{aligned}
B_{n,0}^F(x) &= \frac{n-1}{n^2}F(x)(1 - F(x)) - \frac{1}{n}F(x)(1 - F(x)) = -\frac{1}{n^2}F(x)(1 - F(x)) \\
&= -\frac{1}{n-1}\frac{n-1}{n^2}F(x)(1 - F(x)) = \mathbb{E}\left[-\frac{1}{n(n-1)}\widehat{F}_n(x)(1 - \widehat{F}_n(x))\right].
\end{aligned}$$

Thus, if we define

$$\widehat{B}_{n,0}(\mathbf{x}_n; x) := -\frac{1}{(n-1)n}\widehat{F}_n(\mathbf{x}_n; x)(1 - \widehat{F}_n(\mathbf{x}_n; x)), \quad \mathbf{x}_n \in \mathbb{R}^n$$

for all $x \in \mathbb{R}$, then we have $\mathbb{E}[\widehat{B}_{n,0}(\mathbf{X}_n; x)] = B_{n,0}^F(x)$ for all $x \in \mathbb{R}$. Moreover, $\widehat{B}_{n,0}(\mathbf{x}_n; \cdot)$ is λ -integrable since $\widehat{F}_n(\mathbf{x}_n; \cdot)(1 - \widehat{F}_n(\mathbf{x}_n; \cdot))$ is bounded with bounded support.

Step 2. For every $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{n^2}\sum_{i=1}^n \overline{K}_h(x - X_i)^2 - \frac{1}{n}\widehat{F}_{n,h}(x)^2\right] \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}[\overline{K}_h(x - X_i)^2] - \frac{1}{n}\mathbb{E}[\widehat{F}_{n,h}(x)^2] \\
&= \frac{1}{n}\int \overline{K}_h(x - y)^2 dF(y) - \frac{1}{n}\left(\mathbb{V}\text{ar}[\widehat{F}_{n,h}(x)] + \mathbb{E}[\widehat{F}_{n,h}(x)]^2\right) \\
&= \frac{1}{n}\int \overline{K}_h(x - y)^2 dF(y) - \frac{1}{n}\left(\frac{1}{n}\int \overline{K}_h(x - y)^2 dF(y) - \frac{1}{n}F_h(x)^2 + F_h(x)^2\right) \\
&= \frac{1}{n}\int \overline{K}_h(x - y)^2 dF(y) - \frac{1}{n^2}\int \overline{K}_h(x - y)^2 dF(y) - \frac{n-1}{n^2}F_h(x)^2 \\
&= \frac{n-1}{n^2}\int \overline{K}_h(x - y)^2 dF(y) - \frac{n-1}{n^2}F_h(x)^2 \\
&= \frac{n-1}{n}\left(\frac{1}{n}\int \overline{K}_h(x - y)^2 dF(y) - \frac{1}{n}F_h(x)^2\right),
\end{aligned}$$

where the third = relies on the representations of $\mathbb{V}\text{ar}[\widehat{F}_{n,h}(x)]$ and $\mathbb{E}[\widehat{F}_{n,h}(x)]$ specified in Section A.6. Therefore,

$$\begin{aligned} B_{n,1}^F(x, h) &= \frac{n-1}{n} \left(\frac{1}{n} \int \overline{K}_h(x-y)^2 dF(y) - \frac{1}{n} F_h(x)^2 \right) \\ &\quad - \left(\frac{1}{n} \int \overline{K}_h(x-y)^2 dF(y) - \frac{1}{n} F_h(x)^2 \right) \\ &= -\frac{1}{n} \left(\frac{1}{n} \int \overline{K}_h(x-y)^2 dF(y) - \frac{1}{n} F_h(x)^2 \right) \\ &= -\frac{1}{n-1} \mathbb{E} \left[\frac{1}{n^2} \sum_{i=1}^n \overline{K}_h(x - X_i)^2 - \frac{1}{n} \widehat{F}_{n,h}(x)^2 \right]. \end{aligned}$$

Thus, if we define

$$\widehat{B}_{n,1}(\mathbf{x}_n; x, h) := -\frac{1}{(n-1)n^2} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 + \frac{1}{(n-1)n} \widehat{F}_{n,h}(\mathbf{x}_n; x)^2, \quad \mathbf{x}_n \in \mathbb{R}^n$$

for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, then we have $\mathbb{E}[\widehat{B}_{n,1}(\mathbf{X}_n; x, h)] = B_{n,1}^F(x, h)$ for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$. Since

$$\begin{aligned} |\widehat{B}_{n,1}(\mathbf{x}_n; x, h)| &= \left| \frac{1}{(n-1)n^2} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \frac{1}{(n-1)n} \widehat{F}_{n,h}(\mathbf{x}_n; x)^2 \right| \\ &\leq \frac{1}{n-1} \left| \frac{1}{n^2} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \frac{1}{n} \widehat{F}_{n,h}(\mathbf{x}_n; x)^2 + (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \right. \\ &\quad \left. - \frac{1}{n} \widehat{F}_n(\mathbf{x}_n; x)(1 - \widehat{F}_n(\mathbf{x}_n; x)) \right| \\ &\quad + \frac{1}{n-1} (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 + \frac{1}{n(n-1)} \widehat{F}_n(\mathbf{x}_n; x)(1 - \widehat{F}_n(\mathbf{x}_n; x)) \end{aligned}$$

for all $\mathbf{x}_n \in \mathbb{R}^n$, $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, we can (in view of (9)) use arguments as in (and after) (A.21) to conclude that $\widehat{B}_{n,1}(\mathbf{x}_n; \cdot, h)$ is λ -integrable.

Step 3. Let $k(x, h, x_1) := \overline{K}_h(x - x_1) - \mathbb{1}_{[0, \infty)}(x - x_1)$ for every $x, x_1 \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$. Then, for every $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, we have

$$\begin{aligned} B_{n,2}^F(x, h) &= \mathbb{E}[(\widehat{F}_{n,h}(x) - \widehat{F}_n(x))^2] - (F_h(x) - F(x))^2 \\ &= \mathbb{E}[(\widehat{F}_{n,h}(x) - \widehat{F}_n(x))^2] - \mathbb{E}[\widehat{F}_{n,h}(x) - \widehat{F}_n(x)]^2 \\ &= \mathbb{V}\text{ar}[\widehat{F}_{n,h}(x) - \widehat{F}_n(x)] = \frac{1}{n} \mathbb{V}\text{ar}[\overline{K}_h(x - X_1) - \mathbb{1}_{[0, \infty)}(x - X_1)] \\ &= \frac{1}{n} \mathbb{E}[(\overline{K}_h(x - X_1) - \mathbb{1}_{[0, \infty)}(x - X_1))^2] - \frac{1}{n} \mathbb{E}[\overline{K}_h(x - X_1) - \mathbb{1}_{[0, \infty)}(x - X_1)]^2 \\ &= \frac{n}{n(n-1)} \mathbb{E}[k(x, h, X_1)^2] - \frac{1}{n(n-1)} \mathbb{E}[k(x, h, X_1)^2] \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{n} \mathbb{E}[\overline{K_h}(x - X_1) - \mathbb{1}_{[0,\infty)}(x - X_1)]^2 \\
&= \frac{n}{n(n-1)} \mathbb{E}[k(x, h, X_1)^2] - \frac{1}{n(n-1)} \mathbb{E}[k(x, h, X_1)^2] \\
& \quad - \frac{1}{n(n-1)} \sum_{j=2}^n \mathbb{E}[k(x, h, X_1)] \mathbb{E}[k(x, h, X_j)] \\
&= \frac{n}{n(n-1)} \mathbb{E}[k(x, h, X_1)^2] - \frac{1}{n(n-1)} \sum_{j=1}^n \mathbb{E}[k(x, h, X_1)k(x, h, X_j)] \\
&= \frac{n}{n(n-1)} \mathbb{E}[k(x, h, X_1)^2] - \frac{1}{n^2(n-1)} \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}[k(x, h, X_i)k(x, h, X_j)] \\
&= \frac{n}{n(n-1)} \mathbb{E}[k(x, h, X_1)^2] - \frac{1}{n-1} \mathbb{E}\left[\left(\frac{1}{n} \sum_{i=1}^n k(x, h, X_i)\right)^2\right] \\
&= \frac{1}{n(n-1)} \sum_{i=1}^n \mathbb{E}[(\overline{K_h}(x - X_i) - \mathbb{1}_{[0,\infty)}(x - X_i))^2] - \frac{1}{n-1} \mathbb{E}[(\widehat{F}_{n,h}(x) - \widehat{F}_n(x))^2] \\
&= \mathbb{E}\left[\frac{1}{n(n-1)} \sum_{i=1}^n (\overline{K_h}(x - X_i) - \mathbb{1}_{[0,\infty)}(x - X_i))^2 - \frac{1}{n-1} (\widehat{F}_{n,h}(x) - \widehat{F}_n(x))^2\right].
\end{aligned}$$

Thus, if we define

$$\begin{aligned}
\widehat{B}_{n,2}(\mathbf{x}_n; x, h) &:= \frac{1}{n(n-1)} \sum_{i=1}^n (\overline{K_h}(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \\
& \quad - \frac{1}{n-1} (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2, \quad \mathbf{x}_n \in \mathbb{R}^n
\end{aligned}$$

for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$, then we have $\mathbb{E}[\widehat{B}_{n,2}(\mathbf{X}_n; x, h)] = B_{n,2}^F(x, h)$ for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_{++}$. We have discussed after (A.25) that $(\widehat{F}_{n,h}(\mathbf{x}_n; \cdot) - \widehat{F}_n(\mathbf{x}_n; \cdot))^2$ is λ -integrable. Since $(\overline{K_h}(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 = (\widehat{F}_{1,h}(x_i; x) - \widehat{F}_1(x_i; x))^2$ for every $i = 1, \dots, n$ and $x \in \mathbb{R}$, it follows that $\widehat{B}_{n,2}(\mathbf{x}_n; \cdot, h)$ is λ -integrable.

Conclusion 1. Now, if we set

$$\begin{aligned}
\widehat{B}_n(\mathbf{x}_n; 0) &:= \int \widehat{B}_{n,0}(\mathbf{x}_n; x) \lambda(dx) \\
&= - \int \frac{1}{(n-1)n} \widehat{F}_n(\mathbf{x}_n; x) (1 - \widehat{F}_n(\mathbf{x}_n; x)) \lambda(dx), \quad \mathbf{x}_n \in \mathbb{R}^n
\end{aligned} \tag{A.29}$$

and

$$\begin{aligned}
\widehat{B}_n(\mathbf{x}_n; h) &:= \int (\widehat{B}_{n,1}(\mathbf{x}_n; x, h) + \widehat{B}_{n,2}(\mathbf{x}_n; x, h)) \lambda(dx) \\
&= \int \left(- \frac{1}{(n-1)n^2} \sum_{i=1}^n \overline{K_h}(x - x_i)^2 + \frac{1}{(n-1)n} \widehat{F}_{n,h}(\mathbf{x}_n; x)^2 \right) \lambda(dx)
\end{aligned} \tag{A.30}$$

$$\begin{aligned}
& + \frac{1}{n(n-1)} \sum_{i=1}^n (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \\
& - \frac{1}{n-1} (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx), \quad \mathbf{x}_n \in \mathbb{R}^n
\end{aligned}$$

for all $h \in \mathbb{R}_{++}$, then we can conclude from Steps 1–3 that $\mathbb{E}[\widehat{B}_n(\mathbf{X}_n; h)] = B_n^F(h)$ for all $h \in \mathbb{R}_+$, i.e. that $\widehat{B}_n(\cdot; h)$ is an unbiased estimator of $B_n^F(h)$ for every $h \in \mathbb{R}_+$.

Conclusion 2. If we plug $\widehat{B}_n(\mathbf{x}_n; h)$ (defined by (A.29)–(A.30)) in the right-hand side of (12), then we obtain for every $\mathbf{x}_n \in \mathbb{R}^n$ (by taking (9) into account) that $\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; 0) = (\frac{1}{n} + \frac{1}{(n-1)n}) \int \widehat{F}_n(\mathbf{x}_n; \cdot)(1 - \widehat{F}_n(\mathbf{x}_n; \cdot)) d\lambda = \frac{1}{n-1} \int \widehat{F}_n(\mathbf{x}_n; \cdot)(1 - \widehat{F}_n(\mathbf{x}_n; \cdot)) d\lambda$ and

$$\begin{aligned}
\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; h) &= \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) - \widehat{B}_n(\mathbf{x}_n; h) \\
&= \int \left(\left(\frac{1}{n^2} + \frac{1}{(n-1)n^2} \right) \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \left(\frac{1}{n} + \frac{1}{(n-1)n} \right) \widehat{F}_{n,h}(\mathbf{x}_n; x)^2 \right. \\
&\quad \left. - \frac{1}{n(n-1)} \sum_{i=1}^n (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \right. \\
&\quad \left. + \left(1 + \frac{1}{n-1} \right) (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \right) \lambda(dx) \\
&= \int \left(\frac{1}{n(n-1)} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \frac{1}{n-1} \widehat{F}_{n,h}(\mathbf{x}_n; x)^2 + \frac{n}{n-1} (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \right. \\
&\quad \left. - \frac{1}{n(n-1)} \sum_{i=1}^n (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \right) \lambda(dx)
\end{aligned}$$

for all $h \in \mathbb{R}_{++}$.

A.18. Proof of Lemma 4.3

Let $n \geq 2$ be arbitrary but fixed. For the first claim in Lemma 4.3, we have to show that, for every $\mathbf{x}_n \in \mathbb{R}^n$, the function $\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is well-defined and continuous (see Step 1) and that $\liminf_{h \rightarrow \infty} \widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; h) = \infty$ (see Step 2). For the second claim in Lemma 4.3, we have to show that $h = 0$ is not a minimum point for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$, when the kernel K also satisfies $K_{(1)}^* > K_{(1)}$. For this it suffices to show that $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; \cdot)$ is right-sided differentiable at 0 with right-sided derivative $\widehat{M}'_{n,\text{Emp}}(\mathbf{x}_n; 0) = \frac{1}{n}(K_{(1)} - K_{(1)}^*)$ for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$ (see Step 3).

Step 1. Let $\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^n$ be arbitrary but fixed. Note that we have $\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; h) = \frac{n}{n-1} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) - \frac{1}{n(n-1)} \sum_{i=1}^n \int (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \lambda(dx)$ for all $h \in \mathbb{R}_{++}$, and $\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; 0) = \frac{n}{n-1} \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0)$. Therefore, (A.21) implies

$$\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; h) - \widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; 0)$$

$$\begin{aligned}
&= \frac{n}{n-1} (\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; h) - \widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0)) \\
&\quad - \frac{1}{n(n-1)} \sum_{i=1}^n \int (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \lambda(dx) \\
&= \frac{1}{n-1} \int \left(\frac{1}{n} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \widehat{F}_n(\mathbf{x}_n; x) \right) \lambda(dx) \\
&\quad - \frac{2}{n-1} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x)) \widehat{F}_n(\mathbf{x}_n; x) \lambda(dx) \\
&\quad + \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \\
&\quad - \frac{1}{n(n-1)} \sum_{i=1}^n \int (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \lambda(dx) \\
&=: S_{C,1}(\mathbf{x}_n; h) + S_{C,2}(\mathbf{x}_n; h) + S_{C,3}(\mathbf{x}_n; h) + S_{C,4}(\mathbf{x}_n; h) \tag{A.31}
\end{aligned}$$

for all $h \in \mathbb{R}_{++}$. Since $\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; 0)$ is well-defined (since by Lemma 4.1 $\widehat{M}_{n,\text{Emp}}(\mathbf{x}_n; 0)$ is well-defined) and independent of h , it is proved that the function $\widehat{M}_{n,\text{EmpC}}(\mathbf{x}_n; \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is well-defined and continuous if we can show that, for every $j = 1, 2, 3, 4$, the summand $S_{C,j}(\mathbf{x}_n; \cdot)$, regarded as a real-valued function on $\mathbb{R}_{++}$, is well-defined and continuous and satisfies $\lim_{h \rightarrow 0} S_{C,j}(\mathbf{x}_n; h) = 0$. Since $S_{C,j}(\mathbf{x}_n; \cdot) = \frac{n}{n-1} S_j(\mathbf{x}_n; \cdot)$ for $j = 1, 2, 3$, and we already know from Step 1 in the proof of Lemma 4.1 (see Section A.14) that the functions $S_j(\mathbf{x}_n; \cdot)$, $j = 1, 2, 3$, possess the required properties, it suffices to show that $S_{C,4}(\mathbf{x}_n; \cdot)$ is well-defined and continuous and satisfies $\lim_{h \rightarrow 0} S_{C,4}(\mathbf{x}_n; h) = 0$.

We first note that the transformation theorem and Equation (A.22) (with specifically $n := 1$, $x_1 := y$ and $G(x, h) := G_{v,y}(x, h) := \mathbb{1}_{[y,\infty)}(x - vh) - \mathbb{1}_{[y,\infty)}(x)$, $(x, h) \in \mathbb{R} \times \mathbb{R}_{++}$) imply

$$\begin{aligned}
&\int (\mathbb{1}_{[y,\infty)}(x - uh) - \mathbb{1}_{[y,\infty)}(x)) (\mathbb{1}_{[y,\infty)}(x - vh) - \mathbb{1}_{[y,\infty)}(x)) \lambda(dx) \\
&= -uh \int_{[0,1]} (\mathbb{1}_{[y,\infty)}(y + uht - vh) - \mathbb{1}_{[y,\infty)}(y + uht)) \lambda(dt) \\
&= -uh \int_{[0,1]} (\mathbb{1}_{[0,\infty)}(uht - vh) - \mathbb{1}_{[0,\infty)}(uht)) \lambda(dt) \\
&= -uh \int_{[0,1]} (\mathbb{1}_{[0,\infty)}(ut - v) - \mathbb{1}_{[0,\infty)}(u)) \lambda(dt) \\
&= -uh \left(\int_{[0,1]} \mathbb{1}_{[v,\infty)}(ut) \lambda(dt) - \mathbb{1}_{[0,\infty)}(u) \right) \\
&= -h \left(u \mathbb{1}_{(-\infty,0)}(u) \mathbb{1}_{(-\infty,u]}(v) + v \mathbb{1}_{(-\infty,0)}(v) \mathbb{1}_{(-\infty,v)}(u) \right. \\
&\quad \left. - u \mathbb{1}_{[0,\infty)}(u) \mathbb{1}_{[u,\infty)}(v) - v \mathbb{1}_{(0,\infty)}(v) \mathbb{1}_{(v,\infty)}(u) \right) \tag{A.32}
\end{aligned}$$

for all $(u, h) \in \mathbb{R} \times \mathbb{R}_{++}$, for every $(v, y) \in \mathbb{R}^2$, where it was used for the last step that

$$u \left(\int_{[0,1]} \mathbb{1}_{[v,\infty)}(ut) \lambda(dt) - \mathbb{1}_{[0,\infty)}(u) \right) = \begin{cases} u & , 0 > u \geq v \\ v & , 0 > v > u \\ -u & , v \geq u > 0 \\ -v & , u > v > 0 \\ 0 & , \text{ else} \end{cases}$$

for all $(u, v) \in \mathbb{R}^2$. From the transformation theorem, Fubini's theorem and (A.32) we obtain that $S_{C,4}(\mathbf{x}_n; \cdot)$ is well-defined with

$$\begin{aligned} & S_{C,4}(\mathbf{x}_n; h) \\ &= -\frac{1}{n(n-1)} \sum_{i=1}^n \int (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \lambda(dx) \\ &= -\frac{1}{n(n-1)} \sum_{i=1}^n \int \left(\int K(u) \mathbb{1}_{[x_i,\infty)}(x - uh) \lambda(du) - \mathbb{1}_{[x_i,\infty)}(x) \right)^2 \lambda(dx) \\ &= -\frac{1}{n(n-1)} \sum_{i=1}^n \int \left(\int K(u) (\mathbb{1}_{[x_i,\infty)}(x - uh) - \mathbb{1}_{[x_i,\infty)}(x)) \lambda(du) \right)^2 \lambda(dx) \\ &= -\frac{1}{n(n-1)} \sum_{i=1}^n \iint K(u) K(v) \int (\mathbb{1}_{[x_i,\infty)}(x - uh) - \mathbb{1}_{[x_i,\infty)}(x)) \\ &\quad (\mathbb{1}_{[x_i,\infty)}(x - vh) - \mathbb{1}_{[x_i,\infty)}(x)) \lambda(dx) \lambda(dv) \lambda(du) \\ &= \frac{h}{n-1} \left(\int_{\mathbb{R}_-} \int_{(-\infty, u]} u K(u) K(v) \lambda(dv) \lambda(du) - \int_{\mathbb{R}_+} \int_{[u, \infty)} u K(u) K(v) \lambda(dv) \lambda(du) \right. \\ &\quad \left. + \int_{\mathbb{R}_-} \int_{(-\infty, v]} v K(v) K(u) \lambda(du) \lambda(dv) - \int_{\mathbb{R}_+} \int_{[v, \infty)} v K(v) K(u) \lambda(du) \lambda(dv) \right) \\ &= \frac{h}{n-1} \left(\int_{\mathbb{R}_-} u K(u) \overline{K}(u) \lambda(du) - \int_{\mathbb{R}_+} u K(u) (1 - \overline{K}(u)) \lambda(du) \right. \\ &\quad \left. + \int_{\mathbb{R}_-} v K(v) \overline{K}(v) \lambda(dv) - \int_{\mathbb{R}_+} v K(v) (1 - \overline{K}(v)) \lambda(dv) \right) \\ &= \frac{h}{n-1} \left(\int u (2K(u) \overline{K}(u)) \lambda(du) - 2 \int_{\mathbb{R}_+} u K(u) \lambda(du) \right) \\ &= \frac{h}{n-1} \left(K_{(1)}^* - K_{(1)} - \int |u| K(u) \lambda(du) \right) \end{aligned} \tag{A.33}$$

for all $h \in \mathbb{R}_{++}$. For the well-definedness and the application of Fubini's theorem we have used that the first absolute moment of K^* is finite when the first absolute moment of K is finite. It follows immediately from (A.33) that $S_{C,4}(\mathbf{x}_n; \cdot)$ is continuous with $\lim_{h \rightarrow 0} S_{C,4}(\mathbf{x}_n; h) = 0$.

Step 2. Let $\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^n$ be arbitrary but fixed. We now show that $\lim_{h \rightarrow \infty} \widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h) = \infty$. Since

$$\begin{aligned}
\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h) &= \int \left(\frac{1}{n(n-1)} \sum_{i=1}^n \overline{K}_h(x - x_i)^2 - \frac{1}{n-1} \widehat{F}_{n,h}(\mathbf{x}_n; x)^2 \right. \\
&\quad \left. + \frac{n}{n-1} (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \right. \\
&\quad \left. - \frac{1}{n(n-1)} \sum_{i=1}^n (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \right) \lambda(dx) \\
&= \frac{1}{n-1} \int \left(\int \overline{K}_h(x - y)^2 d\widehat{F}_n(y) - \left(\int \overline{K}_h(x - y) d\widehat{F}_n(y) \right)^2 \right) \lambda(dx) \\
&\quad + \frac{n}{n-1} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \\
&\quad - \frac{1}{n(n-1)} \int \sum_{i=1}^n (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \lambda(dx) \\
&\geq \frac{n}{n-1} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \\
&\quad - \frac{1}{n(n-1)} \int \sum_{i=1}^n (\overline{K}_h(x - x_i) - \mathbb{1}_{[0,\infty)}(x - x_i))^2 \lambda(dx) \\
&= \frac{n}{n-1} S_{C,3}(\mathbf{x}_n; h) + S_{C,4}(\mathbf{x}_n; h)
\end{aligned}$$

for all $h \in \mathbb{R}_{++}$ (where $S_{C,3}$ and $S_{C,4}$ are the functions introduced in (A.31)), it suffices to show that $\lim_{h \rightarrow \infty} (\frac{n}{n-1} S_{C,3}(\mathbf{x}_n; h) + S_{C,4}(\mathbf{x}_n; h)) = \infty$. In Step 2 in Section A.14 it was already shown that $\lim_{h \rightarrow \infty} S_{C,3}(\mathbf{x}_n; h) = \lim_{h \rightarrow \infty} \int (\widehat{F}_{n,h}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) = \infty$. Therefore, it suffices to show that

$$\lim_{h \rightarrow \infty} \frac{S_{C,4}(\mathbf{x}_n; h)}{S_{C,3}(\mathbf{x}_n; h)} = \begin{cases} -\frac{1}{n-1} & , \quad K_{(1)} - K_{(1)}^* + \int |u| K(u) \lambda(du) \neq 0 \\ 0 & , \quad K_{(1)} - K_{(1)}^* + \int |u| K(u) \lambda(du) = 0 \end{cases} \quad (\text{A.34})$$

holds, since in this case we obtain

$$\lim_{h \rightarrow \infty} \left(\frac{n}{n-1} S_{C,3}(\mathbf{x}_n; h) + S_{C,4}(\mathbf{x}_n; h) \right) = \lim_{h \rightarrow \infty} S_{C,3}(\mathbf{x}_n; h) \left(\frac{n}{n-1} + \frac{S_{C,4}(\mathbf{x}_n; h)}{S_{C,3}(\mathbf{x}_n; h)} \right) = \infty.$$

The second case in (A.34) is a direct consequence of the representation $S_{C,4}(\mathbf{x}_n; h) = -\frac{h}{n-1} (K_{(1)} - K_{(1)}^* + \int |u| K(u) \lambda(du))$ for $h \in \mathbb{R}_{++}$ (recall (A.33)). For the first case in (A.34), it suffices to show that $\lim_{h \rightarrow \infty} S_{C,3}(\mathbf{x}_n; h)/h = K_{(1)} - K_{(1)}^* + \int |u| K(u) \lambda(du)$ holds, since in this case we have

$$\lim_{h \rightarrow \infty} \frac{S_{C,4}(\mathbf{x}_n; h)}{S_{C,3}(\mathbf{x}_n; h)} = \lim_{h \rightarrow \infty} \frac{S_{C,4}(\mathbf{x}_n; h)/h}{S_{C,3}(\mathbf{x}_n; h)/h} = -\frac{1}{n-1}.$$

The convergence $\lim_{h \rightarrow \infty} S_{C,3}(\mathbf{x}_n; h)/h = K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du)$ can be shown by using arguments similar to those that led to (A.26). Indeed: In view of $S_{C,3} = \frac{n}{n-1}S_3$ (for the function S_3 defined in (A.21)), the representation (A.25) of S_3 , the dominated convergence theorem and the convergence

$$\lim_{h \rightarrow \infty} (\widehat{F}_n(\mathbf{x}_n; x_i + uht - vh) - \widehat{F}_n(\mathbf{x}_n; x_i + uht)) = \begin{cases} 1 & , v < ut < 0 \\ -1 & , 0 < ut < v \\ 0 & , \text{ else} \end{cases}$$

for $(u, v, t) \in \mathbb{R}^2 \times [0, 1]$ with $ut \neq 0$ and $ut - v \neq 0$ (i.e. for $\lambda^{\otimes 3}$ -a.e. $(u, v, t) \in \mathbb{R}^2 \times [0, 1]$), we obtain

$$\begin{aligned} & \lim_{h \rightarrow \infty} S_{C,3}(\mathbf{x}_n; h)/h \tag{A.35} \\ &= -\frac{n}{n-1} \frac{n-1}{n^2} \sum_{i=1}^n \iint uK(u)K(v) \\ & \quad \int_{[0,1]} \lim_{h \rightarrow \infty} (\widehat{F}_n(\mathbf{x}_n; x_i + uht - vh) - \widehat{F}_n(\mathbf{x}_n; x_i + uht)) \lambda(dt) \lambda(du) \lambda(dv) \\ &= -\frac{1}{n} \sum_{i=1}^n \iint uK(u)K(v) \left(\int_{[0,1]} \mathbb{1}_{(-\infty, 0)}(ut) \mathbb{1}_{[v, \infty)}(ut) \lambda(dt) \right. \\ & \quad \left. - \int_{[0,1]} \mathbb{1}_{[0, \infty)}(ut) \mathbb{1}_{(-\infty, v)}(ut) \lambda(dt) \right) \lambda(du) \lambda(dv) \\ &= - \left(\iint uK(u)K(v) \mathbb{1}_{(-\infty, 0)}(u) \int_{[0,1]} \mathbb{1}_{[v, \infty)}(ut) \lambda(dt) \lambda(du) \lambda(dv) \right. \\ & \quad \left. - \iint uK(u)K(v) \mathbb{1}_{(0, \infty)}(u) \int_{[0,1]} \mathbb{1}_{(-\infty, v)}(ut) \lambda(dt) \lambda(du) \lambda(dv) \right) \\ &= \int_{\mathbb{R}_+} \int_{[u, \infty)} uK(u)K(v) \lambda(dv) \lambda(du) - \int_{\mathbb{R}_-} \int_{(-\infty, u]} uK(u)K(v) \lambda(dv) \lambda(du) \\ & \quad + \int_{\mathbb{R}_+} \int_{[v, \infty)} K(u)vK(v) \lambda(du) \lambda(dv) - \int_{\mathbb{R}_-} \int_{(-\infty, v]} K(u)vK(v) \lambda(du) \lambda(dv), \end{aligned}$$

where the last = follows from

$$u \mathbb{1}_{(-\infty, 0)}(u) \int_{[0,1]} \mathbb{1}_{[v, \infty)}(ut) \lambda(dt) = \begin{cases} u & , 0 > u \geq v \\ v & , 0 > v > u \\ 0 & , v \geq 0 \text{ or } u \geq 0 \end{cases}$$

and

$$-u \mathbb{1}_{(0, \infty)}(u) \int_{[0,1]} \mathbb{1}_{(-\infty, v)}(ut) \lambda(dt) = \begin{cases} -u & , v \geq u > 0 \\ -v & , u > v > 0 \\ 0 & , v \leq 0 \text{ or } u \leq 0 \end{cases}$$

for all $u, v \in \mathbb{R}$. Since $\overline{K}(x) = \int_{(-\infty, x]} K(y) \lambda(dy) = \int_{(-\infty, x)} K(y) \lambda(dy)$ for all $x \in \mathbb{R}$, we can continue with

$$\begin{aligned}
&= \int_{\mathbb{R}_+} uK(u)(1 - \overline{K}(u)) \lambda(du) - \int_{\mathbb{R}_-} uK(u)\overline{K}(u) \lambda(du) \\
&\quad + \int_{\mathbb{R}_+} (1 - \overline{K}(v))vK(v) \lambda(dv) - \int_{\mathbb{R}_-} \overline{K}(v)vK(v) \lambda(dv) \\
&= 2 \int_{\mathbb{R}_+} uK(u) \lambda(du) - \int u(2K(u)\overline{K}(u)) \lambda(du) \\
&= 2 \int_{\mathbb{R}_+} uK(u) \lambda(du) - K_{(1)}^* = K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du),
\end{aligned}$$

where we used for the last step that $2 \int_{\mathbb{R}_+} uK(u) \lambda(du) = K_{(1)} + \int_{\mathbb{R}_+} uK(u) \lambda(du) - \int_{\mathbb{R}_-} uK(u) \lambda(du)$.

Step 3. It remains to show that, for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x}_n \in \mathbb{R}^n$, $\widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; \cdot)$ is right-sided differentiable at 0 with right-sided derivative $\widehat{M}'_{n, \text{Emp}}(\mathbf{x}_n; 0) = \frac{1}{n}(K_{(1)} - K_{(1)}^*)$. In view of (A.31), it suffices to show that $\lim_{h \rightarrow 0} S_{C,4}(\mathbf{x}_n; h)/h = \frac{1}{n-1}(K_{(1)}^* - K_{(1)}) - \frac{1}{n-1} \int |u|K(u) \lambda(du)$ and $\lim_{h \rightarrow 0} (S_{C,1}(\mathbf{x}_n; h) + S_{C,2}(\mathbf{x}_n; h) + S_{C,3}(\mathbf{x}_n; h))/h = \frac{2n-1}{n(n-1)}(K_{(1)} - K_{(1)}^*) + \frac{1}{n-1} \int |u|K(u) \lambda(du)$ for $\mathbb{P}_{X_1}^{\otimes n}$ -a.e. $\mathbf{x} \in \mathbb{R}^n$. The former follows directly from (A.33), while the latter is already known from Step 3 in Section A.14 (taking into account that $S_{C,j} = \frac{n}{n-1}S_j$, $j = 1, 2, 3$).

A.19. Proof of Theorem 4.4

As before, we use $\mathbf{x}_n$ to denote the projection of $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ to the first n coordinates.

The proof of Theorem 4.2 (see Section A.16) carries over verbatim, once we have proved a suitable analogue of Lemma A.5 (see Lemma A.7 below). Unlike Lemma A.5, Lemma A.7 assumes that $\sup_{n \geq 2} h_n < \infty$. Thus, to apply Lemma A.7 in the same way as we applied Lemma A.5 in Section A.16, we have to ensure that $\sup_{n \geq 2} \widehat{h}_n(\mathbf{x}_n) < \infty$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$. We had seen at the end of Step 1 in Section A.16 that $\lim_{n \rightarrow \infty} \frac{1}{n} \int \widehat{F}_n(1 - \widehat{F}_n) d\lambda = \lim_{n \rightarrow \infty} \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; 0) = 0$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$. Since $\widehat{h}_n(\mathbf{x}_n)$ is a minimiser of the function $\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ for every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and $n \geq 2$, it follows that $\lim_{n \rightarrow \infty} \widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; \widehat{h}_n(\mathbf{x}_n)) = 0$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$. On the other hand, Lemma A.6 below shows that, for every $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$ with $\lim_{n \rightarrow \infty} h_n = \infty$, we have $\lim_{n \rightarrow \infty} \widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) = \infty$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$. We can therefore conclude that $\sup_{n \geq 2} \widehat{h}_n(\mathbf{x}_n) < \infty$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$. Indeed: By way of contradiction assume that $\sup_{n \geq 2} \widehat{h}_n(\mathbf{x}_n) = \infty$. In this case we can find a subsequence $(h_{n_j}(\mathbf{x}_{n_j}))_{j \in \mathbb{N}}$ of $(h_n(\mathbf{x}_n))_{n \geq 2}$ with $\lim_{j \rightarrow \infty} h_{n_j}(\mathbf{x}_{n_j}) = \infty$. This implies $\lim_{j \rightarrow \infty} \widehat{M}_{n_j, \text{EmpC}}(\mathbf{x}_{n_j}; h_{n_j}(\mathbf{x}_{n_j})) = \infty$ which contradicts $\lim_{n \rightarrow \infty} \widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; \widehat{h}_n(\mathbf{x}_n)) = 0$.

Lemma A.6 Let K be a kernel that satisfies $K_{[1]} < \infty$ and assume that $\int |x| dF(x) < \infty$. Let $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$ be a sequence with $\lim_{n \rightarrow \infty} h_n = \infty$. Then, $\lim_{n \rightarrow \infty} \widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) = \infty$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$.

Proof Let N be a $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -null set such that $\lim_{n \rightarrow \infty} \|\widehat{F}_n(\mathbf{x}_n; \cdot) - F\|_{\infty} = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n |x_i| = \int |x| dF(x)$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ (the existence of such a null set N is ensured by the classical Glivenko–Cantelli theorem and Kolmogorov’s strong law of large numbers). Let $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ be arbitrary but fixed and let $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$ be a sequence with $\lim_{n \rightarrow \infty} h_n = \infty$. Without loss of generality assume that $h_n > 0$ for all $n \geq 2$. By Step 2 in Section A.18 we have

$$\begin{aligned} \widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) &\geq \frac{n}{n-1} \int (\widehat{F}_{n, h_n}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \\ &\quad - \frac{1}{n(n-1)} \int \sum_{i=1}^n (\overline{K_{h_n}}(x - x_i) - \mathbb{1}_{[0, \infty)}(x - x_i))^2 \lambda(dx) \\ &=: \frac{n}{n-1} T_{n,1}(\mathbf{x}) + T_{n,2}(\mathbf{x}) \end{aligned} \quad (\text{A.36})$$

for all $h \in \mathbb{R}_{++}$ and $n \geq 2$. In view of (A.36), it suffices to show that $\lim_{n \rightarrow \infty} (\frac{n}{n-1} T_{n,1}(\mathbf{x}) + T_{n,2}(\mathbf{x})) = \infty$. For this it suffices to show that $\lim_{n \rightarrow \infty} T_{n,1}(\mathbf{x}) = \infty$ (see Step 1) and $\lim_{n \rightarrow \infty} (T_{n,2}(\mathbf{x})/T_{n,1}(\mathbf{x})) = 0$ (see Step 2) hold true, since in this case we obtain $\lim_{n \rightarrow \infty} (\frac{n}{n-1} T_{n,1}(\mathbf{x}_n) + T_{n,2}(\mathbf{x}_n)) = \lim_{n \rightarrow \infty} T_{n,1}(\mathbf{x}_n) (n/(n-1) + T_{n,2}(\mathbf{x})/T_{n,1}(\mathbf{x})) = \infty$.

Step 1. We first show that $\lim_{n \rightarrow \infty} T_{n,1}(\mathbf{x}) = \infty$. In view of Fatou’s lemma, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} T_{n,1}(\mathbf{x}) &= \liminf_{n \rightarrow \infty} \int (\widehat{F}_{n, h_n}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \\ &\geq \int \liminf_{n \rightarrow \infty} (\widehat{F}_{n, h_n}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x))^2 \lambda(dx) \geq \int (\overline{K}(0) - F(x))^2 \lambda(dx) = \infty. \end{aligned}$$

The second last step can be seen as follows. We have $|(\widehat{F}_{n, h_n}(\mathbf{x}_n; x) - \widehat{F}_n(\mathbf{x}_n; x)) - (\overline{K}(0) - F(x))| \leq \|\widehat{F}_n(\mathbf{x}_n; \cdot) - F\|_{\infty} + \|\widehat{F}_{n, h_n}(\mathbf{x}_n; \cdot) - F_{h_n}\|_{\infty} + |F_{h_n}(x) - \overline{K}(0)|$ for all $n \geq 2$ and $x \in \mathbb{R}$. Because of the choice of the $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -null set N (and since $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$), we have $\lim_{n \rightarrow \infty} \|\widehat{F}_n(\mathbf{x}_n; \cdot) - F\|_{\infty} = 0$. Moreover, by the transformation theorem we have

$$\begin{aligned} \|\widehat{F}_{n, h_n}(\mathbf{x}_n; \cdot) - F_{h_n}\|_{\infty} &= \sup_{x \in \mathbb{R}} \left| \int K_{h_n}(x - y) (\widehat{F}_n(\mathbf{x}_n; y) - F(y)) \lambda(dy) \right| \\ &\leq \|\widehat{F}_n(\mathbf{x}_n; \cdot) - F\|_{\infty} \int |K_{h_n}(x - y)| \lambda(dy) \\ &\leq \|\widehat{F}_n(\mathbf{x}_n; \cdot) - F\|_{\infty} \int |K(u)| \lambda(du) = L \|\widehat{F}_n(\mathbf{x}_n; \cdot) - F\|_{\infty} \end{aligned} \quad (\text{A.37})$$

for all $n \geq 2$, where $L := \int |K(u)| \lambda(du)$ is finite (since K is a kernel). From (A.37) and $\lim_{n \rightarrow \infty} \|\widehat{F}_n(\mathbf{x}_n; \cdot) - F\|_{\infty} = 0$ we obtain $\lim_{n \rightarrow \infty} \|\widehat{F}_{n, h_n}(\mathbf{x}_n; \cdot) - F_{h_n}\|_{\infty} = 0$. Finally, we have $|F_{h_n}(x) - \overline{K}(0)| = |\int (\overline{K_{h_n}}(x - y) - \overline{K}(0)) dF(y)| \leq \int |\overline{K}((x - y)/h_n) - \overline{K}(0)| dF(y)$

and $\bar{K}$ is continuous and bounded (see Section A.2). Since $\lim_{n \rightarrow \infty} (x - y)/h_n = 0$ for all $x, y \in \mathbb{R}$, it follows that $\lim_{n \rightarrow \infty} |F_{h_n}(x) - \bar{K}(0)| = 0$ for all $x \in \mathbb{R}$. Altogether we obtain $\lim_{n \rightarrow \infty} (\hat{F}_{n, h_n}(\mathbf{x}_n; x) - \hat{F}_n(\mathbf{x}_n; x)) = \bar{K}(0) - F(x)$ for all $x \in \mathbb{R}$.

Step 2. We next show that $\lim_{n \rightarrow \infty} T_{n,2}(\mathbf{x})/T_{n,1}(\mathbf{x}) = 0$. For this it suffices to show that for every subsequence $(T_{n_j,2}(\mathbf{x})/T_{n_j,1}(\mathbf{x}))_{j \in \mathbb{N}}$ of $(T_{n,2}(\mathbf{x})/T_{n,1}(\mathbf{x}))_{n \geq 2}$ there exists a subsubsequence $(T_{n_{j_\ell},2}(\mathbf{x})/T_{n_{j_\ell},1}(\mathbf{x}))_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \rightarrow \infty} (T_{n_{j_\ell},2}(\mathbf{x})/T_{n_{j_\ell},1}(\mathbf{x})) = 0$. Let $(T_{n_j,2}(\mathbf{x})/T_{n_j,1}(\mathbf{x}))_{j \in \mathbb{N}}$ be an arbitrary subsequence of $(T_{n,2}(\mathbf{x})/T_{n,1}(\mathbf{x}))_{n \geq 2}$. Then $(T_{n_j,2}(\mathbf{x})/T_{n_j,1}(\mathbf{x}))_{j \in \mathbb{N}}$ depends on $(h_n)_{n \geq 2}$ only via the subsequence $(h_{n_j})_{j \in \mathbb{N}}$ of $(h_n)_{n \geq 2}$ (see (A.36)). We will distinguish the following two cases: either we have $\liminf_{j \rightarrow \infty} h_{n_j}/n_j < \infty$ (*Case 1*) or $\lim_{j \rightarrow \infty} h_{n_j}/n_j = \infty$ (*Case 2*). For this case distinction we use that

$$T_{n,2}(\mathbf{x}) = -\frac{h_n}{n-1} \left(K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du) \right) \quad (\text{A.38})$$

for all $n \geq 2$, which is known from (A.33) (to see this note that, for every fixed $n \geq 2$, $S_{C,4}(\mathbf{x}_n; h_n) = T_{n,2}(\mathbf{x})$ for $S_{C,4}(\mathbf{x}_n; \cdot)$ introduced in (A.31)).

Case 1. Assume that $\liminf_{j \rightarrow \infty} h_{n_j}/n_j < \infty$. Therefore, there exists a subsequence $(h_{n_{j_\ell}})_{\ell \in \mathbb{N}}$ of $(h_{n_j})_{j \in \mathbb{N}}$ with $\limsup_{\ell \rightarrow \infty} h_{n_{j_\ell}}/n_{j_\ell} < \infty$. Then, $(T_{n_{j_\ell},2}(\mathbf{x})/T_{n_{j_\ell},1}(\mathbf{x}))_{\ell \in \mathbb{N}}$ is a subsequence of $(T_{n_j,2}(\mathbf{x})/T_{n_j,1}(\mathbf{x}))_{j \in \mathbb{N}}$. By (A.38) we have $\limsup_{\ell \rightarrow \infty} |T_{n_{j_\ell},2}(\mathbf{x})| = |K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du)| \limsup_{\ell \rightarrow \infty} h_{n_{j_\ell}}/(n_{j_\ell} - 1) < \infty$ (recall that $K_{[1]} < \infty$). Since $\lim_{n \rightarrow \infty} T_{n,1}(\mathbf{x}) = \infty$ by Step 1, we get $\lim_{\ell \rightarrow \infty} T_{n_{j_\ell},2}(\mathbf{x})/T_{n_{j_\ell},1}(\mathbf{x}) = 0$.

Case 2. Assume that $\lim_{j \rightarrow \infty} h_{n_j}/n_j = \infty$. We will show that $\lim_{j \rightarrow \infty} T_{n_j,2}(\mathbf{x})/T_{n_j,1}(\mathbf{x}) = 0$, i.e. that the convergence already holds for the subsequence itself. Recall that we assumed (without loss of generality) that $h_n > 0$ for all $n \geq 2$. By (A.38) we have that $(n-1)T_{n,2}(\mathbf{x})/h_n = -(K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du))$ does not depend on n and is equal to 0 if $K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du) = 0$. Thus, for a kernel K with $K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du) = 0$ we have $T_{n,2}(\mathbf{x})/T_{n,1}(\mathbf{x}) = 0$ for every $n \geq 2$ and therefore $\lim_{j \rightarrow \infty} T_{n_j,2}(\mathbf{x})/T_{n_j,1}(\mathbf{x}) = 0$. Moreover, for a kernel K with $K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du) \neq 0$ it suffices to show that $\lim_{j \rightarrow \infty} (n_j - 1)T_{n_j,1}(\mathbf{x})/h_{n_j} = \infty$. For this it suffices to show that $\lim_{j \rightarrow \infty} T_{n_j,1}(\mathbf{x})/h_{n_j} = K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du)$ (since $K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du) \neq 0$ and $T_{n,1}(\mathbf{x}) \geq 0$ for every $n \geq 2$, see (A.36)).

In view of $T_{n,1}(\mathbf{x}) = \frac{n}{n-1} S_3(\mathbf{x}_n; h_n)$ for every $n \geq 2$ (for the function $S_3(\mathbf{x}_n; \cdot)$ defined in (A.21)), the representation (A.25) of $S_3(\mathbf{x}_n; h_n)$ implies

$$\begin{aligned} & T_{n,1}(\mathbf{x})/h_n \\ &= -\frac{n}{n-1} \frac{n-1}{n^2} \frac{h_n}{h_n} \sum_{i=1}^n \iint uK(u)K(v) \\ & \quad \int_{[0,1]} (\hat{F}_n(\mathbf{x}_n; x_i + uh_n t - vh_n) - \hat{F}_n(\mathbf{x}_n; x_i + uh_n t)) \lambda(dt) \lambda(du) \lambda(dv) \\ &= - \iint uK(u)K(v) \end{aligned} \quad (\text{A.39})$$

$$\int_{[0,1]} \frac{1}{n} \sum_{i=1}^n (\widehat{F}_n(\mathbf{x}_n; x_i + uh_n t - vh_n) - \widehat{F}_n(\mathbf{x}_n; x_i + uh_n t)) \lambda(dt) \lambda(du) \lambda(dv)$$

for every $n \geq 2$. We will show below that

$$\lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{i=1}^{n_j} (\widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + c_1 h_{n_j}) - \widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + c_2 h_{n_j})) = \begin{cases} 1 & , c_1 < 0 < c_2 \\ -1 & , c_1 > 0 > c_2 \\ 0 & , \text{ else} \end{cases} \quad (\text{A.40})$$

for every $(c_1, c_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. From (A.39), (A.40) (with $c_1 := ut - v$ and $c_2 := ut$; note that $\{(u, t, v) \in \mathbb{R}^2 \times [0, 1] : ut = v \text{ or } ut = 0\}$ is a $\lambda^{\otimes 3}$ -null set) and the dominated convergence theorem (recall that $\iint |uK(u)K(v)| \lambda(du) \lambda(dv) = K_{|1|} \int |K| d\lambda < \infty$) we get

$$\begin{aligned} & \lim_{j \rightarrow \infty} (T_{n_j, 1}(\mathbf{x})/h_{n_j}) \\ &= \lim_{j \rightarrow \infty} \left(- \iint uK(u)K(v) \int_{[0,1]} \frac{1}{n_j} \sum_{i=1}^{n_j} (\widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + uh_{n_j}t - vh_{n_j}) \right. \\ & \quad \left. - \widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + uh_{n_j}t)) \lambda(dt) \lambda(du) \lambda(dv) \right) \\ &= - \iint uK(u)K(v) \int_{[0,1]} \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{i=1}^{n_j} (\widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + uh_{n_j}t - vh_{n_j}) \\ & \quad - \widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + uh_{n_j}t)) \lambda(dt) \lambda(du) \lambda(dv) \\ &= - \iint uK(u)K(v) \left(\int_{[0,1]} \mathbb{1}_{(-\infty, 0)}(ut) \mathbb{1}_{[v, \infty)}(ut) \lambda(dt) \right. \\ & \quad \left. - \int_{[0,1]} \mathbb{1}_{[0, \infty)}(ut) \mathbb{1}_{(-\infty, v)}(ut) \lambda(dt) \right) \lambda(du) \lambda(dv) \\ &= K_{(1)} - K_{(1)}^* + \int |u|K(u) \lambda(du), \end{aligned}$$

where the last step can be justified as in the elaborations after (A.35).

It remains to show (A.40). For (A.40) it is sufficient to show that

$$\lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{i=1}^{n_j} \widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + ch_{n_j}) = \begin{cases} 1 & , c > 0 \\ 0 & , c < 0 \end{cases}$$

for every $c \in \mathbb{R} \setminus \{0\}$. For this it suffices to show that $\lim_{j \rightarrow \infty} (2 \max\{|x_1|, \dots, |x_{n_j}|\} - \widetilde{c}h_{n_j}) = -\infty$ for every $\widetilde{c} \in \mathbb{R}_{++}$. Indeed: In this case we can find for every $c \in \mathbb{R} \setminus \{0\}$ a $j_0 = j_0(|c|) \in \mathbb{N}$ such that for every $j \geq j_0$ we have $2 \max\{|x_1|, \dots, |x_{n_j}|\} - |c|h_{n_j} < 0$ and therefore

$$x_i + ch_{n_j} < + \max\{|x_1|, \dots, |x_{n_j}|\} + ch_{n_j} < - \max\{|x_1|, \dots, |x_{n_j}|\} \quad \text{if } c < 0, \quad \text{and}$$

$$x_i + ch_{n_j} > -\max\{|x_1|, \dots, |x_{n_j}|\} + ch_{n_j} > +\max\{|x_1|, \dots, |x_{n_j}|\} \quad \text{if } c > 0$$

for all $i = 1, \dots, n_j$. Thus, for $c < 0$ it follows that $x_i + ch_{n_j} < \min\{x_1, \dots, x_{n_j}\}$ and therefore $\widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + ch_{n_j}) = 0$ for all $i = 1, \dots, n_j$ for every $j \geq j_0$. For $c > 0$ it follows that $x_i + ch_{n_j} > \max\{x_1, \dots, x_{n_j}\}$ and therefore $\widehat{F}_{n_j}(\mathbf{x}_{n_j}; x_i + ch_{n_j}) = 1$ for all $i = 1, \dots, n_j$ for every $j \geq j_0$.

In view of the choice of the $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -null set N and $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$, we have $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n |x_i| = \int |x| dF(x) < \infty$ and therefore $\max\{|x_1|, \dots, |x_{n_j}|\}/n_j$ is bounded above uniformly in $j \in \mathbb{N}$. Since $\lim_{j \rightarrow \infty} h_{n_j}/n_j = \infty$, we can conclude that $\lim_{j \rightarrow \infty} (2 \max\{|x_1|, \dots, |x_{n_j}|\} - \widetilde{c}h_{n_j}) = \lim_{j \rightarrow \infty} n_j(2 \max\{|x_1|, \dots, |x_{n_j}|\}/n_j - \widetilde{c}h_{n_j}/n_j) = -\infty$ for every $\widetilde{c} \in \mathbb{R}_{++}$. $\square$

Lemma A.7 *Let K be a kernel that satisfies $K_{[1]} < \infty$ and assume that $\int |x| dF(x) < \infty$. Then, for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ we have: $\lim_{n \rightarrow \infty} |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) - M_n^F(h_n)| = 0$ for all $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$ with $\sup_{n \geq 2} h_n < \infty$.*

Proof For every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$, $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$ and $n \geq 2$, we have $|\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) - M_n^F(h_n)| \leq |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) - \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; h_n)| + |\widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; h_n) - M_n^F(h_n)|$. By Lemma A.5 we can find a $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -null set $N \in \mathcal{B}(\mathbb{R})^{\otimes \mathbb{N}}$ such that the latter summand converges to 0 as $n \rightarrow \infty$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$. Thus, it suffices to show that $\lim_{n \rightarrow \infty} |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) - \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; h_n)| = 0$ for all $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$ with $\sup_{n \geq 2} h_n < \infty$.

Let $\mathbf{x} \in \mathbb{R}^{\mathbb{N}} \setminus N$ and $(h_n)_{n \geq 2} \in \mathbb{R}_+^{\mathbb{N}}$ with $\sup_{n \geq 2} h_n < \infty$ be arbitrary but fixed. For every $n \geq 2$ and $j = 1, \dots, 4$, let $S_{C,j}$ be as introduced in (A.31). Recall that $S_j(\mathbf{x}_n, \cdot) = \frac{n-1}{n} S_{C,j}(\mathbf{x}_n, \cdot)$ for all $j = 1, 2, 3$ and $n \geq 2$, where S_j is as introduced in (A.21). Thus, (A.21) and (A.31) imply that

$$\begin{aligned} & |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; h_n) - \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; h_n)| \\ &= \left| S_{C,1}(\mathbf{x}_n; h_n) + S_{C,2}(\mathbf{x}_n; h_n) + S_{C,3}(\mathbf{x}_n; h_n) + S_{C,4}(\mathbf{x}_n; h_n) + \widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; 0) \right. \\ &\quad \left. - \left(\frac{n-1}{n} S_{C,1}(\mathbf{x}_n; h_n) + \frac{n-1}{n} S_{C,2}(\mathbf{x}_n; h_n) + \frac{n-1}{n} S_{C,3}(\mathbf{x}_n; h_n) + \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; 0) \right) \right| \\ &\leq \left| \frac{1}{n} S_{C,1}(\mathbf{x}_n; h_n) + \frac{1}{n} S_{C,2}(\mathbf{x}_n; h_n) + \frac{1}{n} S_{C,3}(\mathbf{x}_n; h_n) + S_{C,4}(\mathbf{x}_n; h_n) \right| \\ &\quad + |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; 0) - \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; 0)| \\ &\leq \frac{1}{n} |S_{C,1}(\mathbf{x}_n; h_n)| + \frac{1}{n} |S_{C,2}(\mathbf{x}_n; h_n)| + \frac{1}{n} |S_{C,3}(\mathbf{x}_n; h_n)| + |S_{C,4}(\mathbf{x}_n; h_n)| \\ &\quad + |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; 0) - \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; 0)| \end{aligned}$$

for all $n \geq 2$. Thus, it suffices to show that

- (a) the sequence $(S_{C,j}(\mathbf{x}_n; h_n))_{n \geq 2}$ is bounded for every $j = 1, 2, 3$,
- (b) $\lim_{n \rightarrow \infty} S_{C,4}(\mathbf{x}_n; h_n) = 0$,

$$(c) \lim_{n \rightarrow \infty} |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; 0) - \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; 0)| = 0.$$

In the rest of the proof we will show that assertions (a)–(c) are indeed true.

(a): For every $n \geq 2$, we have $|S_{C,1}(\mathbf{x}_n; h_n)| \leq |K_{(1)}^*| \sup_{n \geq 2} h_n$ (see (A.23)), $|S_{C,2}(\mathbf{x}_n; h_n)| \leq 2K_{|1|} \sup_{n \geq 2} h_n$ (see (A.24)) and $|S_{C,3}(\mathbf{x}_n; h_n)| \leq K_{|1|} \int |K| d\lambda \sup_{n \geq 2} h_n$ (see (A.25)). So the assertion is ensured by $\sup_{n \geq 2} h_n < \infty$ and $K_{|1|} < \infty$ (recall that then also $|K_{(1)}^*| < \infty$).

(b): The assertion is a direct consequence of (A.33), $\sup_{n \geq 2} h_n < \infty$ and $K_{|1|} < \infty$.

(c): Note that, by (12) and (A.29), we have

$$\begin{aligned} & |\widehat{M}_{n, \text{EmpC}}(\mathbf{x}_n; 0) - \widehat{M}_{n, \text{Emp}}(\mathbf{x}_n; 0)| \\ &= \frac{1}{(n-1)n} \left| \int \widehat{F}_n(\mathbf{x}_n; x) (1 - \widehat{F}_n(\mathbf{x}_n; x)) \lambda(dx) \right| \\ &\leq \frac{1}{(n-1)n} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x) - (\widehat{F}_n(\mathbf{x}_n; x)^2 - F(x)^2)| \lambda(dx) \\ &\quad + \frac{1}{(n-1)n} \int (F(x) - F(x)^2) \lambda(dx) \\ &= \frac{1}{(n-1)n} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \cdot |1 - (\widehat{F}_n(\mathbf{x}_n; x) + F(x))| \lambda(dx) \\ &\quad + \frac{1}{(n-1)n} \int F(x)(1 - F(x)) \lambda(dx) \\ &\leq \frac{3}{(n-1)n} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \lambda(dx) + \frac{1}{(n-1)n} \int F(x)(1 - F(x)) \lambda(dx). \quad (\text{A.41}) \end{aligned}$$

In the proof of Lemma A.5 it can be seen that the null set N can be chosen such that $\lim_{n \rightarrow \infty} \int |\widehat{F}_n(\mathbf{x}_n; x) - F(x)| \lambda(dx) = 0$ if $\int |x| dF(x) < \infty$ (recall that $\mathbf{x} \in \mathbb{R}^N \setminus N$). Moreover, $\int |x| dF(x) < \infty$ implies that $\int F(1 - F) d\lambda < \infty$. Thus, the upper bound in (A.41) converges to 0 as $n \rightarrow \infty$. $\square$

A.20. Proofs of Theorems 5.1–5.3

Step 1. We first prove Theorem 5.2. For every $\mathbf{x} \in \mathbb{R}^N$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} & \|(\widehat{F}_{n, \widehat{h}_n(\mathbf{x}_n)}(\mathbf{x}_n; \cdot) - F(\cdot))\phi(\cdot)\|_\infty \\ &\leq \|(\widehat{F}_{n, \widehat{h}_n(\mathbf{x}_n)}(\mathbf{x}_n; \cdot) - F_{\widehat{h}_n(\mathbf{x}_n)}(\cdot))\phi(\cdot)\|_\infty + \|(F_{\widehat{h}_n(\mathbf{x}_n)}(\cdot) - F(\cdot))\phi(\cdot)\|_\infty. \end{aligned}$$

The second summand on the right-hand side of this inequality converges to 0 as $n \rightarrow \infty$ for $\mathbb{P}_{X_1}^{\otimes N}$ -a.e. $\mathbf{x} \in \mathbb{R}^N$ by Lemma 2.6 (take into account that we have assumed that $\|F'\phi\|_\infty$, $K_{|1|}$ and $\sup_{x, y \in \mathbb{R}: |x-y| \leq 1} \phi(x)/\phi(y)$ are finite and that $\lim_{n \rightarrow \infty} \widehat{h}_n(\mathbf{x}_n) = 0$ for $\mathbb{P}_{X_1}^{\otimes N}$ -a.e. $\mathbf{x} \in \mathbb{R}^N$). For the first summand we have

$$\|(\widehat{F}_{n, \widehat{h}_n(\mathbf{x}_n)}(\mathbf{x}_n; \cdot) - F_{\widehat{h}_n(\mathbf{x}_n)}(\cdot))\phi(\cdot)\|_\infty$$

$$\begin{aligned}
&= \sup_{x \in \mathbb{R}} \left| \int K_{\hat{h}_n(\mathbf{x}_n)}(x-y) (\hat{F}_n(\mathbf{x}_n; y) - F(y)) \lambda(dy) \right| \phi(x) \\
&\leq \|(\hat{F}_n(\mathbf{x}_n; \cdot) - F(\cdot))\phi(\cdot)\|_\infty \sup_{x \in \mathbb{R}} \int |K_{\hat{h}_n(\mathbf{x}_n)}(x-y)| \frac{\phi(x)}{\phi(y)} \lambda(dy) \\
&= \|(\hat{F}_n(\mathbf{x}_n; \cdot) - F(\cdot))\phi(\cdot)\|_\infty \sup_{x \in \mathbb{R}} \int |K(z)| \frac{\phi(x)}{\phi(x - \hat{h}_n(\mathbf{x}_n)z)} \lambda(dz) \\
&\leq \|(\hat{F}_n(\mathbf{x}_n; \cdot) - F(\cdot))\phi(\cdot)\|_\infty L_\phi \int |K| d\lambda
\end{aligned}$$

for every $n \in \mathbb{N}$ with $\hat{h}_n(\mathbf{x}_n) \leq 1$, where $L_\phi := \sup_{x,y \in \mathbb{R}: |x-y| \leq 1} \phi(x)/\phi(y)$. For the last step we used that K vanishes outside $[-1, 1]$. Note that L_ϕ is finite by assumption and that $\int |K| d\lambda$ is finite by the λ -integrability of K . Thus, since $\lim_{n \rightarrow \infty} \|(\hat{F}_n(\mathbf{x}_n; \cdot) - F(\cdot))\phi(\cdot)\|_\infty = 0$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ by Theorem 2.3 in [7] (for which we need the assumption $\int \phi dF < \infty$), the assertion of Theorem 5.2 follows.

Step 2. Theorem 5.1 can be proved in the same way with the obvious modifications. In particular, Lemma 2.6 has to be replaced by Lemma 2.5 (and Theorem 2.3 in [7] can be replaced by the classical Glivenko–Cantelli theorem).

Step 3. Finally we prove Theorem 5.3. For every $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int |\hat{F}_{n, \hat{h}_n(\mathbf{x}_n)}(\mathbf{x}_n; x) - F(x)| \lambda(dx) \\
&\leq \int |\hat{F}_{n, \hat{h}_n(\mathbf{x}_n)}(\mathbf{x}_n; x) - F_{\hat{h}_n(\mathbf{x}_n)}(x)| \lambda(dx) + \int |F_{\hat{h}_n(\mathbf{x}_n)}(x) - F(x)| \lambda(dx).
\end{aligned}$$

The first summand on the right-hand side converges to 0 as $n \rightarrow \infty$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ by assertion (e) in the proof of Lemma A.5 (with $h_n := \hat{h}_n(\mathbf{x}_n)$). The second summand on the right-hand side converges to 0 as $n \rightarrow \infty$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ by Lemma 2.4 (take into account that $K|_{[1]} < \infty$ and $\lim_{n \rightarrow \infty} \hat{h}_n(\mathbf{x}_n) = 0$ for $\mathbb{P}_{X_1}^{\otimes \mathbb{N}}$ -a.e. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$).

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