# A Online Appendix: Multiself consistency and Proof of Proposition 4.1

I first extend the modified multi-self consistency to the setup here. Recall the expression $f(q|\theta)$ given by (5) and beliefs $p(q)$ and $p(q,s)$ given by (7).

**Definition A.1.** A SFSA $M$ satisfies *modified multi-self consistency* under $\mathbf{P}_0$ if

1. for each memory state $q \in Q$ with $\sum_\theta \mathbf{P}_0(\theta) f(q|\theta) > 0$, each signal $s$, and any $q'$ such that $\tau(q,s;q') > 0$,

$$p(q,s)V_{q'}(H)+[1-p(q,s)]V_{q'}(L) \geq p(q,s)V_{q''}(H)+[1-p(q,s)]V_{q''}(L) \text{ for all } q'' \in Q; \tag{33}$$

2. for each memory state $q \in Q$ with $\sum_\theta \mathbf{P}_0(\theta) f(q|\theta) > 0$ and each action $a$ with $d(a;q) > 0$,

$$p(q)u(a,H) + [1-p(q)]u(a,L) \geq p(q)u(a',H) + [1-p(q)]u(a',L) \text{ for all } a' \in A. \tag{34}$$

The following result is crucial for the proof of Proposition 4.1.

**Proposition A.1.** *Suppose that $M$ is an optimal SFSA under prior $\mathbf{P}_0$ among those with $|Q| \leq K$.*

1. *(Modified Multi-self Consistency) It satisfies modified multi-self consistency under prior $\mathbf{P}_0$.*

2. *(Revelation Principle) For any $q, q' \in Q$,*

$$p(q)V_q(H) + [1-p(q)]V_q(L) \geq p(q)V_{q'}(H) + [1-p(q)]V_{q'}(L). \tag{35}$$

*Proof.* For any memory states $q$ and $q'$, define the set

$$W_{q,q'} = \bigcup_{n=1}^{\infty} W_{q,q'}^n,$$

where for each $n = 1, 2, ....,$

$$W_{q,q'}^n = \{\mathbf{w} = (q, s_1; q_1, s_2; ....; q_{n-1}, s_n; q') : s_i \in \mathcal{S}, q_i \in Q\},$$

that is, the set of possible state transitions from $q$ to $q'$. Given a state of nature $\theta$ and $\mathbf{w} \in W^n_{q,q'}$, define

$$\mathbb{P}_\theta(\mathbf{w}) = \eta(1-\eta)^{n-1} \times \prod_{i=1}^{n} \mu^\theta_{s_i} \tau(q_{i-1}, s_i; q_i),$$

where $q_0 = q$ and $q_n = q'$. The expected payoff from the SFSA is then

$$V = \sum_{q \in Q, a \in A} \left\{ \sum_\theta \mathbf{P}_0(\theta) \sum_{\mathbf{w} \in W_{q^o,q}} \mathbb{P}_\theta(\mathbf{w}) \right\} d(a; q) u(a, \theta). \tag{36}$$

We now prove (33) and (34).

First, consider (34). Suppose, by contradiction, that for some memory state $\hat{q}$ with $\sum_\theta \mathbf{P}_0(\theta) f(\hat{q}|\theta) > 0$ such that (34) does not hold, and hence there are actions $a, a' \in \{0, 1\}$ with $d(a, \hat{q}) > 0$ but has the inequality in (34) reversed with a strict inequality. By (5), $f(\hat{q}|\theta) = \sum_{\mathbf{w} \in W_{q^o,\hat{q}}} \mathbb{P}_\theta(\mathbf{w})$, this then implies that

$$\sum_\theta \mathbf{P}_0(\theta) \sum_{\mathbf{w} \in W_{q^o,\hat{q}}} \mathbb{P}_\theta(\mathbf{w}) u(a, \theta) < \sum_\theta \mathbf{P}_0(\theta) \sum_{\mathbf{w} \in W_{q^o,\hat{q}}} \mathbb{P}_\theta(\mathbf{w}) u(a', \theta). \tag{37}$$

Now, consider the alternative SFSA $M'$, which differs from $M$ only in that $d'(a; \hat{q}) = 0$ and $d'(a'; \hat{q}) = d(a, \hat{q}) + d(a'; \hat{q})$. From (36) and (37) it follows that $M'$ gives a strictly higher expected payoff than $M$, a contradiction to the optimality of $M$.

Now consider (33). Suppose, by contradiction, that $\tau(q, s; q') > 0$ and that for some $q'' \neq q'$,

$$p(q, s)V_{q'}(H) + [1 - p(q, s)]V_{q'}(L) < p(q, s)V_{q''}(H) + [1 - p(q, s)]V_{q''}(L). \tag{38}$$

We denote $p' = \tau(q, s; q')$ and $p'' = \tau(q, s; q'')$. Now, fix all other transition probabilities other than $p'$ and $p''$, each term $\mathbb{P}_\theta(\mathbf{w})$ in $V$ given by (36) is a polynomial of $(p', p'')$ and, since $\eta \in (0, 1)$, $V$ is differentiable w.r.t. $(p', p'')$. Since $M$ is optimal and $p' = \tau(q, s; q') > 0$, the FOCs require that $\frac{\partial}{\partial p'} V \geq \frac{\partial}{\partial p''} V$. However, we show below that (38) implies that

$$\frac{\partial}{\partial p''} V > \frac{\partial}{\partial p'} V, \tag{39}$$

a contradiction to the optimality of $M$.

To prove (39), it is straightforward to verify that

$$\frac{\partial}{\partial p'} V = \sum_{\hat{q} \in Q, a \in A} \left\{ \sum_\theta \mathbf{P}_0(\theta) \sum_{\mathbf{w} \in W_{q^o,\hat{q}}(q,s;q')} \varphi_{(q,s;q')}(\mathbf{w}) \frac{\mathbb{P}_\theta(\mathbf{w})}{p'} \right\} d(a, \hat{q}) u(a, \theta), \tag{40}$$

where

$$W_{q^o,\hat{q}}(q,s;q') = \{\mathbf{w} \in W_{q^o,\hat{q}} : (q,s,q') \text{ occurs in } \mathbf{w}\}$$

and $\varphi_{(q,s;q')}(\mathbf{w})$ is the number of repetitions of $(q,s;q')$ within $\mathbf{w}$.

Now, we show that $\frac{\partial}{\partial p'}V$ is proportional to $p(q,s)V_{q'}(H) + [1-p(q,s)]V_{q'}(L)$:

$$\begin{aligned}
&\left[\sum_\theta \mathbf{P}_0(\theta) f(q|\theta)\mu_s^\theta\right][p(q,s)V_{q'}(H) + [1-p(q,s)]V_{q'}(L)] = \sum_\theta \mathbf{P}_0(\theta) f(q|\theta)\mu_s^\theta V_{q'}(\theta) \\
= \;& \sum_\theta \mathbf{P}_0(\theta) \sum_{\hat{q}\in Q, a\in A} \left\{\left[\sum_{\mathbf{w}_q \in W_{q^o,q}} \mathbb{P}_\theta(\mathbf{w}_q)\right]\mu_s^\theta \left[\sum_{\mathbf{w}_{q'}\in W_{q',\hat{q}}} \mathbb{P}_\theta(\mathbf{w}_{q'})\right]\right\} d(a,\hat{q})u(a,\theta) \\
= \;& \sum_\theta \mathbf{P}_0(\theta) \sum_{\hat{q}\in Q, a\in A} \left\{\sum_{\mathbf{w}_q\in W_{q^o,q}, \mathbf{w}_{q'}\in W_{q',\hat{q}}} \frac{\mathbb{P}_\theta[(\mathbf{w}_q, s; \mathbf{w}'_q)]}{\tau(q,s;q')}\right\} d(a,\hat{q})u(a,\theta) \\
= \;& \sum_\theta \mathbf{P}_0(\theta) \sum_{\hat{q}\in Q, a\in A} \left\{\sum_{\mathbf{w}\in W_{q^o,\hat{q}}} \varphi_{(q,s;q')}(\mathbf{w}) \frac{\mathbb{P}_\theta(\mathbf{w})}{p'}\right\} d(a,\hat{q})u(a,\theta) = \frac{\partial}{\partial p'}V,
\end{aligned}$$

where the last equality follows from (40) and the second last equality follows from $p' = \tau(q,s;q')$ and the fact that for any $\mathbf{w}_q \in W_{q^o,q}$ and any $\mathbf{w}_{q'} \in W_{q',\hat{q}}$, $(\mathbf{w}_q, s; \mathbf{w}_{q'}) \in W_{q^o,\hat{q}}(q,s;q')$ and that each $\mathbf{w} \in W_{q^o,\hat{q}}(q,s;q')$ is counted $\varphi_{(q,s;q')}(\mathbf{w})$ times in that list. We have analogous expression for $\frac{\partial}{\partial p''}V$, and hence (38) implies that (39).

Now I prove (35). By modified multi-self consistency, for any $s \in \mathcal{S}$ and any $q_1, q_2$ with $\tau(q,s;q_1) > 0$ and $\tau(q,s;q_2) > 0$ and any $q_3 \in Q$,

$$\begin{aligned}
[p(q,s)V_{q_1}(H) + [1-p(q,s)]V_{q_1}(L)] &= [p(q,s)V_{q_2}(H) + [1-p(q,s)]V_{q_2}(L)] \\
&\geq [p(q,s)V_{q_3}(H) + [1-p(q,s)]V_{q_3}(L)],
\end{aligned}$$

By (7), this implies that

$$\sum_{\theta=H,L} \mathbf{P}_0(\theta) f(q|\theta)\mu_s^\theta V_{q_1}(\theta) = \sum_{\theta=H,L} \mathbf{P}_0(\theta) f(q|\theta)\mu_s^\theta V_{q_2}(\theta) \geq \sum_{\theta=H,L} \mathbf{P}_0(\theta) f(q|\theta)\mu_s^\theta V_{q_3}(\theta). \tag{41}$$

Thus, (here I assume that the decision rule is deterministic, with no loss of generality because of (34))

$$
\begin{aligned}
& p(q)V_q(H) + [1-p(q)]V_q(L) \\
= \;& p(q)\left\{\eta u[d(q),H] + (1-\eta)\left[\sum_{s\in\mathcal{S},q''\in Q} \mu_s^H \tau(q,s;q'')V_{q''}(H)\right]\right\} \\
+ \;& [1-p(q)]\left\{\eta u[d(q),L] + (1-\eta)\left[\sum_{s\in\mathcal{S},q''\in Q} \mu_s^L \tau(q,s;q'')V_{q''}(L)\right]\right\} \\
= \;& \eta\{p(q)u[d(q),H] + [1-p(q)]u[d(q),L]\} \\
+ \;& (1-\eta)\sum_{s\in\mathcal{S}}\left\{\sum_{q''\in Q} \frac{\sum_{\theta=H,L}\mathbf{P}_0(\theta)f(q|\theta)\mu_s^\theta V_{q''}(\theta)}{\sum_{\theta'=H,L}\mathbf{P}_0(\theta')f(q|\theta')}\tau(q,s;q'')\right\} \\
\geq \;& \eta\{p(q)u[d(q'),H] + [1-p(q)]u[d(q'),L]\} \\
+ \;& (1-\eta)\sum_{s\in\mathcal{S}}\left\{\sum_{q''\in Q} \frac{\sum_{\theta=H,L}\mathbf{P}_0(\theta)f(q|\theta)\mu_s^\theta V_{q''}(\theta)}{\sum_{\theta'=H,L}\mathbf{P}_0(\theta')f(q|\theta')}\tau(q',s;q'')\right\} \\
= \;& p(q)V_{q'}(H) + [1-p(q)]V_{q'}(L),
\end{aligned}
$$

where the first equality follows from the recursive equation for $V_q(\theta)$ for each $\theta = H, L$, the second follows from (7), the inequality follows term by term, first the terms starting with $\eta$ follow from (34), the terms starting with $(1-\eta)$ follows from (41), again term by term for each $s$: any term with $q''$ with $\tau(q,s;q'') > 0$ has the same value in the inequality above, and that value is no less than that for the corresponding term with $\tau(q',s;q'') > 0$, and the last equality follows from the recursive equation for $V_{q'}(\theta)$. □

Now we are ready to prove Proposition 4.1.

**(1)** Let $i < j$ be given. By (35),

$$
p(q_j)\Delta V_{j,i}^H + [1-p(q_j)]\Delta V_{j,i}^L \geq 0, \text{ and } p(q_i)\Delta V_{i,j}^H + [1-p(q_i)]\Delta V_{i,j}^L \geq 0. \tag{42}
$$

Since there are no equivalent states, either $\Delta V_{i,j}^H > 0$ or $\Delta V_{i,j}^H < 0$. By our convention it must be $\Delta V_{j,i}^H > 0$. By the second inequality in (42), $\Delta V_{i,j}^L \geq 0$. Now, if this last inequality is an equality, then we can replace all the transition to $q_i$ to transition to $q_j$ and obtain a higher ex ante payoff, which is a contradiction to the optimality of the SFSA. Now, let $i < j < k$. Again, by (35), we have

$$
p(q_j)\Delta V_{j,i}^H + [1-p(q_j)]\Delta V_{j,i}^L \geq 0, \text{ and } p(q_j)\Delta V_{j,k}^H + [1-p(q_j)]\Delta V_{j,k}^L \geq 0, \tag{43}
$$

and hence

$$\frac{\Delta V_{i,j}^{L}}{\Delta V_{j,i}^{H}} \leq \frac{p(q_j)}{1-p(q_j)} \leq \frac{\Delta V_{j,k}^{L}}{\Delta V_{k,j}^{H}}.$$

**(2)** For part (a), (35) implies that

$$p(q_i)V_{q_i}(H) + [1-p(q_i)]V_{q_i}(L) \geq p(q_i)V_{q_{i+1}}(H) + [1-p(q_i)]V_{q_{i+1}}(L)$$

and hence, by rearranging terms, we have $\rho(q_i) \leq \bar{\rho}_i$. A similar argument holds for $\rho(q_i) \geq \bar{\rho}_{i-1}$.

For (b), let $q \in Q$ be given. By (33), $\tau(q, s; q_i) > 0$ only if

$$p(q,s)V_{q_i}(H) + [1-p(q,s)]V_{q_i}(L) \geq p(q,s)V_{q_j}(H) + [1-p(q,s)]V_{q_j}(L)$$

for both $j = i-1$ and $j = i+1$. This then implies (9). Conversely, it is straightforward to verify that if (9) holds, then

$$p(q,s)V_{q_i}(H) + [1-p(q,s)]V_{q_i}(L) \geq p(q,s)V_{q_j}(H) + [1-p(q,s)]V_{q_j}(L)$$

for any $j = 0, ..., K-1$, where $q_0 = q_L$ and $q_{K-1} = q_H$. Note that we need the fact that $\bar{\rho}_i$ increases with $i$ for this, as proved in part (1). Moreover, if $\rho(q,x) \in (\bar{\rho}_{i-1}, \bar{\rho}_i)$, then the above inequality is strict for any $j \neq i$ and hence $\tau(q, s; q_i) = 1$.

Finally, (c) follows from (34) and a similar argument.