

Online Appendix B

The purpose of this appendix is to obtain uniqueness as claimed in Proposition 6 regarding the rates of Proposition 5.

Preliminaries for the proof of Proposition 6.

Consider the competition over the portfolio of the specific informed firm $i \in \mathcal{N} = \{1, 2, \dots, n\}$, the portfolios of firm j being isomorphic. Denote $\delta_i(r)$ as the probability mass placed by firm i 's strategy at rate r . Recall the minimum poaching rate $\underline{r}_j$ defined in equation (6) and $\bar{r} = c/q$ where firms break even on the high risk types h . Denote the strategy of firm i for its low-type agents l by $L(r)$ and the strategy for its high-type agents h by $H(r)$. The strategy of uninformed firm j who offers a rate for the agents of firm i is $U_j(r)$.

The uninformed firms may differ regarding their strategies. An uninformed firm, for example, has the choice to participate or not. In any equilibrium there is a set $\mathcal{A} = \{j, \dots, m\} \subseteq \mathcal{N}$, $j \neq i$, of active uninformed firms who quote a rate to the l and h agents of firm i , where $1 \leq |\mathcal{A}| \leq \mathcal{N} - 1$. The minimum poaching rate offered by the uninformed firms to the agents of firm i follows the distribution

$$U_{\min}(r) = 1 - \prod_{j \neq i}^m [1 - U_j(r)]$$

By assumption, agents go for the lowest rate on offer. The distribution of the best outside offer is $U_{\min}(r)$. Furthermore, define

$$U_{\min}^k(r) = 1 - \prod_{k \neq i, j}^m [1 - U_k(r)]$$

to be the distribution of the lowest rate offered by uninformed firms other than uninformed firm j . Furthermore, for another uninformed firm k , define the price-setting strategies of the other uninformed firms

$$U_{\min}^{j \neq k} = 1 - \prod_{j \neq i, k}^m [1 - U_j(r)]$$

The expected return for firm i on its high-type agents h is

$$E[\pi_i^h(r_i^h)] = [1 - U_{\min}(r_i^h)] w(r_i^h) h$$

Firm i expects to make on its low-type agents l

$$E[\pi_i^l(r_i^l)] = [1 - U_{\min}(r_i^l)] v(r_i^l) l \tag{19}$$

The expected return for uninformed firm j on the agents of firm i is

$$E [\pi_j (r_j^u)] = (1 - U_{\min}^k (r_j^u)) [1 - L (r_j^u)] v (r_j^u) l + (1 - U_{\min}^k (r_j^u)) [1 - H (r_j^u)] w (r_j^u) h \quad (20)$$

Recall that both $v(r)$ and $w(r)$ are linear and increasing in the rate r .

The proof of Proposition 6 is structured as follows. We first determine the support of the equilibrium strategies in Lemmas 10 - 21. Next, we characterize the equilibrium strategies and prove uniqueness in Lemmas 22 - 23.

Proof of Proposition 6.

Lower Bound of the Support

Lemma 10. *Informed firm i does not quote rates below $\bar{r}$ to its high-type agents h , that is $H (r_i^h < \bar{r}) = 0$.*

Proof. Since the informed firm i knows its h clientele and $w (r_i^h < \bar{r}) < 0$, it follows that $H (r_i^h < \bar{r}) = 0$. $\square$

Lemma 11. *Below $\bar{r}$, if r_j^u is such that j attracts the l , it also attracts the h .*

Proof. Since uninformed firms cannot discriminate between agent types and if the l switch to j , it means that j has set the lowest rate among the uninformed firms. Thus, the h have no incentive to switch to an uninformed firm other than j . By Lemma 10 the informed firm does not compete over the h agents below $\bar{r}$. $\square$

Lemma 12. *No uninformed firm offers a rate below $\underline{r}$.*

Proof. If $r_j^u < \underline{r}$ firm j attracts the low-type agents l , and by Lemma 11 j also gets the high-type agents h . But for $r_j^u < \underline{r}$, by the definition of $\underline{r}$ the sum $v (r_j^u) l + w (r_j^u) h < 0$. Hence firm j has no incentive to quote $r_j^u < \underline{r}$. $\square$

Lemma 13. *Informed firm i sets rates r_i^l for its low-risk clientele such that $r_i^l \geq \underline{r}$.*

Proof. At $r_i^l = \underline{r}$ firm i retains all its low-type agents with certainty. This holds as no agents switch if this quote is matched by some uninformed firm, due to Assumption 1, and because no uninformed firm quotes rates below $\underline{r}$ by Lemma 12. Thus firm i can guarantee itself $v(\underline{r})l$. Because $v(r)$ is increasing in r , firm i has no incentive to set $r_i^l < \underline{r}$. $\square$

Before we move to the upper-bound of the support, we show that the strategies U and L cannot have gaps on the support.

Lemma 14. *The $U_{\min}(r)$ and $L_i(r)$ contain no gaps, i.e., are monotone and increasing throughout the support.*

Proof. Suppose that $U_{\min}(r)$ has a gap in the distribution between $r \in (x, x + \varepsilon)$, this implies that $U_{\min}(x + \varepsilon) - U_{\min}(x) = 0$. According to equation 19, the return for the informed lender if it moves probability mass from x to $x + \varepsilon$ then equals $v(x + \varepsilon)l - v(x)l > 0$, with the same probability. The postulated L_i would not be an equilibrium strategy as a result. A similar argument shows that $L_i(r)$ cannot have gaps. Note that the postulated strategies in Proposition 5 are indeed monotone and increasing. $\square$

Upper Bound of the Support

We first show that strategies with mass points above $\bar{r}$, including pure strategies, cannot exist in equilibrium.

Suppose a distribution function $U_j(s)$ of some j has some probability mass at x denoted by $\delta_j(x)$. Due to this $U_{\min}(s)$ jumps up at x . By convention the distribution function is right continuous at x . Due to the tie-rule in assumption (1) by which existing agents do not switch if firm i quotes the same rate x as an uninformed firm, we need the left limit of the distribution function to identify the probability that agents stay with firm i (agents opt for the lowest price on offer). Thus, let

$$U_{\min}(x^-) \equiv \lim_{s \uparrow x} U_{\min}(s)$$

Lemma 15. *The strategies $U_j(r)$ for any $j \in \mathcal{A}$, and by implication $U_{\min}(r)$ contain no atoms in the open interval $(\bar{r}, \infty)$.*

Proof. Note that both $v(r) > 0$ and $w(r) > 0$ are continuous for $r > \bar{r}$ and increase monotonically. Suppose $U_j(r)$ has a mass point $\delta_j(x) > 0$ at some $x > \bar{r}$. Then $[1 - U_{\min}(x)]$ has a downward jump at x . The payoff to the informed firm i at x is

$$[1 - U_{\min}(x^-)] [v(x)l + w(x)h]$$

since whatever the mass that the uninformed places at x , does not affect the probability that agents switch due to Assumption 1. We deal with two cases. First, x is not an endpoint $U_{\min}(x) < 1$ and second, x is an endpoint $U_{\min}(x) = 1$. Start with the first case. For the no

endpoint case, there exists $\varepsilon > 0$ such that

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} [1 - U_{\min}(x + \varepsilon)] [v(x + \varepsilon)l + w(x + \varepsilon)h] \\ & < [1 - U_{\min}(x^-)] [v(x)l + w(x)h] \end{aligned}$$

by continuity of $v(r)l + w(r)h$ and the discontinuity in $U_{\min}(r)$ at x . Therefore, the informed firm i does not place any mass in some ε -neighborhood above x . But then it is not an equilibrium strategy for uninformed firm j to place mass at x , as it could make more by moving the mass point slightly to the right (and have the same probability of winning over the agents from i).¹⁷

Next, consider the second case, where x is the endpoint of the support of $U_{\min}(x)$, so that $U_{\min}(x) = 1$, but $U_{\min}(x^-) < 1$. In that case for the j -th firm

$$[1 - U_{\min}^k(x)] [(1 - L(x))v(x)l + (1 - H(x))w(x)h] = 0$$

If over some open interval $(d, e) \subset (\bar{r}, x)$ it holds for some $r \in (d, e)$ that $U_{\min}^k(r) < 1$, and at least $L(r) < 1$ or $H(r) < 1$ for $r \in (d, e)$, as otherwise the j -th firm cannot win over agents, then

$$[1 - U_{\min}^k(r)] [(1 - L(r))v(r)l + (1 - H(r))w(r)h] > 0$$

Thus the j -th firm would be better off by moving the mass from x to the lower interval (d, e) . $\square$

Lemma 16. *The strategies $L(r)$ and $H(r)$ contain no atoms on the open interval $(\bar{r}, \infty)$.*

Proof. In analogy with Lemma 15 one shows that if $L(r)$ or $H(r)$ have a mass point at r , the uninformed firms have no incentive to place any mass in the neighborhood to the right

¹⁷Other uninformed firms also remove mass above x . Consider uninformed firm $k \neq i, j$, and denote

$$U_{\min}^{j \neq k}(r) = 1 - \prod_{j \neq i, k}^m [1 - U_j(r)]$$

The expected payoff to k is

$$\begin{aligned} E[\pi_k(r)] &= [1 - U_{\min}^{j \neq k}(r)] [1 - L(r)]v(r)l \\ &\quad + [1 - U_{\min}^{j \neq k}(r)] [1 - H(r)]w(r)h \end{aligned}$$

due to the discrete jump upwards at x in $U_j(r)$, the term $1 - U_{\min}^{j \neq k}(r)$ jumps downwards. As a result $\exists \varepsilon, \delta > 0$ for which

$$E[\pi_k(r + \varepsilon)] < E[\pi_k(r - \delta)]$$

hence uninformed firm k also removes any mass in some right-hand side ε neighborhood.

of this mass point. So suppose that either $L(r)$ or $H(r)$, or both have a mass point at x . Due to Assumption 1, we can now directly use the right continuity of $L(x)$ and $H(x)$. Thus, the expected payoff to an uninformed firm j offering x is

$$[1 - U_{\min}^k(x)] [(1 - L(x)) v(x) l + (1 - H(x)) w(x) h]$$

Because for some $\varepsilon > 0$

$$\begin{aligned} & [1 - U_{\min}^k(x)] [(1 - L(x)) v(x) l + (1 - H(x)) w(x) h] < \\ & [1 - U_{\min}^k(x - \varepsilon)] [(1 - L(x - \varepsilon)) v(x - \varepsilon) l + (1 - H(x - \varepsilon)) w(x - \varepsilon) h] \end{aligned}$$

due to the discrete jump downwards at x of $1 - L(r)$ or $1 - H(r)$, or both, while $v(r)$ and $w(r)$ are continuous at x . Hence, the uninformed firms do not place any mass in some right neighborhood of x , and the informed would be better off to move mass up. $\square$

Combining the two cases in Lemma 15 and Lemma 16, it follows that there cannot occur any mass point in the potential mixing strategies, i.e., $\delta(r) = 0$ for $r > \bar{r}$. By these lemmas, one shows analogously that $U(r)$ and $L(r)$ cannot have mass points in the half open interval $[\underline{r}, c/q)$. We leave this proof to the reader.

The following lemmas prove that the strategies $L(r_i > \bar{r}) > 0$, $H(r_i > \bar{r}) > 0$ and $U_{\min}(r_i > \bar{r}) > 0$ are independently distributed.

Lemma 17. *For $r > \bar{r}$, the probability that the informed firm quotes the same rate to its two types of agents over some open interval is zero.*

Proof. Suppose $U_{\min}(r_j)$ is continuous and monotone on the open interval (d, e) , where $d > \bar{r}$, so there are no gaps and recall Lemma 15. The expected payoff to the informed firm if it quotes rate t to its low-type agents l and s to its high-type agents h is

$$E[\pi_i(t; s)] = [1 - U_{\min}(t)] v(t) l + [1 - U_{\min}(s)] w(s) h \quad (21)$$

Suppose $t = s \in (d, e)$. Then by the assumed continuity and monotonicity of $U_{\min}(r)$ on (d, e) , payoffs must be constant, equal to $E[\pi_i]$ say. Hence for every $r \in (d, e)$ it holds that

$$[1 - U_{\min}(r)] v(r) l + [1 - U_{\min}(r)] w(r) h = E[\pi_i]$$

As a result we can solve for the conjectured mixing strategy of the minimum rate quoted by the uninformed firms

$$1 - U_{\min}(r) = \frac{E[\pi_i]}{v(r) l + w(r) h} \quad (22)$$

Substituting (22) in (21) gives

$$E[\pi_i(t; s)] = \frac{E[\pi_i]}{v(t)l + w(t)h} v(t)l + \frac{E[\pi_i]}{v(s)l + w(s)h} w(s)h$$

Thus $E[\pi_i(t = r; s = r)] = E[\pi_i]$. Consider a deviation $t \neq s$ in such a way that $t = r - \varepsilon$ and $s = r + \varepsilon$ for some $\varepsilon > 0$. Then

$$\frac{E[\pi_i]}{v(r - \varepsilon)l + w(r - \varepsilon)h} v(r - \varepsilon)l > \frac{E[\pi_i]}{v(r)l + w(r)h} v(r)l$$

To show this, bring the denominators to the other sides

$$[v(r) - p\varepsilon]l[v(r)l + w(r)h] > v(r)l[v(r)l - lp\varepsilon + w(r)h - q\varepsilon h]$$

and simplify this to

$$-p\varepsilon l[v(r)l + w(r)h] > -[p\varepsilon l + q\varepsilon h]v(r)l$$

which reduces further to

$$(p - q)c > 0$$

and this holds by assumption. In a similar way one shows that

$$\frac{E[\pi_i]}{v(r + \varepsilon)l + w(r + \varepsilon)h} w(r + \varepsilon)h > \frac{E[\pi_i]}{v(r)l + w(r)h} w(r)h$$

which also reduces to $(p - q)c > 0$. It follows that firm i does not want to quote the same rate $s = t = r$ to both type of agents. $\square$

Note that the Lemma 17 does not rule out that the strategies $L(s)$ and $H(t)$ have the same support. If the two mixing distributions are independent and continuous over the same support, then the probability of quoting the same rate to both types is still zero.

In the next series of lemmas, we consider the following cases: I/ all three strategies U , L and H have an finite upper bound above $\bar{r}$; II/ two of the three strategies have an upper bound above $\bar{r}$; III/ just one of the three strategies has an upper bound above $\bar{r}$; and IV/ all strategies have unbounded support. Lemma 18 shows that case I cannot be part of a Nash equilibrium. The text under Lemma 18 explains why case II cannot be part of a Nash equilibrium and Lemma 19 and Lemma 20 analyze case III. Finally Lemma 21 explains why case IV/ cannot be part of a Nash equilibrium. We start with the first case I.

Lemma 18. *Strategies $L(\cdot)$, $H(\cdot)$, $U(\cdot)$ with a finite endpoint x , where $x \in (\bar{r}, \infty)$, cannot be part of a Nash equilibrium.*

Proof. Assume the upper bound $x > \bar{r}$ is finite and $U_j(x) = 1$. In this case the informed firm does not place any mass above x , as this generates no clients from either type for sure. So the upper bounds for $H(\cdot)$ and $L(\cdot)$ cannot be above x either. At x , the payoff for the j -th uninformed firm is (with $k \neq j$)

$$[1 - U_{\min}^k(x)] \{[1 - L(x)] v(x) l + [1 - H(x)] w(x) h\} = 0$$

since $L(x) = H(x) = 1$ and as there are no mass points in the strategies L and H above $\bar{r}$ by Lemma 16, regardless of whether or not the upper boundaries for $L(\cdot)$ and $H(\cdot)$ are at or below x . Suppose first that $L(\cdot)$ and $H(\cdot)$ share this upper bound x . Then $L(x - \varepsilon) < 1$, $H(x - \varepsilon) < 1$ and $U_{\min}^k(x - \varepsilon) < 1$ for some $\varepsilon > 0$ (also recall Lemma 15). But then

$$\begin{aligned} & [1 - U_{\min}^k(x - \varepsilon)] \times [1 - L(x - \varepsilon)] v(x - \varepsilon) l \\ & + [1 - U_{\min}^k(x - \varepsilon)] \times [1 - H(x - \varepsilon)] w(x - \varepsilon) h > 0 \end{aligned}$$

Thus it is optimal for j to move the upper bound of its strategy down from x .

The same argument holds for uninformed firm k . If $U_j(x) = 1$, the expected return for firm k at x is

$$[1 - U_{\min}^{j \neq k}(x)] [(1 - L(x)) v(x) l + (1 - H(x)) w(x) h] = 0$$

where $U_{\min}^{j \neq k}(x)$ is the lowest rate offered by the other uninformed firms, not being firm k . Because $U_j(x) = 1$, and hence $U_{\min}^{j \neq k}(x) = 1$, the expected return for uninformed firm k is zero. If uninformed firm k quotes a rate $x - \varepsilon$, its expected return is

$$\begin{aligned} & [1 - U_{\min}^{j \neq k}(x - \varepsilon)] \times [1 - L(x - \varepsilon)] v(x - \varepsilon) l \\ & + [1 - U_{\min}^{j \neq k}(x - \varepsilon)] \times [1 - H(x - \varepsilon)] w(x - \varepsilon) h > 0 \end{aligned}$$

for some $\varepsilon > 0$ if $L(x - \varepsilon) < 1$ or $H(x - \varepsilon) < 1$, and $U_{\min}^{j \neq k}(x - \varepsilon) < 1$. As a result, uninformed firm k is also better off deviating to a lower rate than x .

If the strategies $L(\cdot)$ and $H(\cdot)$ have an upper bound (possibly different) lower than x , then it is in the interest of j to move x down to at least just below the highest of these two upper bounds. Suppose for example that the highest upper bound is for $H(\cdot)$, say at $s < x$,

so that $H(s) = 1$, then the expected return for uninformed firm j is

$$[1 - U_{\min}^k(s)] [1 - H(s)] w(s) h = 0$$

but

$$[1 - U_{\min}^k(s - \varepsilon)] [1 - H(s - \varepsilon)] w(s - \varepsilon) h > 0$$

if $U_{\min}^k(s - \varepsilon) < 1$, i.e., if the other uninformed also have the upper bound at s . In either of these cases as the j -th uninformed firm moves the upper bound of its strategy down, the informed firm wants to follow suit. As a result of this leap frogging, any finite $x > \bar{r}$ is not part of a Nash equilibrium. A similar reasoning applies if the highest upper bound is for $L(\cdot)$. $\square$

In the same manner case II in which two of the three strategies U , L and H have a finite upper bound above $\bar{r}$ can be ruled out. If U and L , or U and H have an upper bound, then the informed firm does not offer rates above the upper bound of U for neither of its agents, as otherwise these agents would switch. Similarly, if L and H have an upper bound, the uninformed firm does not make offers above the highest of the two bounds, as it would not win over any agents. In all these cases, it pays for i or j to lower the upper bound to $\bar{r}$, in analogy to the arguments used in the proof to Lemma 18.

We turn to case III, in which just one of the three strategies has an upper bound. The case in which just U has an upper bound can be ruled out on the same grounds as above, because the informed would not be able to retain its agents by pricing above this bound. The other two cases in which either H or L has an upper bound require more scrutiny. The next two lemmas assume that the supports of H and L do not overlap. Thus, if L has an upper bound, then the support of H can only be to the right of this upper bound and vice versa. This ensures that Lemma 17 is satisfied.

The last Lemma 21 shows two things. For the case III when either H or L have an upper bound and have partly overlapping supports, Lemma 21 shows this cannot be the case in a Nash equilibrium. Moreover, Lemma 21 shows that in the case of unbounded supports, the resulting H and L are also not Nash equilibrium strategies, i.e., case IV cannot occur. But we first deal with the cases in which the supports of H and L do not overlap.

Lemma 19. *Consider the case where $U_{\min}(\cdot)$ and $H(\cdot)$ have supports with unbounded upper end and lower bound at x , but $L(x) = 1$, where $\bar{r} \leq x < \infty$. This configuration cannot be part of a Nash equilibrium.*

Proof. Suppose that $L(x) = 1$, where $x \geq \bar{r}$. Then the expected payoffs for $r \geq x$ are

respectively

$$E[\pi_j(r)] = [1 - U_{\min}^k(r)] [1 - H(r)] w(r) h$$

for the j -th uninformed. For the informed firm i we have

$$E[\pi_i^h(r)] = [1 - U_{\min}^k(r)] [1 - U_j(r)] w(r) h$$

and $E[\pi_i^l] = v(x)l$. Specifically at the lower bound x it thus holds that

$$E[\pi_j] = [1 - U_{\min}^k(x)] [1 - H(x)] w(x) h = w(x) h$$

and

$$E[\pi_i^h] = [1 - U_{\min}^k(x)] [1 - U_j(x)] w(x) h = w(x) h$$

given the assumption that the lower bounds of the supports of $U_{\min}(\cdot)$ and $H(\cdot)$ are at x , and given that $E[\pi_j]$ and $E[\pi_i^l]$ must be constant on the support, since Lemmas 15 and 16 show that $H(\cdot)$ and $U_{\min}(\cdot)$ do not contain mass points. So that

$$E[\pi_i^h] = E[\pi_j] = w(x) h$$

In the case that $x > \bar{r}$ we can solve these equations (if $x = \bar{r}$ then $w(x) = 0$). With at least one uninformed being active and the identical expected rents, i.e., $m \geq 0$, division shows that

$$1 - H(r) = 1 - U_j(r) = \left(\frac{w(x)}{w(r)} \right)^{1/(m+1)} \quad (23)$$

and use that $[1 - U_{\min}(x)] = [1 - U_j(x)]^{(m+1)}$. Note that kinks in the strategies such as in Baye *et al.* (1992), whereby some firms become inactive at some point $t > x$ and load all remaining mass at the upper bound of the support, would be inconsistent with the supposition of an unbounded support, there can be no mass at infinity. The distributions in equation (23) resemble the Baye and Morgan (1999) type of unbounded Bertrand equilibria, since it is in all firm's interest to choose x as high as possible.

But in this case, these distributions cannot be part of a Nash equilibrium by the following arguments. If $x > \bar{r}$, it is in the interest of an uninformed firm to move all mass down to ε below x , but ε sufficiently small such that $x - \varepsilon > \bar{r}$, directly violating the supposition of this lemma. The uninformed firm then captures the l type agents for certain, implying a discrete gain of

$$v(x - \varepsilon)l + [w(x - \varepsilon) - w(x)] h > 0$$

for some ε sufficiently small. If $x = \bar{r}$ and $E[\pi_j] = 0$ as $w(\bar{r}) = 0$, then the uninformed still

could gain by moving all mass down to ε below $\bar{r}$. Note that with $L(\bar{r}) = 1$ the informed firm loads all mass at $\bar{r}$ to maximize its payoff. Therefore the uninformed would gain

$$v(\bar{r} - \varepsilon)l + w(\bar{r} - \varepsilon)h$$

The slight loss on the h agents is more than compensated for by the discrete gain of $v(\bar{r} - \varepsilon)l$. But then the informed firm moves its offers to the high-type agents h down as well, which is against the supposition that the supports of H and L do not overlap. $\square$

The other case in which just H has an upper bound is similar.

Lemma 20. *Consider the case where $U_{\min}(\cdot)$ and $L(\cdot)$ have supports with an unbounded upper end and lower bound at x , but $H(x) = 1$, where $\bar{r} \leq x < \infty$. This configuration cannot be part of a Nash equilibrium.*

Proof. Suppose that $H(x) = 1$, where $x \geq \bar{r}$. Then the expected payoffs for $r \geq x$ are respectively

$$E[\pi_j(r)] = [1 - U_{\min}^k(r)] [1 - L(r)] v(r) l$$

for the j -th uninformed. For the informed we have

$$E[\pi_i^l(r)] = [1 - U_{\min}^k(r)] [1 - U_j(r)] v(r) l$$

Specifically, at the lower bound x it thus holds that

$$E[\pi_j] = [1 - U_{\min}^k(x)] [1 - L(x)] v(x) l = v(x) l$$

and

$$E[\pi_i^l] = [1 - U_{\min}^k(x)] [1 - U_j(x)] v(x) l = v(x) l$$

given the assumption that the lower bounds of the supports of $U_{\min}(\cdot)$ and $L(\cdot)$ are at x , and given that $E[\pi_j]$ and $E[\pi_i^l]$ must be constant on the support. Lemmas 15 and 16 show that $L(\cdot)$ and $U_{\min}(\cdot)$ do not contain mass points. Division shows as in the proof of Lemma 19 that with $m + 1$ active uninformed firms

$$1 - L(r) = 1 - U_j(r) = \left(\frac{v(x)}{v(r)} \right)^{1/(m+1)} \quad (24)$$

Note that kinks in the strategies such as in Baye *et al.* (1992), whereby some firms become inactive at some point $t > x$ and load all remaining mass at the upper bound of the support, would be inconsistent with the supposition of an unbounded support, as there can be no mass

at infinity. The distributions (24) resemble the Baye and Morgan (1999) type of unbounded Bertrand equilibria.

But in this case, these distributions cannot be Nash equilibria by the following arguments. If $x > \bar{r}$, then it is not in the interest of firm i to put any mass below x . Instead, the informed places all mass at x , as this maximizes profits on its high-risk agents h . So, suppose that the informed has $H(x) = 1$, regardless of whether $x = \bar{r}$, or $x > \bar{r}$. If, however $x > \bar{r}$, it is in the interest of an uninformed firm j to move all mass down to ε below x , directly violating the supposition of this lemma. The uninformed then captures the h and l type agents for certain, implying a discrete gain of

$$w(x - \varepsilon)h + [v(x - \varepsilon) - v(x)]l > 0$$

for some ε sufficiently small. But then the informed firm moves its pricing for its low-risk agents l down as well, violating the assumption that the supports of H and L do not overlap. If $x = \bar{r}$, no uninformed firm has an incentive to deviate below $\bar{r}$, as capturing the high-type agents below $\bar{r}$ is loss making. But in this case the informed has an incentive to move the upper bound for H above $\bar{r}$ by at least an $\varepsilon > 0$. As this gives i by (24) (apart from the $v(\bar{r})l$ that he is expected to make on its low-risk types)

$$w(\bar{r} + \varepsilon)h \left(\frac{v(\bar{r})}{v(\bar{r} + \varepsilon)} \right)^{1/(m+1)} > w(\bar{r})h = 0$$

On the left-hand side all terms are positive, hence it pays to move x up from $\bar{r}$. But in this case the supports of H and L would overlap, which is against the supposition in the claim. $\square$

To conclude, the configurations where $L(r)$ or $H(r)$ has an endpoint $x \in (\bar{r}, \infty)$, but the other two do not, cannot be part of a Nash equilibrium according to Lemma 19 and Lemma 20. Lastly the configuration where $U(r)$ has a finite endpoint, but $L(r)$ and $H(r)$ are unbounded cannot hold according to the reasoning just below Lemma 18. So all three distributions $L(\cdot)$, $H(\cdot)$ and $U(\cdot)$ must have unbounded supports or at least partially overlap, if there is to exist a valid mixing equilibrium for r in the interval $(\bar{r}, \infty)$. We consider this case IV in the next lemma.

Lemma 21. *There does not exist a configuration of rates s, t , where the informed firm quotes t to its low-risk agents l and s to the high-risk agents h , where $t \in [d_1, \infty)$ and / or $s \in [d_2, \infty)$, i.e., one of the two may have a finite upper bound but supports do partially overlap, for some $d_1 \geq \underline{r}$ and $d_2 \geq \underline{r}$, and where $E[\pi_i^l]$, $E[\pi_i^h]$ and $E[\pi_i(t, s)]$ from (21) are constant.*

Proof. Suppose that expected profits for the informed are indeed constant $E[\pi_i(t, s)] = \pi > 0$. From (21) we have

$$(1 - U_{\min}(t))v(t)l + (1 - U_{\min}(s))w(s)h = \pi \quad (25)$$

where at least one of the two t or s is unbounded from above (given Lemma's 18, 19 and 20), but with partial overlapping supports. For (25) to hold, it must be the case that on some open intervals for t and s

$$(1 - U_{\min}(t))v(t)l = Z(t) \geq 0 \text{ and } (1 - U_{\min}(s))w(s)h = D(s) \geq 0$$

where $Z(t) + D(s) = \pi(t, s) = \pi > 0$. The functions Z and D may depend on each other, but from Lemma 17 we already know that we cannot have $t = s$ on any open interval. Suppose first Z and D are *not* constant and $(1 - U_{\min}(t))v(t)$ varies with t and $(1 - U_{\min}(s))w(s)$ varies with s . Then varying t and s both would require one of the two parts to decrease and the other part to increase, otherwise the sum $\pi(t, s)$ cannot be constant and equal to π . But then it would be optimal to hold one of the two, t or s , fixed for the part that is decreasing and solely increase the other part. This would raise the expected payoff locally above π , contradicting an equilibrium. Thus, both Z and D are necessarily constant.

Then we can express the conjectured strategy of the uninformed firm as

$$1 - U_{\min}(t) = \frac{Z}{v(t)l} \text{ and } 1 - U_{\min}(s) = \frac{D}{w(s)h} \quad (26)$$

But as long as $Z, D > 0$, the two are incompatible. Lemmas 19 and 20 showed that if one of the two variables t or s has a finite upper bound and their supports do not overlap, then this cannot be part of a Nash equilibrium. This leaves the possibility that even if one of the two t or s is bounded from above, their supports partially overlap. Clearly if both are unbounded from above, the supports do overlap.

Suppose $d_1 < d_2$. Consider an open set $(t, s) \in (t \geq \underline{t}, s \geq \underline{s})$ such that for t sufficiently above the lower bound d_1 , t rises above the lower bound of s , d_2 , so that there exists at least partial overlap of the supports. Let (t_1, t_2) where $t_1 < t_2$, be such that these are in an open interval where the two supports overlap. Let (s_1, s_2) be such that $s_1 \neq t_1$ and $s_2 \neq t_2$. Then for

$$[1 - U_{\min}(t_1)]v(t_1)l = Z$$

there is at least one s_1 such that

$$[1 - U_{\min}(s_1)]w(s_1)h = D$$

Similarly, for

$$[1 - U_{\min}(t_2)]v(t_2)l = Z$$

there is at least one s_2 such that

$$[1 - U_{\min}(s_2)]w(s_2)h = D$$

Hence, t_1 and t_2 imply

$$\frac{[1 - U_{\min}(t_1)]v(t_1)l}{[1 - U_{\min}(t_2)]v(t_2)l} = 1 \quad (27)$$

and s_1 and s_2 imply

$$\frac{[1 - U_{\min}(s_1)]w(s_1)h}{[1 - U_{\min}(s_2)]w(s_2)h} = 1 \quad (28)$$

But since the two supports overlap, there exist t_1 and t_2 that are both in the support of s . Then by 28 it must also hold that

$$\frac{[1 - U_{\min}(t_1)]w(t_1)}{[1 - U_{\min}(t_2)]w(t_2)} = 1 \quad (29)$$

By combining the equations (27) and (29), the following must hold

$$\frac{p(t_1) - c}{p(t_2) - c} = \frac{q(t_1) - c}{q(t_2) - c} \quad (30)$$

But this can only hold if $t_1 = t_2$. Hence, even if t and s differ at any point in the part where the supports overlap, the two expressions for Z and D in equation (26) are incompatible. $\square$

We have shown that there does not exist a pure strategy or mixed strategy equilibrium with unbounded support in which both informed and uninformed earn positive returns. We turn to the interval $[\underline{r}, \bar{r}]$. Note that we already have established that $H(r < \bar{r}) = 0$ in Lemma 10. We also showed with Lemma 18 that $H(\cdot)$ cannot have a bounded endpoint $x \in (\bar{r}, \infty)$ and that $H(\cdot)$ cannot be unbounded by Lemma 19, thus we conclude that $H(\bar{r}) = 1$.

Equilibrium Strategy and Uniqueness

Lemma 22. (i) $\delta_L(\underline{r}) = 0$, $\delta_L(\bar{r}) = 0$ for $L(r)$ and $\delta_j(\underline{r}) = 0$ for $U_{\min}(r)$; and (ii) $\delta_j(\bar{r}) > 0$ for $U_{\min}(r)$

Proof. (i) Suppose $\delta_j(\underline{r}) > 0$. Then there exists an $\varepsilon > 0$ such that for firm i

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} [1 - U_{\min}(\underline{r} + \varepsilon)] v(\underline{r} + \varepsilon) l \\ & < [1 - U_{\min}(\underline{r}^-)] v(\underline{r}) l \end{aligned}$$

by continuity of $v(r)l$ and the discontinuity in $U_{\min}(r)$ at $\underline{r}$. Therefore, the informed firm i does not place any mass in some open ε -neighborhood above $\underline{r}$. But then it is not an equilibrium strategy for the uninformed firm j to place mass at $\underline{r}$, as it could make more by moving the mass point slightly to the right. Thus $\delta_j(\underline{r}) > 0$ is not part of an equilibrium. If firm i has a mass point at the lower bound $\delta_L(\underline{r}) > 0$, firm j prefers to remove mass, in an ε -neighborhood to the right of $\underline{r}$, since

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \{[1 - L(\underline{r} + \varepsilon)] v(\underline{r} + \varepsilon) l + w(\underline{r} + \varepsilon) h\} \\ & = [1 - L(\underline{r})] v(\underline{r}) l + w(\underline{r}) h < 0 \end{aligned}$$

as $1 - L(\underline{r}) < 1$ by the tie rule 1. But if j puts no mass in this right-side neighborhood, i wants to move mass up, a contradiction.

If $\delta_L(\bar{r}) > 0$, it must be that $\lim_{\varepsilon \downarrow 0} [1 - L(\bar{r} - \varepsilon)] > 0$, so that in some ε -neighborhood to the left of $\bar{r}$ firm j can make positive profits

$$\begin{aligned} & [1 - L(\bar{r} - \varepsilon)] v(\bar{r} - \varepsilon) l + w(\bar{r} - \varepsilon) h = \\ & [1 - L(\bar{r} - \varepsilon)] [lp\bar{r} - lp\varepsilon - cl] - hq\varepsilon \end{aligned}$$

Taking limits gives

$$\lim_{\varepsilon \downarrow 0} [1 - L(\bar{r} - \varepsilon)] v(\bar{r}) l > 0$$

So, firm j would remove all probability mass from $\bar{r}$, and firm i no longer wants to put mass at $\bar{r}$.

(ii) Because the distribution of the lowest rate of the uninformed firms has support on $r \geq \underline{r}$, and because of Assumption 1, firm i earns at least $v(\underline{r}) > 0$ on its low-type agents l . This implies firm i has to outbid the uninformed firms with positive probability. This can only occur at the upper bound $\bar{r}$ if the distribution of the minimum rate of the uninformed firms has probability mass at the upper bound of the support, thus $\delta_j(\bar{r}) > 0$. $\square$

Lemma 14 and lemma 22 imply that the $U_{\min}(r)$ and $L(r)$ are continuous on the half open interval $[\underline{r}, \bar{r})$. Thus both firms must make their equilibrium expected profits on this

interval. As $\underline{r}$ is in both supports, $E[\pi_j(r)] = 0$ and $E[\pi_i^l(r)] = v(\underline{r})l$. Given this, $1 - L(r)$ and $1 - U_{\min}(r)$ are uniquely determined by equations (19) and (20).

Lemma 23. *The equilibrium rate strategies $U_{\min}(r) = 1 - v(\underline{r})/v(r)$ and $L(r) = 1 + w(r)h/v(r)l$ are unique.*

Proof. Given that $E[\pi_i^l(r)] = v(\underline{r})l$; this combined with (19) implies

$$U_{\min}(r) = 1 - \frac{v(\underline{r})}{v(r)}$$

Turning to $L(r)$, consider the expected return for uninformed firm j

$$E[\pi_j] = [1 - U_{\min}^k(r)] \{(1 - L(r))v(r)l + w(r)h\} \quad (31)$$

Using that $w(\bar{r}) = 0$, we get

$$E[\pi_j(\bar{r})] = [1 - L(\bar{r})] [1 - U_{\min}^k(\bar{r})] v(\bar{r})l$$

But at $\bar{r}$, $L(\bar{r}) = 1$ and $U_{\min}^k(\bar{r}) = 1$ so that $E[\pi_j(\bar{r})] = 0$. Substitution of $E[\pi_j] = 0$ in equation (31), gives the unique solution for the strategy for the l -type agents of the informed

$$L(r) = 1 + \frac{w(r)h}{v(r)l}$$

where $w(r)/v(r) < 0$. □

It is also evident that at least one uninformed firm quotes rates to the agents of firm i , i.e., $\mathcal{A} \neq \emptyset$. If no uninformed firm is active, $\mathcal{A} = \emptyset$, then it is in the interest of the monopolist firm i to quote rates as high as can be absorbed. In such a case at least the rates offered to the agents of i would be above $\underline{r}$. But then any uninformed firm has an incentive to enter, undercut and obtain all agents l and h .