

Supplementary Appendix

“Competing Sellers in Security-Bid Auctions under Risk-Averse Bidders”

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This Online Appendix contains additional technical results, proofs, and supplementary numerical details omitted from the main text. Section I establishes auxiliary properties of survival functions and order statistics underlying Proposition 6, and introduces a scalar measure of intensity in steepness that complements the ordinal steepness order used in the paper. Section II presents additional simulations for the homogeneous and heterogeneous risk-aversion environments.

I Additional Results

Lemma I.1. *Let Λ be an absolutely continuous distribution with density λ , survival function $\bar{\Lambda} := 1 - \Lambda$, and hazard rate $\psi(x) := \lambda(x)/\bar{\Lambda}(x)$. Then:*

- i) if ψ is strictly decreasing, then $\bar{\Lambda}$ is both strictly convex and strictly log-convex;*
- ii) if ψ is strictly increasing, then $\bar{\Lambda}$ is strictly log-concave. If, additionally, there exists $x^* \in (x_L, x_H)$ such that $\psi'(x) > \psi(x)^2$ for all $x > x^*$, then $\bar{\Lambda}$ is strictly concave on (x^*, x_H) . At any point where $\psi'(x) \leq \psi(x)^2$, $\bar{\Lambda}$ is convex.*

Proof. Write the survival function in terms of the hazard rate as

$$\bar{\Lambda}(x) = \exp \left\{ - \int_{x_L}^x \psi(t) dt \right\}.$$

Thus $(\log \bar{\Lambda})'(x) = -\psi(x)$ and $(\log \bar{\Lambda})''(x) = -\psi'(x)$. Hence, a strictly decreasing hazard rate implies that $\bar{\Lambda}$ is log-convex, while a strictly increasing hazard rate implies that $\bar{\Lambda}$ is log-concave.

Moreover, since $\bar{\Lambda}'(x) = -\psi(x)\bar{\Lambda}(x)$,

$$\bar{\Lambda}''(x) = \bar{\Lambda}(x) [\psi(x)^2 - \psi'(x)].$$

Because $\bar{\Lambda}(x) > 0$, the sign of $\bar{\Lambda}''(x)$ is determined by $\psi(x)^2 - \psi'(x)$. If ψ is strictly decreasing, then $\psi'(x) < 0$, so $\bar{\Lambda}''(x) > 0$ and $\bar{\Lambda}$ is strictly convex.

If instead ψ is strictly increasing and there exists $x^* \in (x_L, x_H)$ such that $\psi'(x) > \psi(x)^2$ for all $x > x^*$, then $\bar{\Lambda}''(x) < 0$ for all $x > x^*$, so $\bar{\Lambda}$ is strictly concave on (x^*, x_H) . Otherwise, at any point where $\psi'(x) \leq \psi(x)^2$, the survival function is convex. ■

Intensity in steepness

The steepness order ranks families of securities ordinally. For the analysis below, it is useful to associate each family $\mathcal{S}$ with a scalar index $\alpha_{\mathcal{S}}$ that measures its intensity in steepness relative to the debt family $\mathcal{S}^d$. We define

$$\alpha_{\mathcal{S}} = \sup_{v \in \mathcal{V}} \inf_{s \in [s_L, s_H]} \left\{ \frac{\partial P_{\mathcal{S}}(v, s)}{\partial v} - \frac{\partial P_{\mathcal{S}^d}(v, s^d)}{\partial v} : P_{\mathcal{S}}(v, s) = P_{\mathcal{S}^d}(v, s^d) \text{ for some } s^d \in S^d \right\}.$$

Thus, $\alpha_{\mathcal{S}}$ provides a one-dimensional measure of how steep a family is, relative to the debt benchmark, while remaining consistent with the primitive steepness order.

Lemma I.2. *For any two families $\mathcal{S}'$ and $\mathcal{S}''$ if $\mathcal{S}'$ is steeper than $\mathcal{S}''$, then $\alpha_{\mathcal{S}'} > \alpha_{\mathcal{S}''}$.*

Proof. Suppose $\mathcal{S}'$ and $\mathcal{S}''$ are two families such that $\mathcal{S}'$ is steeper than $\mathcal{S}''$. Then, we have that

$$\begin{aligned} & \sup_{v \in \mathcal{V}} \left\{ \frac{\partial P_{\mathcal{S}'}(v, s')}{\partial v} - \frac{\partial P_{\mathcal{S}''}(v, s'')}{\partial v} : P_{\mathcal{S}'}(v, s') = P_{\mathcal{S}''}(v, s'') \right\} \\ &= \sup_{v \in \mathcal{V}} \left\{ \frac{\partial P_{\mathcal{S}'}(v, s')}{\partial v} - \frac{\partial P_{\mathcal{S}''}(v, s'')}{\partial v} : P_{\mathcal{S}'}(v, s') = P_{\mathcal{S}''}(v, s'') = P_{\mathcal{S}^d}(v, s^d) \text{ for some } s^d \in S^d \right\} \\ &= \sup_{v \in \mathcal{V}} \left\{ \left(\frac{\partial P_{\mathcal{S}'}(v, s')}{\partial v} - \frac{\partial P_{\mathcal{S}^d}(v, s^d)}{\partial v} \right) - \left(\frac{\partial P_{\mathcal{S}''}(v, s'')}{\partial v} - \frac{\partial P_{\mathcal{S}^d}(v, s^d)}{\partial v} \right) : \begin{array}{l} P_{\mathcal{S}'}(v, s') = P_{\mathcal{S}''}(v, s'') \\ = P_{\mathcal{S}^d}(v, s^d), s^d \in S^d \end{array} \right\} \\ &\geq \sup_{v \in \mathcal{V}} \left\{ \left(\frac{\partial P_{\mathcal{S}'}(v, s')}{\partial v} - \frac{\partial P_{\mathcal{S}^d}(v, s^d)}{\partial v} \right) : P_{\mathcal{S}'}(v, s') = P_{\mathcal{S}^d}(v, s^d) \text{ for some } s^d \in S^d \right\} \\ &\quad - \sup_{v \in \mathcal{V}} \left\{ \left(\frac{\partial P_{\mathcal{S}''}(v, s'')}{\partial v} - \frac{\partial P_{\mathcal{S}^d}(v, s^d)}{\partial v} \right) : P_{\mathcal{S}''}(v, s'') = P_{\mathcal{S}^d}(v, s^d) \text{ for some } s^d \in S^d \right\} \\ &\geq \sup_{v \in \mathcal{V}} \inf_{s' \in [s_L, s_H]} \left\{ \left(\frac{\partial P_{\mathcal{S}'}(v, s')}{\partial v} - \frac{\partial P_{\mathcal{S}^d}(v, s^d)}{\partial v} \right) : P_{\mathcal{S}'}(v, s') = P_{\mathcal{S}^d}(v, s^d) \text{ for some } s^d \in S^d \right\} \\ &\quad - \sup_{v \in \mathcal{V}} \inf_{s'' \in [s_L, s_H]} \left\{ \left(\frac{\partial P_{\mathcal{S}''}(v, s'')}{\partial v} - \frac{\partial P_{\mathcal{S}^d}(v, s^d)}{\partial v} \right) : P_{\mathcal{S}''}(v, s'') = P_{\mathcal{S}^d}(v, s^d) \text{ for some } s^d \in S^d \right\} \end{aligned}$$

where the last inequality holds because $\mathcal{S}'$ is steeper than $\mathcal{S}''$. ■

Therefore, the use of intensity in steepness gives a way to capture the degree of steepness between different families in a one-dimensional way.

II Numerical Simulation

In this section, we numerically explore how firms' risk-aversion levels affect equilibrium strategies in all stages of the game.

II.1 Simulation Algorithm

Homogeneously risk-averse bidders We describe the simulation algorithm using backward induction.

- **Step 3: Interim expected utility in the auction stage.** For each number of bidders I and each number $1 \leq k \leq I - 1$, we simulate the empirical distribution of the highest order statistic $v^{(k:k)}$ for bidders' signals for each family $\mathcal{S}$, by making one million draws from the uniform distribution. This empirical distribution is necessary to compute the maximum highest bid from a group of k participants. With this empirical distribution on hand, we compute the winning probability and the expected utility for any bidder i with risk aversion r .
- **Step 2: Bidders' entry strategies.** For any seller's strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$, we compute bidders' entry probability strategies $\gamma'(r \mid \mathcal{S}', \mathcal{S}'')$ using the interim expected utility obtained in Step 3. Specifically, we solve the non-linear equation defined in (MS).
- **Step 1: Sellers' expected revenue.** Having computed $\gamma'(r \mid \mathcal{S}', \mathcal{S}'')$ in the previous step, we construct an *auxiliary* random index ω to implement the entry strategy prescribed by $\gamma'(r \mid \mathcal{S}', \mathcal{S}'')$. The random index ω is drawn from a uniform distribution with support $[0, 1]$, and is used to randomize bidders across auctions. Specifically, we draw an index ω_i for each bidder i and then follow a threshold rule: if $\omega_i \leq \gamma'(r \mid \mathcal{S}', \mathcal{S}'')$, then bidder i goes to the auction with the steeper family; otherwise, she goes to the auction with the flatter family. Once we distribute bidders across auctions in a random fashion, we compute the

sellers' revenue using equation (5). We repeat this process one million times and compute sellers' expected revenue in (6) as the average revenue over all iterations.

- **Step 0: Sellers' choice of design.** We repeat Steps 1–3 for each possible strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$. Then, we obtain the Nash equilibrium relative to the computed matrix of payoffs.

Heterogeneously risk-averse bidders

- **Step 1: Jumping strategies.** Fix a sellers' strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$ and an arbitrary bidder i with a fixed risk-aversion level $r_i \in \mathcal{R}^+$. Then, for each I and k such that $1 \leq k \leq I - 1$, we assume that the strategy of bidder i 's opponents is determined by the rule of Proposition 5 relative to a cutoff determined precisely by bidder i 's risk-aversion level r_i .¹ Then, we make one thousand draws of signals and risk-aversion levels. For any realization of bidder i 's signal v_i , we compute the distribution of losing bids against the type $\theta_i = (v_i, r_i)$. With this distribution in hand, we compute the ex-ante expected utility of joining each auction, i.e., $\mathcal{U}_{\mathcal{S}'}(r_i, r_i \mid \eta)$ and $\mathcal{U}_{\mathcal{S}''}(r_i, r_i \mid \eta)$ (cf. equation (10)). These correspond to the expected utilities of a bidder i with risk aversion r_i from joining each auction when all other bidders “jump” exactly at the value r_i .
- **Step 2: Candidate equilibrium cutoff.** We repeat Step 1 for every $r_i \in \mathcal{R}^+$ until finding a type $\hat{r}$ of indifference, which is a type such that $\mathcal{U}_{\mathcal{S}'}(\hat{r}, \hat{r} \mid \eta) = \mathcal{U}_{\mathcal{S}''}(\hat{r}, \hat{r} \mid \eta)$. The value $\hat{r}$ constitutes the candidate equilibrium cutoff, i.e., $\hat{r} = r(\mathcal{S}', \mathcal{S}'')$.
- **Step 3: Verification of the proposed equilibrium.** To check the validity of $\hat{r}$ as an equilibrium candidate, we first compute Step 1 relative to the cutoff value $\hat{r}$. Then, we verify that the rule embedded in Proposition 5 is satisfied in an ϵ -neighborhood, for $\epsilon > 0$ sufficiently small. That is, we check that for any $r_i \neq \hat{r}$,

$$|\mathcal{U}_{\mathcal{S}'}(r_i, \hat{r} \mid \eta) - \mathcal{U}_{\mathcal{S}''}(r_i, \hat{r} \mid \eta)| < \epsilon.$$

- **Step 4: Sellers' expected revenue.** For a fixed sellers' strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$, we

¹In the case of a strategy profile where both sellers choose the same family—i.e., $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \langle \mathcal{S}, \mathcal{S} \rangle$ for some $\mathcal{S}$ —we define a focal point so that all bidders $j \neq i$ with risk aversion $r_j \leq r_i$ join the $\mathcal{S}_1$ -auction and all bidders with risk aversion $r_j > r_i$ join the $\mathcal{S}_2$ -auction.

distribute bidders across auctions following the rule of Proposition 5, relative to the cutoff $r(\mathcal{S}', \mathcal{S}'')$ found in Step 2. We make one million draws of signals and risk-aversion levels and compute each seller's expected revenue for the corresponding family. Sellers' expected revenue is computed as the average revenue over all iterations.

- **Step 5: Sellers' Nash equilibrium.** We repeat Step 4 for each possible strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$. Then, we obtain the Nash equilibrium relative to the computed matrix of payoffs.

II.2 Example

Assume the environment in Section 6. There are two sellers that compete for $I \geq 2$ bidders. The signal of bidder i , denoted by v_i , represents the net present value (NPV) of the project. Signals are independently distributed across bidders according to a uniform distribution with support $\mathcal{V} = [0, 2]$. The upfront investment is set at $X = 0.8$. If the project is undertaken by a bidder with signal v_i , it yields a stochastic revenue z_i , drawn from a log-normal distribution with mean $X + v_i$ and volatility $(X + 1)/2 = 0.9$.

To secure payment from the winning bidder, sellers can use one of three standard families of securities to organize their auctions. Thus, $\mathcal{S} = \{\mathcal{S}^d, \mathcal{S}^e, \mathcal{S}^{co}\}$. Hence, we consider a larger set of securities relative to the simulation in [Gorbenko and Malenko \(2011\)](#), which makes the characterization of equilibria more challenging.

Example 1: Homogeneously risk-averse agents Bidders are homogeneously risk-averse and seek to maximize their expected utility. Bidders' Bernoulli utility function is given by $u(m, r) = (1 - e^{-rm})/r$, where m denotes monetary payoffs and r denotes the common level of absolute risk aversion. To make the contrast with the risk-neutral case more transparent, we use degenerate distributions that make the risk-aversion parameter fixed and commonly known. The description of the algorithm used to simulate the game is presented in the Appendix.²

²It is worth highlighting that, unlike in [Gorbenko and Malenko \(2011\)](#), in which any equilibrium can be characterized solely by the probability with which a bidder enters a given auction, our algorithm must compute the interim expected utility of each bidder for each possible pair of families in $\mathcal{S}$, making the implementation more challenging.

Entry probabilities We run our algorithm for different numbers of bidders to analyze how the market size affects bidders' entry probabilities for any given sellers' family choices.

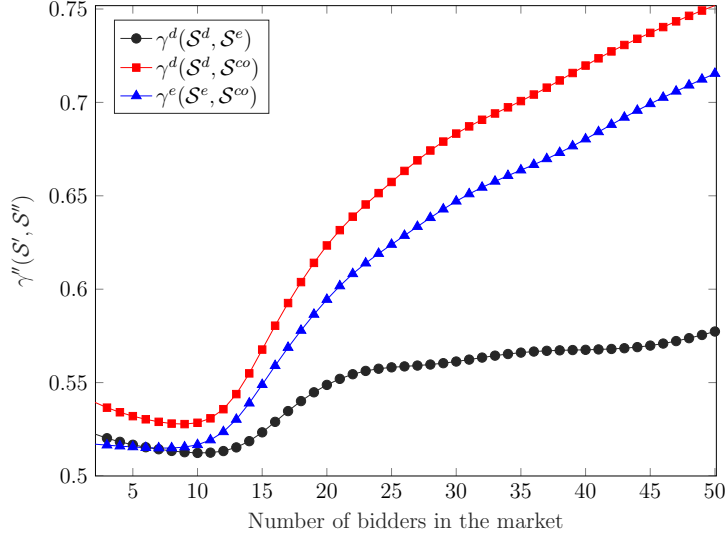


Figure II.1: Bidders' entry probability as a function of the market size when they are risk averse.

Figure II.1 shows that when the number of bidders is lower than 13, for any seller's profile, the probability of entering the auction under the flatter family is always between 0.5 and 0.55. This implies that when the number of bidders is low, the competition effect dominates because bidders try to balance the opposition they face in both auctions. The probability of entering the flatter auction increases with the number of bidders, although not monotonically. This reflects the fact that, as the competition gets sufficiently high in both auctions, bidders' entry probabilities are driven by their relative valuation of the insurance and the extraction effects. Furthermore, the gap between the entry probability to the flatter auction increases in the difference of steepness between the two families chosen by the seller. Thus, it is maximal when one seller chooses a call option and the other chooses debt.

Seller's expected revenues We now inspect the sellers' choice of security when the market is large.³ Similar to our determination of entry probabilities, we distinguish between the cases when bidders are risk-neutral and when they are risk-averse.

Figure II.2 depicts the expected revenue and the entry probability corresponding to a seller

³For illustration purposes, we depict the case when there are 50 bidders.

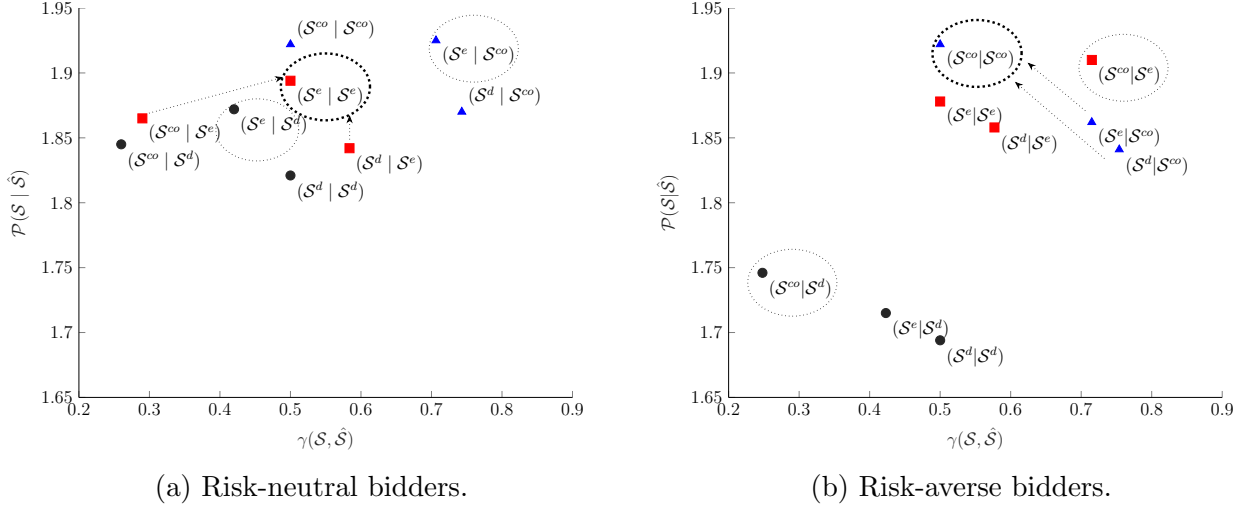


Figure II.2: Seller's expected revenue and bidder's entry probability.

that chooses family S' , conditional on the opposing seller choosing family S'' . The dashed circles depict those situations in which the chosen family is a best response to the opponent's choice. In particular, the circles in thick black correspond to the Nash equilibrium. Furthermore, arrows indicate the direction of profitable deviations. When bidders are risk-neutral—i.e., Panel II.2a—the equilibrium is given by both sellers choosing equity. Nonetheless, as soon as bidders become risk-averse—i.e., Panel II.2b—the insurance provided by steeper securities reinforces sellers' incentives to choose a steeper security (due to the greater surplus extraction), and, thus, the equilibrium transitions to one in which both sellers choose the call-option family S^{co} , which is the steepest family.

Interestingly, the equilibrium under risk-averse bidders yields greater expected revenue for sellers, even though bidders shade their bids relative to the risk-neutral benchmark. The reason is that the insurance provided by steeper securities makes risk-averse bidders more likely to enter auctions run under steeper families and, consequently, enables sellers to coordinate on the steepest family. Furthermore, the more aggressive bidding observed under risk neutrality partially offsets the loss in surplus extraction generated by the use of flatter securities.

Example 2: Heterogeneously risk-averse agents We now consider heterogeneously risk-averse bidders whose risk-aversion parameter follows a uniform distribution with support $[0.5, 1.5]$. We compute bidders' equilibrium in cutoff strategies following a novel algorithm presented in the Appendix.

	$\mathcal{S}^d, \mathcal{S}^d$	$\mathcal{S}^e, \mathcal{S}^e$	$\mathcal{S}^d, \mathcal{S}^{co}$	Median[r]	$\mathcal{S}^{co}, \mathcal{S}^{co}$	$\mathcal{S}^d, \mathcal{S}^e$	$\mathcal{S}^e, \mathcal{S}^{co}$
$r(\cdot, \cdot)$	0.63	0.76	0.78	1	1.14	1.24	1.35

Table II.1: Bidders' pure-strategy cutoff equilibrium.

Cutoff values In Table II.1, we depict how bidders separate in equilibrium following a cutoff pure strategy. First, when sellers choose the same family of securities, bidders separate in an unbalanced way, with a skewness towards the right tail of the distribution—i.e., the mass of more-risk-averse bidders that join a given auction is greater. The reason is that both auctions offer the same insurance, but less-risk-averse bidders are more competitive; thus, the auction in which they participate must feature lower participation to balance the payoffs. This contrasts with the outcome in the homogeneous case, in which bidders join both auctions in an even fashion whenever both sellers choose the same family. Second, when sellers continue to choose the same family, but are now steeper, the equilibrium cutoff moves to the right. This follows from the result in Proposition 2, since, as the steepness increases, more-risk-averse bidders become relatively more aggressive, and so, the mass of competitors must decrease to balance payoffs. Finally, when sellers choose different families, there is always an equilibrium in which bidders separate themselves in both auctions following the rule in Proposition 5: the more-risk-averse bidders join the steeper auction, while the less-risk-averse join the flatter auction. Furthermore, notice that, as the difference in steepness between the families chosen by sellers enlarges, the proportion of the market taken by the seller who chooses the steeper family increases.

Sellers' expected revenues We now depict sellers' expected revenue for all possible combinations of securities in $\mathcal{S}$ when the market is large.⁴ In this case, the unique Bayes-Nash equilibrium corresponds to one in which one seller chooses a call-option family and the other chooses an equity family. Meanwhile, bidders endogenously separate themselves, so, that, those with a risk-aversion parameter lower than $r(\mathcal{S}^e, \mathcal{S}^{co})$ join the equity auction, while those with a risk aversion greater than $r(\mathcal{S}^e, \mathcal{S}^{co})$ join the call-option auction. The threshold $r(\mathcal{S}^e, \mathcal{S}^{co})$ is computed according to Proposition 5.

⁴The necessity of a large market is due only to the fact that, for tractability in the simulation, we impose a uniform distribution of the signals. The same results can be obtained in a small market under asymmetric distributions, provided that the sufficient conditions in Proposition 6 are met.

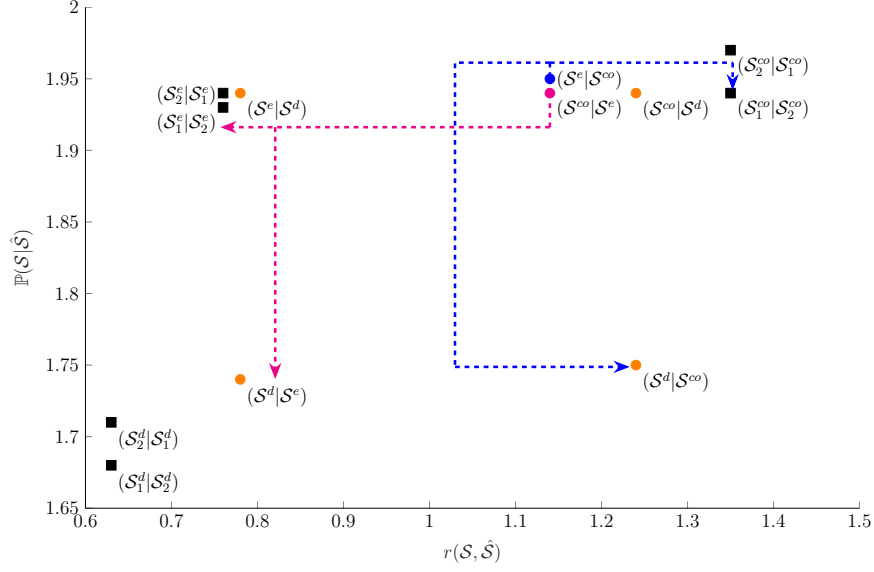


Figure II.3: Sellers' expected revenue for different security-design choices when bidders are heterogeneously risk-averse.

It is worth noticing that the asymmetry of bidders' risk aversion induces asymmetry in the sellers' payoffs even when both sellers choose the same family. This, in turn, affects the possible profitable deviations of each seller, as depicted in Figure II.3. In the equilibrium above, we assume that if the seller who chooses equity deviates from the call option, he will obtain the lowest payoff among the ones available. This can be justified by assuming a focal equilibrium in line with the literature that compares behaviors relative to an "industry benchmark." Interestingly, in this equilibrium, the seller choosing the steepest family obtains lower expected revenue.

References

Gorbenko, A. S. and A. Malenko (2011). Competition among Sellers in Securities Auctions. *American Economic Review* 101(5), 1806–41.