Supplementary Information to the Paper "Transient Forward Harmonic Adjoint Sensitivity Analysis"

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1 Derivation Steps

We recall the integral to calculate the transient adjoint sensitivity over a time interval ($t_0 = 0$ stated as a variable for the derivation):

$$\int_{t_0}^{t_\mathrm{m}} \frac{\mathrm{d}U}{\mathrm{d}p}(t)\ \mathrm{d}t = \int_{t_0}^{t_\mathrm{m}} \boldsymbol{\lambda}^\mathsf{T}(t, t_\mathrm{m}) \left(\frac{\mathrm{d}\boldsymbol{J}_\mathrm{C}(t)}{\mathrm{d}p} \dot{\boldsymbol{x}}(t) + \frac{\mathrm{d}\boldsymbol{J}_\mathrm{G}(t)}{\mathrm{d}p} \boldsymbol{x}(t) \right)\ \mathrm{d}t. \tag{1}$$

To obtain the sensitivity for the voltage at a certain time step, the generalized version of Leibniz integral rule can be applied to the term at the right hand side [1]. It states that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t_\mathrm{m}} \int_{t_0}^{t_\mathrm{m}} G(\boldsymbol{x}, \dot{\boldsymbol{x}}, t, t_\mathrm{m})\ \mathrm{d}t \\
&= G(\boldsymbol{x}, \dot{\boldsymbol{x}}, t_\mathrm{m}) \frac{\mathrm{d}t_\mathrm{m}}{\mathrm{d}t_\mathrm{m}} - G(\boldsymbol{x}, \dot{\boldsymbol{x}}, t_0) \frac{\mathrm{d}t_0}{\mathrm{d}t_\mathrm{m}} + \int_{t_0}^{t_\mathrm{m}} \frac{\partial G(\boldsymbol{x}, \dot{\boldsymbol{x}}, t, t_\mathrm{m})}{\partial t_\mathrm{m}}\ \mathrm{d}t. \\
&= G(\boldsymbol{x}, \dot{\boldsymbol{x}}, t_\mathrm{m}) - G(\boldsymbol{x}, \dot{\boldsymbol{x}}, t_0) \frac{\mathrm{d}t_0}{\mathrm{d}t_\mathrm{m}} + \int_{t_0}^{t_\mathrm{m}} \frac{\partial G(\boldsymbol{x}, \dot{\boldsymbol{x}}, t, t_\mathrm{m})}{\partial t_\mathrm{m}}\ \mathrm{d}t.
\end{aligned} \tag{2}$$

In equation (1) $\mathrm{d}U/\mathrm{d}p$ does not depend on t_m. Resultingly, the integral is eliminated by the derivation without application of Leibniz integral rule to the left hand side of Eq. (1). Applied to equation (1), this yields

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}p}(t = t_\mathrm{m}) = &\boldsymbol{\lambda}^\mathsf{T}(t_\mathrm{m}, t_\mathrm{m}) \left(\frac{\mathrm{d}\boldsymbol{J}_\mathrm{C}(t_\mathrm{m})}{\mathrm{d}p} \dot{\boldsymbol{x}}(t_\mathrm{m}) + \frac{\mathrm{d}\boldsymbol{J}_\mathrm{G}(t_\mathrm{m})}{\mathrm{d}p} \boldsymbol{x}(t_\mathrm{m}) \right) \\
&+ \boldsymbol{\lambda}^\mathsf{T}(t_0, t_\mathrm{m}) \left(\frac{\mathrm{d}\boldsymbol{J}_\mathrm{C}(t_0)}{\mathrm{d}p} \dot{\boldsymbol{x}}(t_0) + \frac{\mathrm{d}\boldsymbol{J}_\mathrm{G}(t_0)}{\mathrm{d}p} \boldsymbol{x}(t_0) \right) \frac{\mathrm{d}t_0}{\mathrm{d}t_\mathrm{m}} \\
&+ \int_{t_0}^{t_\mathrm{m}} \frac{\partial \boldsymbol{\lambda}^\mathsf{T}(t, t_\mathrm{m})}{\partial t_\mathrm{m}} \left(\frac{\mathrm{d}\boldsymbol{J}_\mathrm{C}(t)}{\mathrm{d}p} \dot{\boldsymbol{x}}(t) + \frac{\mathrm{d}\boldsymbol{J}_\mathrm{G}(t)}{\mathrm{d}p} \boldsymbol{x}(t) \right)\ \mathrm{d}t.
\end{aligned} \tag{3}$$

Because the adjoint solution $\boldsymbol{\lambda}$ is 0 at time t_m per definition, the first term on the right hand side is 0. As $t_0 = 0$ was stated, The second term on the right hand side vanishes. The sensitivity for the QoI at time instance t_m is therefore given by

$$\frac{\mathrm{d}U}{\mathrm{d}p}(t = t_\mathrm{m}) = \int_{t_0}^{t_\mathrm{m}} \frac{\partial \boldsymbol{\lambda}^\mathsf{T}(t, t_\mathrm{m})}{\partial t_\mathrm{m}} \left(\frac{\mathrm{d}\boldsymbol{J}_\mathrm{C}(t)}{\mathrm{d}p} \dot{\boldsymbol{x}}(t) + \frac{\mathrm{d}\boldsymbol{J}_\mathrm{G}(t)}{\mathrm{d}p} \boldsymbol{x}(t) \right)\ \mathrm{d}t. \tag{4}$$

References

[1] Flanders, H.: Differentiation under the integral sign. The American Mathematical Monthly **80**(6), 615–627 (1973). https://doi.org/10.2307/2319163