

Supplementary Mathematica notebook:

Bounds for survival probabilities in supercritical Galton-Watson Processes and applications to population genetics

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1. Important definitions

1.1. Generating functions

The notation is essentially the same as in the ms

```
In[*]:= 
$$\begin{aligned}\varphi_{\text{FL}}[p\_ , r\_ ] [x\_ ] &:= \frac{r + x (1 - p - r)}{1 - p x}; \\ \varphi_{\text{Poi}}[m\_ ] [x\_ ] &:= e^{-m (1-x)}; \\ \varphi_{\text{Bin}}[n\_ , p\_ ] [x\_ ] &:= (1 - p + p x)^n; \\ \varphi_{\text{NB}}[r\_ , p\_ ] [x\_ ] &:= \frac{p^r}{(1 - (1 - p) x)^r}; \\ \varphi_{\text{F3}}[p0\_ , p2\_ , p3\_ ] [x\_ ] &:= p0 + (1 - p0 - p2 - p3) x + p2 x^2 + p3 x^3;\end{aligned}$$

```

```
In[*]:= 
$$\begin{aligned}\varphi_{\text{GP}}[\mu\_ , \theta\_ ] [x\_ ] &:= e^{-\mu (1-x)}; \\ \varphi_{\text{GP}}[\mu\_ , \theta] [x\_ ] &:= e^{-\mu (1-x)}; \\ \varphi_{\text{GP}}[\mu\_ , \lambda\_ ] [x\_ ] &:= \text{Exp}\left[-\mu \left(1 + \frac{\text{ProductLog}[-x \lambda \text{Exp}[-\lambda]]}{\lambda}\right)\right]\end{aligned}$$

```

In some publications the generating function φ_{GP} is incorrect (essentially, with μ and λ swapped); it is correct, for instance, in the book by Johnson and Kotz.

1.2. Definitions of ultimate extinction and survival probabilities, of $\gamma = \varphi'(\text{Pinf})$, and of mean and variance

```
In[*]:= PinfFL[p_, r_] :=  $\frac{r}{p}$ ;
gammaFL[p_, r_] :=  $\frac{1-p}{1-r}$ ;
meanFL[p_, r_] :=  $\frac{1-r}{1-p}$ ;
varFL[p_, r_] :=  $\frac{(1-r)(p+r)}{(1-p)^2}$ 
```

```
In[*]:= Simplify[Solve[ $\varphi\text{Poi}[m][x] == x$ , x]]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

```
Out[*]=  $\left\{ \left\{ x \rightarrow -\frac{\text{ProductLog}[-e^{-m}m]}{m} \right\} \right\}$ 
```

```
In[*]:= PinfPoi[m_] :=  $-\frac{\text{ProductLog}[-e^{-m}m]}{m}$ ;
gammaPoi[m_] :=  $-\text{ProductLog}[-e^{-m}m]$ ;
SinfPoi[m_] :=  $1 - \text{PinfPoi}[m]$ ;
```

```
In[*]:= PinfF3[p0_, p2_, p3_] :=  $\frac{\sqrt{4 p_0 p_3 + (p_2 + p_3)^2} - (p_2 + p_3)}{2 p_3}$ ;
gammaF3[p0_, p2_, p3_] :=  $1 - \frac{(p_2 + 3 p_3) \sqrt{4 p_0 p_3 + (p_2 + p_3)^2} - 4 p_0 p_3 - (p_2 + p_3)^2}{2 p_3}$ ;
meanF3[p0_, p2_, p3_] :=  $1 - p_0 + p_2 + 2 p_3$ ;
varF3[p0_, p2_, p3_] :=  $-p_0^2 + p_2 + 4 p_3 - (p_2 + 2 p_3)^2 + p_0 (1 + 2 p_2 + 4 p_3)$ ;
```

In the following cases, Pinf needs to be calculated numerically:

```
In[*]:= PinfBin[n_, p_] := x /. FindRoot[ $\varphi\text{Bin}[n, p][x] == x$ , {x, 0.5}];
gammaBin[n_, p_] :=  $\frac{n p \text{PinfBin}[n, p]}{(1 - p + p \text{PinfBin}[n, p])}$ ;
meanBin[n_, p_] :=  $n p$ ;
varBin[n_, p_] :=  $n p (1 - p)$ ;
SinfBin[n_, p_] :=  $1 - \text{PinfBin}[n, p]$ 
```

```
In[*]:= PinfNB[r_, p_] := x /. FindRoot[φNB[r, p][x] == x, {x, 0.5}];
gammaNB[r_, p_] := -r (-1 + p) p^r (1 + (-1 + p) PinfNB[r, p])^{-1-r};
meanNB[r_, p_] := r \frac{1-p}{p};
varNB[r_, p_] := r \frac{1-p}{p^2};
SinfNB[r_, p_] := 1 - PinfNB[r, p]
```

```
In[*]:= PinfGP[μ_, λ_] := x /. FindRoot[φGP[μ, λ][x] == x, {x, 0.5}];
gammaGP[μ_, 0] := gammaPoi[μ];
gammaGP[μ_, λ_] := -\frac{e^{-\mu - \frac{\mu \text{ProductLog}[-e^{-\lambda} \text{PinfGP}[\mu, \lambda] \lambda]}{\lambda}} \mu \text{ProductLog}[-e^{-\lambda} \text{PinfGP}[\mu, \lambda] \lambda]}{\text{PinfGP}[\mu, \lambda] \lambda (1 + \text{ProductLog}[-e^{-\lambda} \text{PinfGP}[\mu, \lambda] \lambda])};
meanGP[μ_, λ_] := \frac{\mu}{1-\lambda};
varGP[μ_, λ_] := \frac{\mu}{(1-\lambda)^3};
SinfGP[μ_, λ_] := 1 - PinfGP[μ, λ];
```

1.3. The (prospective) bounding fractional linear generating functions

First determine the parameters π and ρ :

```
In[*]:= Simplify[Solve[PinfFL[p, r] == PinfPoi[m] && gammaFL[p, r] == gammaPoi[m], {p, r}]]
Out[*]:= \left\{ \left\{ p \rightarrow \frac{m (1 + \text{ProductLog}[-e^{-m} m])}{m - \text{ProductLog}[-e^{-m} m]^2}, r \rightarrow \frac{\text{ProductLog}[-e^{-m} m] (1 + \text{ProductLog}[-e^{-m} m])}{-m + \text{ProductLog}[-e^{-m} m]^2} \right\} \right\}
```

```
In[*]:= substprPoi[m_] :=
\left\{ p \rightarrow \frac{m (1 + \text{ProductLog}[-e^{-m} m])}{m - \text{ProductLog}[-e^{-m} m]^2}, r \rightarrow -\frac{\text{ProductLog}[-e^{-m} m] (1 + \text{ProductLog}[-e^{-m} m])}{m - \text{ProductLog}[-e^{-m} m]^2} \right\};
```

```
In[*]:= FullSimplify[Flatten[Solve[φFL[p, r][PinfF3[p0, p2, p3]] == PinfF3[p0, p2, p3] &&
(D[φFL[p, r][x], x] /. x → PinfF3[p0, p2, p3]) == gammaF3[p0, p2, p3], {p, r}]]]
```

 **Solve:** There may be values of the parameters for which some or all solutions are not valid.

```
Out[*]:= \left\{ p \rightarrow \frac{1}{-\frac{1}{2} + \frac{1}{\sqrt{4 p_0 p_3 + (p_2 + p_3)^2}} + \frac{-p_2 + \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}{2 p_3}}, r \rightarrow \frac{2 p_0 \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}{p_2 + p_3 + (1 + 2 p_0) \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}} \right\}
```

```
In[*]:= substprF3[p0_, p2_, p3_] :=
\left\{ p \rightarrow \frac{2 p_3 \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}{2 (1 + 2 p_0) p_3 - (p_2 + p_3) (-p_2 - p_3 + \sqrt{4 p_0 p_3 + (p_2 + p_3)^2})}, \right.
r \rightarrow \left. \frac{2 p_0 \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}{p_2 + p_3 + (1 + 2 p_0) \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}} \right\};
```

The following are computed numerically:

```
In[*]:= substprBin[nBin_, pBin_] :=
  Flatten[NSolve[ $\varphi$ FL[p, r][PinfBin[nBin, pBin]] == PinfBin[nBin, pBin] &&
    (D[ $\varphi$ FL[p, r][x], x] /. x  $\rightarrow$  PinfBin[nBin, pBin]) == gammaBin[nBin, pBin], {p, r}]]
```

```
In[*]:= substprNB[rNB_, pNB_] :=
  Flatten[NSolve[ $\varphi$ FL[p, r][PinfNB[rNB, pNB]] == PinfNB[rNB, pNB] &&
    (D[ $\varphi$ FL[p, r][x], x] /. x  $\rightarrow$  PinfNB[rNB, pNB]) == gammaNB[rNB, pNB], {p, r}]]
```

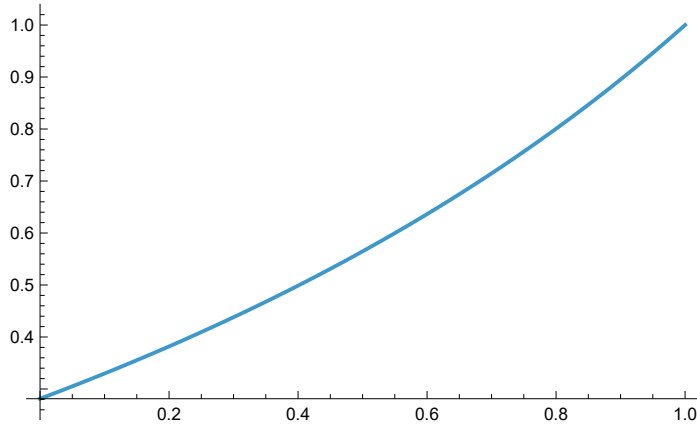
```
In[*]:= substprGP[ $\mu$ _,  $\lambda$ _] := Flatten[NSolve[ $\varphi$ FL[p, r][PinfGP[ $\mu$ ,  $\lambda$ ]] == PinfGP[ $\mu$ ,  $\lambda$ ] &&
  (D[ $\varphi$ FL[p, r][x], x] /. x  $\rightarrow$  PinfGP[ $\mu$ ,  $\lambda$ ]) == gammaGP[ $\mu$ ,  $\lambda$ ], {p, r}]]
```

Here are fractional linear generating functions that approximate the given one :

```
In[*]:=  $\varphi$ PoiFL[m_][x_] := ( $\varphi$ FL[p, r] /. substprPoi[m])[x];
 $\varphi$ BinFL[n_, pp_][x_] := ( $\varphi$ FL[p, r] /. substprBin[n, pp])[x];
 $\varphi$ NBFL[rr_, pp_][x_] := ( $\varphi$ FL[p, r] /. substprNB[rr, pp])[x];
 $\varphi$ F3FL[p0_, p2_, p3_][x_] := ( $\varphi$ FL[p, r] /. substprF3[p0, p2, p3])[x];
 $\varphi$ GPFL[ $\mu$ _,  $\lambda$ ][x_] := ( $\varphi$ FL[p, r] /. substprGP[ $\mu$ ,  $\lambda$ ])[x];
```

The functions φ where (p, r) is evaluated numerically can be plotted only after applying Evaluate to φ :

```
In[*]:= Plot[Evaluate[ $\varphi$ BinFL[10, 0.11][x]], {x, 0, 1}]
Out[*]=
```



2. Poisson distribution

We recall that the basic definitions are given in Section 1

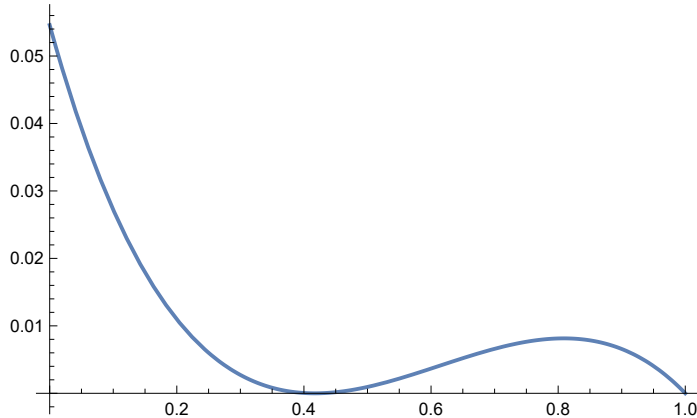
2.1. $f_{\text{Poi}}(x)$ and Figure 4.1

We define

```
In[*]:= fPoi[m_][x_] := Log[ $\varphi$ Poi[m][x]] - Log[ $\varphi$ PoiFL[m][x]]
```

```
In[ ]:= Plot[fPoi[1.5][x], {x, 0, 1}, PlotRange -> All]
```

```
Out[ ]:=
```



2.2. Some elements of the proof of Theorem 4.1

```
In[ ]:= Simplify[D[fPoi[m][x], {x, 1}] /. x -> 1]
```

```
Out[ ]:=
```

$$m + \frac{1}{\text{ProductLog}[-e^{-m}m]}$$

```
In[ ]:= Simplify[m + \frac{1}{\text{ProductLog}[-e^{-m}m]} - (m - \text{gammaPoi}[m]^{(-1)})]
```

```
Out[ ]:=
```

0

```
In[ ]:= Factor[D[fPoi[m][x], {x, 2}] /. x -> PinfPoi[m]]
```

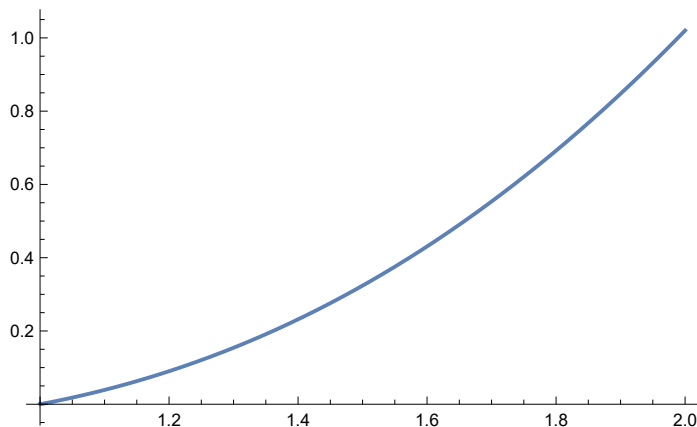
```
Out[ ]:=
```

$$-\frac{m^2 (2 - m + \text{ProductLog}[-e^{-m}m])}{m + \text{ProductLog}[-e^{-m}m]}$$

The following is not a proof, but illustrates what should be proved :

```
In[ ]:= Plot[-\frac{m^2 (2 - m + \text{ProductLog}[-e^{-m}m])}{m + \text{ProductLog}[-e^{-m}m]}, {m, 1.0000001, 2}]
```

```
Out[ ]:=
```



The following is the left hand side of (4.10) :

```
In[*]:= Factor[p + 2 r /. substprPoi[m]]
Out[*]=
```

$$\frac{(m - 2 \text{ProductLog}[-e^{-m} m]) (1 + \text{ProductLog}[-e^{-m} m])}{m - \text{ProductLog}[-e^{-m} m]^2}$$

This is from where it comes :

```
In[*]:= Simplify[D[-Log[φFL[p, r][x]], {x, 2}] /. x → PinfFL[p, r]]
Out[*]=
```

$$\frac{(-1 + p) p^2 (-1 + p + 2 r)}{(-1 + r)^2 r^2}$$

Finally, we confirm (4.12):

```
In[*]:= d[x_] := 3/4 (2 p x (1 - p - r) - (1 - p - r - r p))^2 + 1/4 (1 - p)^2 (1 - r)^2
Out[*]=
```

$$\text{Simplify}\left[D[\text{Log}[\varphi\text{FL}[p, r][x]], \{x, 3\}] - \frac{2(1-p)(1-r)d[x]}{(r(1-x) + (1-p)x)^3(1-px)^3}\right]$$

0

2.3. Comparison with Pollak's and Agresti's methods applied to the Poisson distribution

2.3.1. Pollak's (1971) method and application to the Poisson distribution

In our notation, Pollak's eq (5.2) yields (3.18) in the ms :

$$\frac{\text{Pinf} - P_n}{\gamma^n} \leq \frac{2(1-\gamma)\text{Pinf}}{2(1-\gamma) + (1-\gamma^n)\text{Pinf} \varphi''[\text{Pinf}]/\gamma} = : (\bar{d})^{(n)}$$

This leads to the following lower bound for P_n (where $\text{gam} = \gamma = \varphi'[\text{Pinf}]$ and $\text{scndder} = \varphi''[\text{Pinf}]$):

```
In[*]:= boundPnPollak[Pinf_, gam_, scndder_, n_] :=
  Pinf (1 - gam^n) (2 gam (1 - gam) + Pinf scndder) /
  (2 gam (1 - gam) + (1 - gam^n) Pinf scndder)
```

For the Poisson distribution we have

```
In[*]:= FullSimplify[
  {PinfPoi[m], gammaPoi[m], D[φPoi[m][x], {x, 2}] /. x → PinfPoi[m]}, Assumptions → m > 1]
Out[*]=
```

$$\left\{-\frac{\text{ProductLog}[-e^{-m} m]}{m}, -\text{ProductLog}[-e^{-m} m], -m \text{ProductLog}[-e^{-m} m]\right\}$$

Therefore, we obtain (4.13) in the ms :

```
In[*]:= FullSimplify[2(1-γ)Pinf / (2(1-γ) + (1-γ^n)Pinf scndder / γ) /. scndder → m γ /. m → γ / Pinf]
Out[*]=
```

$$\frac{2 \text{Pinf} (-1 + \gamma)}{-2 + \gamma + \gamma^{1+n}}$$

This equals (4.13) :

```
In[*]:= Simplify[
$$\frac{2 \text{Pinf} (-1 + \gamma)}{-2 + \gamma + \gamma^{1+n}} - \frac{\text{Pinf}}{1 + \frac{\gamma (1-\gamma^n)}{2 (1-\gamma)}}$$
]
Out[*]=
0
```

Pollak' s lower bound for Pn becomes $\frac{\text{PinfPoi} (1-\gamma\text{Poi}^n)}{1 - \frac{\gamma\text{Poi}^{n+1}}{2-\gamma\text{Poi}}}$:

```
In[*]:= FullSimplify[boundPnPollak[PinfPoi, γPoi, γPoi^2 / PinfPoi, n] - 
$$\frac{\text{PinfPoi} (1 - \gamma\text{Poi}^n)}{1 - \frac{\gamma\text{Poi}^{n+1}}{2-\gamma\text{Poi}}}$$
]
Out[*]=
0
```

2.3.2. Agresti' s (1974) method for deriving bounds for the supercritical case from bounds for the subcritical case, and application to the Poisson distribution

Agresti suggests to derive bounds for the supercritical case from bounds for the subcritical case by the relation

$\varphi_{\text{sub}}[x] = \varphi[x \text{Pinf}] / \text{Pinf}$, where φ_{sub} is the subcritical pgf:

```
In[*]:= φPoisub[m_][x_] := 
$$\frac{\varphi\text{Poi}[m][x \text{PinfPoi}[m]]}{\text{PinfPoi}[m]}$$

```

where $m > 1$ and $\varphi\text{Poisub}[m][x]$ is a subcritical generating function (our notation differs from his)

The starting point is his Theorem 2 (bounds for the Poisson distribution in the (sub)critical case; $\lambda = \text{mean} \leq 1$); the same bound was derived by Seneta 1967, p.474).

The relation between his λ and our m is the following . His λ is:

```
In[*]:= FullSimplify[D[φPoisub[m][x], x] /. x -> 1]
Out[*]=
-ProductLog[-e-m m]
```

This is precisely our gamma:

```
In[*]:= gammaPoi[m]
Out[*]=
-ProductLog[-e-m m]
```

Therefore, we define:

```
In[*]:= λAg[m_] := -ProductLog[-e-m m]
```

Agresti' s lower bound for Pn (already derived by Seneta 1967, p. 474):

We follow Agresti's definitions in his Theorem 2 for the subcritical case, where (b1,c1) and (b2,c2) are the parameters that he uses for the parameterization of φ_{FL} for the lower and upper bound, respectively (see his eq 2.1):

```
In[*]:= b1AgrestPoi[λ_] := λ  $\left(\frac{2}{\lambda + 2}\right)^2$ ;
c1AgrestPoi[λ_] :=  $\frac{\lambda}{\lambda + 2}$ ;
```

Agresti's upper bound for P_n :

```
In[*]:= b2AgrestPoi[λ_] :=  $\frac{(1 - \text{Exp}[-\lambda])^2}{\lambda}$ ;
c2AgrestPoi[λ_] :=  $\frac{\lambda + \text{Exp}[-\lambda] - 1}{\lambda}$ ;
```

Now we compute our parameter (π, ρ) for φ_{FL} , already in terms of m :

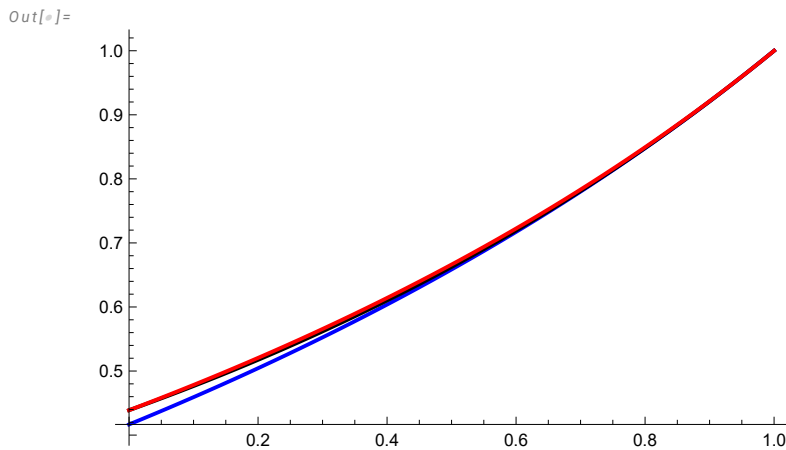
```
In[*]:= pFL1Poi[m_] := c1AgrestPoi[λAg[m]];
rFL1Poi[m_] :=  $\frac{1 - b1AgrestPoi[λAg[m]] - c1AgrestPoi[λAg[m]]}{1 - c1AgrestPoi[λAg[m]]}$ ;
pFL2Poi[m_] := c2AgrestPoi[λAg[m]];
rFL2Poi[m_] :=  $\frac{1 - b2AgrestPoi[λAg[m]] - c2AgrestPoi[λAg[m]]}{1 - c2AgrestPoi[λAg[m]]}$ ;
```

The Agresti's lower and upper bounds for P_n in the subcritical case are

```
In[*]:= φFLAg1Poi[m_][x_] := φFL[pFL1Poi[m], rFL1Poi[m]][x];
φFLAg2Poi[m_][x_] := φFL[pFL2Poi[m], rFL2Poi[m]][x];
```

The mean in the subcritical case is gammaPoi:

```
In[*]:= Plot[{φFLAg1Poi[1.2][x], φPoi[gammaPoi[1.2]][x], φFLAg2Poi[1.2][x]},
{x, 0, 1}, PlotStyle -> {Blue, Black, Red}]
```



From this, we obtain by the duality relation above, the lower and upper bounds for $\varphi_{Poi}[m][x]$:

```
In[*]:= φFLAg1PoiSup[m_][z_] := PinfPoi[m] × φFLAg1Poi[m][z / PinfPoi[m]];
φFLAg2PoiSup[m_][z_] := PinfPoi[m] × φFLAg2Poi[m][z / PinfPoi[m]];
```

In the ms , $P^{\infty} \varphi_{sub,L}(x / P^{\infty}) = \varphi_{FLAg1PoiSup}(x)$

$\varphi_{FLAg1PoiSup}[m][z]$ and $\varphi_{Poi}[m][z]$ have the same 1st derivative at 1 and the same 1st and 2nd

derivatives at $z = \text{PinfPoi}$ (for $\phi\text{FLAG2PoiSup}[m][z]$ and $\phi\text{Poi}[m][z]$ this is not the case for the 2nd derivative):

```
In[*]:= Simplify[D[φFLAG1PoiSup[m][z], {z, 1}] /. z → PinfPoi[m]]
Out[*]= -ProductLog[-e-m m]

In[*]:= FullSimplify[D[φPoi[m][z], {z, 1}] /. z → PinfPoi[m]]
Out[*]= -ProductLog[-e-m m]

In[*]:= FullSimplify[D[φFLAG1PoiSup[m][z], {z, 2}] /. z → PinfPoi[m], Assumptions → m > 1]
Out[*]= -m ProductLog[-e-m m]

In[*]:= FullSimplify[D[φPoi[m][z], {z, 2}] /. z → PinfPoi[m], Assumptions → m > 1]
Out[*]= -m ProductLog[-e-m m]

In[*]:= {PinfPoi[m], gammaPoi[m]}
Out[*]= {

$$-\frac{\text{ProductLog}[-e^{-m} m]}{m}, -\text{ProductLog}[-e^{-m} m] \}$$

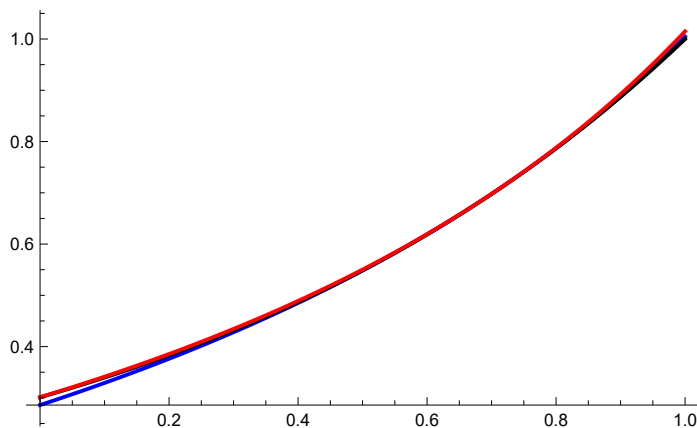

In[*]:= FullSimplify[D[φFLAG1Poi[m][z], {z, 1}] /. z → 1]
Out[*]= -ProductLog[-e-m m]

In[*]:= FullSimplify[D[φFLAG2PoiSup[m][z], {z, 1}] /. z → PinfPoi[m]]
Out[*]= -ProductLog[-e-m m]

In[*]:= FullSimplify[D[φFLAG2PoiSup[m][z], {z, 2}] /. z → PinfPoi[m], Assumptions → m > 1]
Out[*]= 2 m  $\left( -1 + \frac{\text{ProductLog}[-e^{-m} m]}{-1 + e^{\text{ProductLog}[-e^{-m} m]}} \right)$ 
```

The upper bound is red, the lower bound is blue:

```
In[*]:= Plot[{φFLAG1PoiSup[1.2][x], φPoi[1.2][x], φFLAG2PoiSup[1.2][x]},
{x, 0, 1}, PlotStyle → {Blue, Black, Red}]
Out[*]=
```



`In[ ]:=  $\varphi$ FLAg1PoiSup and  $\varphi$ FLAg2PoiSup are not generating functions because`

`In[ ]:= { $\varphi$ FLAg1PoiSup[1.2][1],  $\varphi$ FLAg2PoiSup[1.2][1]}`

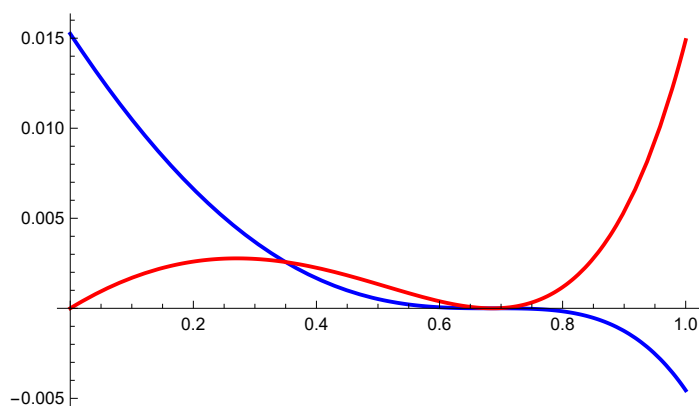
`Out[ ]:=`

`{1.00455, 1.01489}`

Plot differences :

`In[ ]:= Plot[{ $\varphi$ Poi[1.2][x] -  $\varphi$ FLAg1PoiSup[1.2][x],  $\varphi$ FLAg2PoiSup[1.2][x] -  $\varphi$ Poi[1.2][x]},`
`{x, 0, 1}, PlotStyle -> {Blue, Red}]`

`Out[ ]:=`

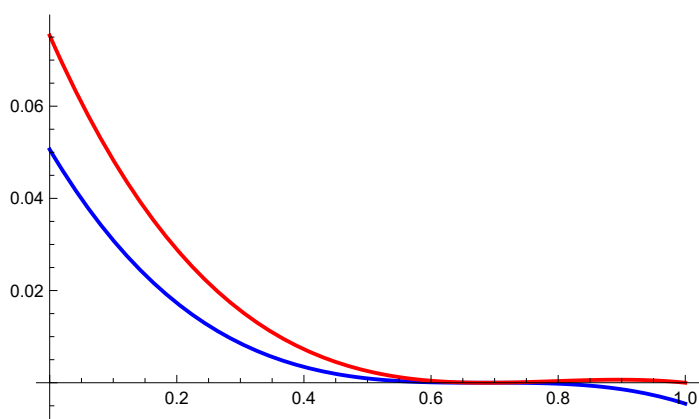


Obviously, φ FLAg1PoiSup is a lower bound only if $x \leq \text{Pinf}$

Comparison of relative errors of Agresti's lower bound (blue) with our lower bound (red) for φ Poi:

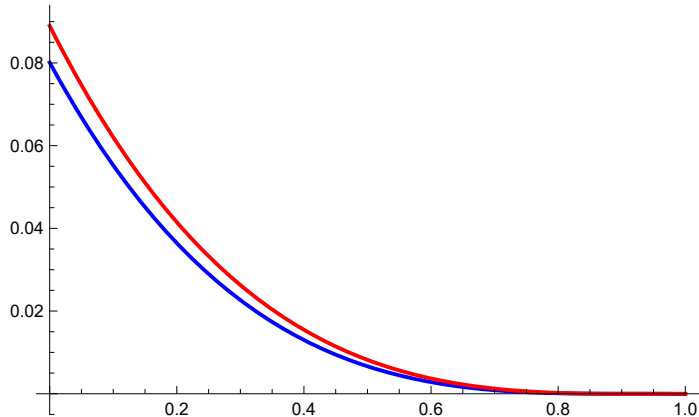
`In[ ]:= Plot[{( $\varphi$ Poi[1.2][x] -  $\varphi$ FLAg1PoiSup[1.2][x]) /  $\varphi$ Poi[1.2][x],`
`( $\varphi$ Poi[1.2][x] -  $\varphi$ PoiFL[1.2][x]) /  $\varphi$ Poi[1.2][x]}, {x, 0, 1}, PlotStyle -> {Blue, Red}]`

`Out[ ]:=`



```
In[*]:= Plot[{(φPoi[1.05][x] - φFLAG1PoiSup[1.05][x]) / φPoi[1.05][x],
              (φPoi[1.05][x] - φPoiFL[1.05][x]) / φPoi[1.05][x]}, {x, 0, 1}, PlotStyle -> {Blue, Red}]
```

```
Out[*]=
```



Numerical evaluation of the extinction probabilities up to generation n
(exact = φPoi , our bound = φPoiFL , Agresti' s lower bound = $\varphi\text{FLAG1PoiSup}$):

```
In[*]:= NestList[φPoi[1.2], 0, 20]
```

```
Out[*]=
```

```
{0, 0.301194, 0.43233, 0.506007, 0.552782, 0.584697, 0.607524,
 0.624396, 0.637166, 0.647005, 0.65469, 0.660755, 0.665582, 0.669448,
 0.672561, 0.675078, 0.67712, 0.678782, 0.680136, 0.681243, 0.682148}
```

```
In[*]:= NestList[φPoiFL[1.2], 0, 20]
```

```
Out[*]=
```

```
{0, 0.278503, 0.413114, 0.49128, 0.541565, 0.576075, 0.600823,
 0.619134, 0.632998, 0.643681, 0.652023, 0.658605, 0.663841, 0.668035,
 0.671411, 0.67414, 0.676354, 0.678155, 0.679623, 0.680822, 0.681802}
```

```
In[*]:= NestList[φFLAG1PoiSup[1.2], 0, 20]
```

```
Out[*]=
```

```
{0, 0.285948, 0.420447, 0.497474, 0.546616, 0.580158, 0.60412,
 0.6218, 0.63516, 0.645437, 0.653453, 0.659771, 0.664795, 0.668815,
 0.67205, 0.674664, 0.676784, 0.678508, 0.679913, 0.68106, 0.681998}
```

```
In[*]:= NestList[φPoi[1.02], 0, 20]
```

```
Out[*]=
```

```
{0, 0.360595, 0.520902, 0.613436, 0.674155, 0.717228, 0.749441,
 0.774475, 0.794505, 0.810905, 0.824583, 0.836168, 0.846108, 0.85473,
 0.86228, 0.868946, 0.874874, 0.88018, 0.884957, 0.889279, 0.893209}
```

```
In[*]:= NestList[φPoiFL[1.02], 0, 20]
```

```
Out[*]=
```

```
{0, 0.327453, 0.490877, 0.588819, 0.654056, 0.700618, 0.735513,
 0.762634, 0.784314, 0.802039, 0.816797, 0.829273, 0.839957, 0.849207,
 0.857293, 0.864419, 0.870746, 0.876401, 0.881483, 0.886075, 0.890243}
```

```
In[*]:= NestList[φFLAG1PoiSup[1.02], 0, 20]
```

```
Out[*]=
```

```
{0, 0.328833, 0.492411, 0.590274, 0.655389, 0.701828, 0.736614,
 0.763638, 0.785234, 0.802884, 0.817578, 0.829998, 0.840631, 0.849837,
 0.857883, 0.864973, 0.871268, 0.876892, 0.881948, 0.886515, 0.890661}
```

2.3.3. Comparison of the relative errors of S_n resulting from the different bounds

Our lower bound (3.7) for $P_n\text{Poi}$ is $\frac{\text{PinfPoi}(1-\gamma\text{Poi}^n)}{1-\gamma\text{Poi}^n \text{PinfPoi}}$

```
In[*]:= boundPnRB[Pinf_, gam_, n_] := 
$$\frac{\text{Pinf}(1 - \text{gam}^n)}{1 - \text{gam}^n \text{Pinf}}$$

```

The simple bound resulting from (3.15) in the ms is :

```
In[*]:= boundPnSimple[Pinf_, gam_, n_] := Pinf(1 - gam^n)
```

Pollak's and Agresti's lower bounds are indeed identical (as already noted by Agresti for the subcritical case):

```
In[*]:= FullSimplify[Table[Nest[φFLAG1PoiSup[m], 0, k] -  
    boundPnPollak[PinfPoi[m], gammaPoi[m], m gammaPoi[m], k], {k, 1, 10}]]  
Out[*]=  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

In principle, the iterates of $\phi\text{FLAG1PoiSup}[m]$ can be computed explicitly using eq (2.11) in the main text. However, we abstain from it because Agresti already noted equality with Pollak's bound.

We define the relative errors of the survival probabilities S_n for the three different methods (recall that $\phi\text{itPoi}[m, n][0] = P_n$, the n -fold iteration of ϕPoi):

```
In[*]:= relErrSnSimple[m_, nmax_] :=  
    (φitPoi[m, nmax][0] - Table[boundPnSimple[PinfPoi[m], gammaPoi[m], k],  
        {k, 1, nmax}]) / (1 - φitPoi[m, nmax][0]);  
relErrSnRB[m_, nmax_] :=  
    (φitPoi[m, nmax][0] - Table[boundPnRB[PinfPoi[m], gammaPoi[m], k], {k, 1, nmax}]) /  
    (1 - φitPoi[m, nmax][0]);  
relErrSnPollak[m_, nmax_] :=  
    (φitPoi[m, nmax][0] - Table[boundPnPollak[PinfPoi[m], gammaPoi[m],  
        m gammaPoi[m], k], {k, 1, nmax}]) / (1 - φitPoi[m, nmax][0]);
```

Data for Table 4.1 :

$m = 1.5$:

```
In[*]:= {relErrSnSimple[1.5, 200][{1, 5, 10, 20, 50, 100, 200}],  
    relErrSnRB[1.5, 200][{1, 5, 10, 20, 50, 100, 200}],  
    relErrSnPollak[1.5, 200][{1, 5, 10, 20, 50, 100, 200}]}  
Out[*]=  
{ {0.0862577, 0.0294636, 0.00306645, 0.000284699,  $2.22394 \times 10^{-11}$ , 0., 0.},  
    {0.0152574, 0.00346553, 0.000340607,  $3.14371 \times 10^{-6}$ ,  $2.45566 \times 10^{-12}$ , 0., 0.},  
    {0.00616883, 0.000977183, 0.0000900595,  $8.25513 \times 10^{-7}$ ,  $6.44823 \times 10^{-13}$ , 0., 0.} }
```

$m = 1.1$:

```
In[*]:= {relErrSnSimple[1.1, 200][{1, 5, 10, 20, 50, 100, 200}],
          relErrSnRB[1.1, 200][{1, 5, 10, 20, 50, 100, 200}],
          relErrSnPollak[1.1, 200][{1, 5, 10, 20, 50, 100, 200}]}

Out[*]= {{0.383188, 0.986854, 0.94154, 0.473267, 0.0282291, 0.000207065, 1.09912 × 10-8},
          {0.0420279, 0.037154, 0.020838, 0.00704969, 0.00034832, 2.5303 × 10-6, 1.34303 × 10-10},
          {0.0341988, 0.026153, 0.0132144, 0.00403586, 0.000187971, 1.36103 × 10-6, 7.224 × 10-11}}
```

m = 1.02:

```
In[*]:= {relErrSnSimple[1.02, 200][{1, 5, 10, 20, 50, 100, 200}],
          relErrSnRB[1.02, 200][{1, 5, 10, 20, 50, 100, 200}],
          relErrSnPollak[1.02, 200][{1, 5, 10, 20, 50, 100, 200}]}

Out[*]= {{0.534289, 2.214, 3.71055, 5.40513, 5.58914, 2.8017, 0.432376},
          {0.051832, 0.0587394, 0.0443885, 0.0277688, 0.0104987, 0.00316334, 0.000395488},
          {0.0496735, 0.0544576, 0.039935, 0.0238573, 0.00826737, 0.0023323, 0.000281982}}
```

3. Binomial distribution

We recall that the basic definitions are given in Section 1

3.1. We switch to the parameters n and z = PinfBin^{1/n}

The z here corresponds to ξ in the ms.

The we obtain:

```
In[*]:= Simplify[Solve[(1 - p + p z^n) == z, p]]

Out[*]= {{p -> \frac{-1 + z}{-1 + z^n}}}
```

$$\text{In[*]:= } pzn = \frac{1 - z}{1 - z^n};$$

```
In[*]:= Simplify[D[phiBin[n, pzn][x], {x, 1}] /. x -> z^n]

Out[*]= \frac{n (-1 + z) z^{-1+n}}{-1 + z^n}
```

Therefore, we have

$$\text{In[*]:= } \text{gammaBinz} = \frac{n (1 - z) z^n}{z (1 - z^n)};$$

We use (3.3) and (3.4) in the ms to calculate the parameters π and ρ for the fractional linear bound :

```
In[*]:= Simplify[Solve[phiFL[p, r][z^n] == z^n && (D[phiFL[p, r][x], x] /. x -> z^n) == gammaBinz, {p, r}]]

Out[*]= {{p -> z^{-n}, r -> 1}, {p -> \frac{z - n z^n + (-1 + n) z^{1+n}}{z - n z^{2n} - z^{1+n} + n z^{1+2n}}, r -> \frac{z^n (z - n z^n + (-1 + n) z^{1+n})}{z - n z^{2n} - z^{1+n} + n z^{1+2n}}}}
```

```
In[*]:= substprBinz = {p ->  $\frac{z (1 - z^n + n z^n) - n z^n}{z (1 - z^n + n z^{2n}) - n z^{2n}}$ , r ->  $z^n \frac{z (1 - z^n + n z^n) - n z^n}{z (1 - z^n + n z^{2n}) - n z^{2n}}$ };
```

3.2. Proof of Theorem 4.4 (Appendix A in the ms)

We define (see eq (A.1) in the ms):

```
In[*]:= xit = Sum[z^k, {k, 0, n - 1}];
v[n_, z_] [x_] :=  $\frac{(x - z^n) (1 - z)}{z (1 - z^n)}$ 
```

```
In[*]:= Simplify[v[n, z] [x] -  $\frac{x - z^n}{z \text{xit}}$ ]
```

```
Out[*]=
```

0

```
In[*]:= Simplify[{v[n, z] [0], v[n, z] [z^n], v[n, z] [1]}]
```

```
Out[*]=
```

$\left\{-\frac{(-1 + z) z^{-1+n}}{-1 + z^n}, 0, -1 + \frac{1}{z}\right\}$

```
In[*]:= Simplify[Solve[v[n, z] [x] == v, x]]
```

```
Out[*]=
```

$\left\{\left\{x \rightarrow z^n + \frac{v z (-1 + z^n)}{-1 + z}\right\}\right\}$

The following confirms (A.4) :

```
In[*]:= FullSimplify[ $\phi\text{Bin}\left[n, \frac{1 - z}{1 - z^n}\right] \left[z^n + \frac{v z (-1 + z^n)}{-1 + z}\right]$ ]
```

```
Out[*]=
```

$(1 + v) z^n$

The following confirms (A.5):

```
In[*]:= FullSimplify[ $\left(\phi\text{FL}[p, r] \left[z^n + \frac{v z (-1 + z^n)}{-1 + z}\right] /. \text{substprBinz}\right) -$   

 $z^n \frac{(1 - z) (1 - z^n) + v (n (1 - z) - z (1 - z^n))}{(1 - z) (1 - z^n) + v (n z^n (1 - z) - z (1 - z^n))}$ ]
```

```
Out[*]=
```

0

The following is fBin(v) in eq (A.6) :

```
In[*]:= fBin[n_, z_] [v_] :=  $(1 + v)^n ((1 - z) (1 - z^n) + v (n z^n (1 - z) - z (1 - z^n))) -$   

 $((1 - z) (1 - z^n) + v (n (1 - z) - z (1 - z^n)))$ 
```

The following is fBin in the first line of eq (A.7) :

```
In[*]:= fBinA[n_, z_] [v_] :=  $((((1 + v)^n - 1) (1 - (1 + v) z) (1 - z^n)) - n v (1 - z) (1 - (1 + v)^n z^n))$ 
```

```
In[*]:= Simplify[fBin[n, z][v] - fBinA[n, z][v]]
Out[*]=
0
```

Defining

```
In[*]:= fhatBin[n_, z_][v_] :=
  ((1 + v)^n - 1) * Sum[z^k, {k, 0, n - 1}] / (1 - z) - (n v / (1 - z)) * Sum[(1 + v)^k z^k, {k, 0, n - 1}]
```

as in (A.8), we obtain (A.7):

```
In[*]:= FullSimplify[fBinA[n, z][v] - (1 - z)^2 (1 - (1 + v) z) fhatBin[n, z][v]]
Out[*]=
0
```

Define the following coefficients of v^j occurring in (A.9b) and (A.9c):

```
In[*]:= coeff1serz[n_, j_] := Binomial[n, j + 1] (Sum[(k + 1) z^k, {k, 0, n - 1}] + n z^n / (1 - z));
coeff2serz[n_, j_] :=
  n (Sum[Binomial[j + k + 1, k] z^(j + k), {k, 0, n - j - 1}] + Binomial[n, j + 1] z^n * (1 / (1 - z)));
```

Then fhatBin in (A.9b) and (A.9c) is

```
In[*]:= fhatBinA[n_, z_][v_] :=
  v Sum[v^j coeff1serz[n, j], {j, 0, n - 1}] - v Sum[v^j coeff2serz[n, j], {j, 0, n - 1}]
```

The following confirms this:

```
In[*]:= Simplify[fhatBin[5, z][v] - fhatBinA[5, z][v]]
Out[*]=
0
```

```
In[*]:= Simplify[fhatBin[16, z][v] - fhatBinA[16, z][v]]
Out[*]=
0
```

Finally, we define the coefficient of $v^j z^k$ occurring in (A.10):

```
In[*]:= coeffvza[n_, j_, k_] := Binomial[n, j] (k + 1) - n Binomial[k + 1, j]
```

```
In[*]:= fhatBinB[n_, z_][v_] :=
  v Sum[v^j Sum[z^k coeffvza[n, j + 1, k], {k, 0, n - 1}], {j, 0, n - 1}];
```

```
In[*]:= Simplify[fhatBinB[15, z][v] - fhatBinfinal[15, z][v]]
Out[*]=
0
```

The following is (A.11)

```
In[*]:= coeffvz[n_, j_, k_] := Binomial[n, j + 2] (k + 1) - n Binomial[k + 1, j + 2];
fhatBinfinal[n_, z_][v_] :=
  v^2 Sum[v^j Sum[z^k coeffvz[n, j, k], {k, 0, n - 2}], {j, 0, n - 2}];
```

Check:

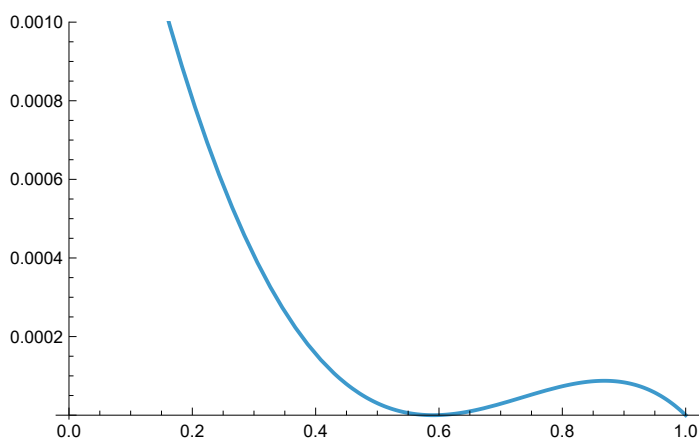
```
In[*]:= Simplify[fhatBin[15, z][v] - fhatBinfinal[15, z][v]]
Out[*]=
0
```

```
In[*]:= Factor[fhatBin[10, z][v] - fhatBinfinal[10, z][v]]
Out[*]=
0
```

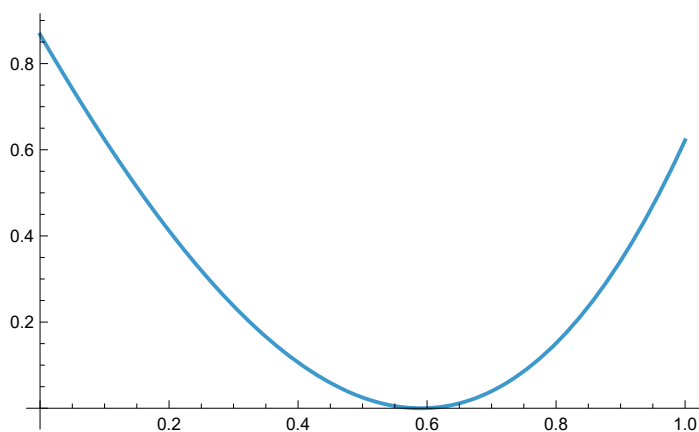
Graphs:

```
In[*]:= 0.9^5
Out[*]=
0.59049
```

```
In[*]:= Plot[fBin[5, 0.9][v[5, 0.9][x]], {x, 0, 1}, PlotRange → {-10^(-5), 10^(-3)}]
Out[*]=
```



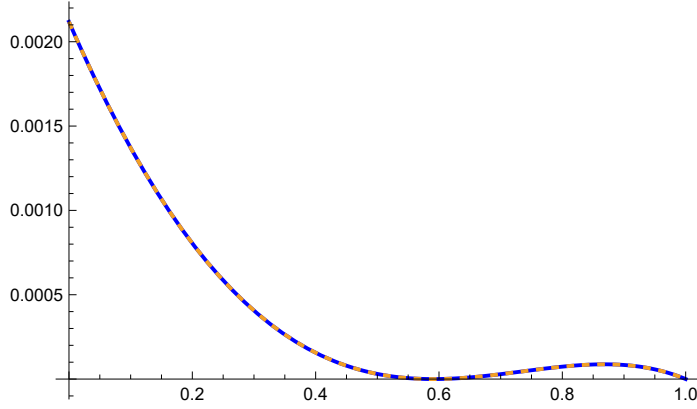
```
In[*]:= Plot[fhatBinfinal[5, 0.9][v[5, 0.9][x]], {x, 0, 1}, PlotRange → All]
Out[*]=
```



Ultimate numerical check :

```
In[*]:= Plot[{fBin[5, 0.9][v[5, 0.9][x]],
  (1 - 0.9)^2 (1 - (1 + v[5, 0.9][x]) 0.9) fhatBinfinal[5, 0.9][v[5, 0.9][x]]},
  {x, 0, 1}, PlotRange -> All, PlotStyle -> {Blue, Dashed}]
```

Out[*]=



The following yields (A.12):

```
In[*]:= coefffvzratio[n_, j_, k_] := 
$$\frac{\text{Binomial}[n, j + 2] (k + 1)}{n \text{Binomial}[k + 1, j + 2]}$$

coefffvzsimp[n_, j_, k_] := 
$$\frac{\text{Product}[n - 1 - i, \{i, 0, j\}]}{\text{Product}[k - i, \{i, 0, j\}]}$$

```

```
In[*]:= Simplify[
  Table[Table[coefffvzratio[10, j, k] - coefffvzsimp[10, j, k], {j, 0, k - 1}], {k, 1, 8}]]
```

Out[*]=

```
{ {}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0} }
```

Assume $v < 0$

We use $v > -1/n$. The following confirm (A.14) and (A.15):

```
In[*]:= FullSimplify[ 
$$\left( \text{coefffvz}[n, j, k] - \frac{1}{n} \text{coefffvz}[n, j + 1, k] \right) -$$


$$\left( \text{Binomial}[n, j + 2] (k + 1) \left( 1 - \frac{1}{n} \frac{n - j - 2}{j + 3} \right) - n \text{Binomial}[k + 1, j + 2] \left( 1 - \frac{1}{n} \frac{k - j - 1}{j + 3} \right) \right) ]$$

```

Out[*]=

0

```
In[*]:= fac2[n_, j_, k_] := 
$$\frac{n (j + 3) - n + j + 2}{n (j + 3) - k + j + 1}$$

```

```
In[*]:= FullSimplify[

$$\frac{\text{Binomial}[n, j + 2] (k + 1) \left( 1 - \frac{1}{n} \frac{n - j - 2}{j + 3} \right)}{n \text{Binomial}[k + 1, j + 2] \left( 1 - \frac{1}{n} \frac{k - j - 1}{j + 3} \right)} \Big/ (\text{coefffvzsimp}[n, j, k] \times \text{fac2}[n, j, k]) ]$$

```

Out[*]=

1

```
In[*]:= Series[ $\frac{1}{v^2}$  fhatBinfinal[7, z][v], {v, 0, 7}]
```

```
Out[*]= (21 + 35 z + 42 z^2 + 42 z^3 + 35 z^4 + 21 z^5) + (35 + 70 z + 98 z^2 + 112 z^3 + 105 z^4 + 70 z^5) v +  
(35 + 70 z + 105 z^2 + 133 z^3 + 140 z^4 + 105 z^5) v^2 + (21 + 42 z + 63 z^2 + 84 z^3 + 98 z^4 + 84 z^5) v^3 +  
(7 + 14 z + 21 z^2 + 28 z^3 + 35 z^4 + 35 z^5) v^4 + (1 + 2 z + 3 z^2 + 4 z^3 + 5 z^4 + 6 z^5) v^5 + O[v]^8
```

The following confirms (A.16) :

```
In[*]:= Simplify[Reduce[ $\frac{n-1}{k}$  fac2[n, j, k] > 1 && 1 ≤ k ≤ n - 2 && 0 ≤ j ≤ k - 1 && n ≥ 2],  
Assumptions → 1 ≤ k ≤ n - 2 && 0 ≤ j ≤ k - 1 && n ≥ 2]
```

```
Out[*]= True
```

4. Negative binomial distribution

We recall that the basic definitions are given in Section 1

4.1. We switch to the parameters r and $\xi = \text{PinfNB}^{1/r}$

The we obtain:

```
Simplify[Solve[φNB[r, p][ξ^r] == ξ^r, p]]
```

```
Out[*]= {{p →  $\frac{\xi (-1 + \xi^n)}{-1 + \xi^{1+n}}$ }}
```

```
In[*]:= pξr =  $\frac{\xi (\xi^r - 1)}{\xi^{1+r} - 1}$  ;
```

```
In[*]:= FullSimplify[D[φNB[r, pξr][x], {x, 1}] /. x → ξ^r, Assumptions → ξ > 0 && r > 1]
```

```
Out[*]=  $\frac{r (-1 + \xi) \xi^r}{-1 + \xi^r}$ 
```

Therefore, we have

```
In[*]:= gammaNBξ =  $\frac{r (1 - \xi) \xi^r}{1 - \xi^r}$  ;
```

We use (4.3) and (4.4) in the ms to calculate the parameters π and ρ for the fractional linear bound :

```
In[*]:= FullSimplify[  
Solve[φFL[πNB, ρNB][ξ^r] == ξ^r && (D[φFL[πNB, ρNB][x], x] /. x → ξ^r) == gammaNBξ,  
{πNB, ρNB}], Assumptions → ξ > 0 && r > 1]
```

```
Out[*]= {{πNB → ξ^-r, ρNB → 1}, {πNB →  $\frac{1 + (-1 + r (-1 + \xi)) \xi^r}{1 + \xi^r (-1 + r (-1 + \xi) \xi^r)}$ , ρNB →  $\frac{\xi^r + (-1 + r (-1 + \xi)) \xi^{2r}}{1 + \xi^r (-1 + r (-1 + \xi) \xi^r)}$ }}
```

```
In[*]:= substprBNξ = {πNB →  $\frac{1 - (1 + r(1 - \xi)) \xi^r}{1 - \xi^r(1 + r(1 - \xi) \xi^r)}$ , ρNB →  $\xi^r \frac{1 - (1 + r(1 - \xi)) \xi^r}{1 - \xi^r(1 + r(1 - \xi) \xi^r)}$ };
```

4.2. Proof of Theorem 4.6 (Appendix B in the ms)

As in (B.1) we define

```
In[*]:= ξit[r_, ξ_] := Sum[ξ^k, {k, 0, r - 1}];
yr[r_, ξ_] [x_] :=  $\frac{\xi^{r-x}}{\xiit[r, \xi]}$ 
```

We note that

```
In[*]:= Simplify[{yr[r, ξ] [0], yr[r, ξ] [ξ^r], yr[r, ξ] [1]}]
```

```
Out[*]= { $\frac{(-1 + \xi) \xi^r}{-1 + \xi^r}$ , 0, -1 + ξ}
```

and we obtain (B.3) and (B.4):

```
In[*]:= FullSimplify[(φFL[πNB, ρNB] [x] /. substprBNξ) -  $\left( \xi^r \frac{1 - x - r yr[r, \xi] [x]}{1 - x - r \xi^r yr[r, \xi] [x]} \right)$ ,
Assumptions → 0 < ξ < 1 && r > 1 && 0 < x < 1]
```

```
Out[*]= 0
```

```
In[*]:= FullSimplify[ $\frac{p \xi^r}{(1 - (1 - p \xi^r) x)} - \frac{\xi}{(1 + yr[r, \xi] [x])}$ ,
Assumptions → 0 < ξ < 1 && r > 1 && 0 < x < 1]
```

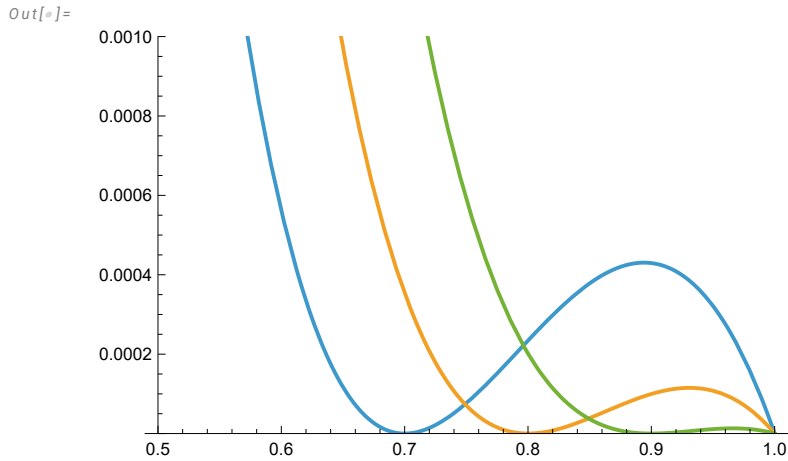
```
Out[*]= 0
```

The following is $\xi^r (\varphi_{FL}^{-1} - \varphi_{NB}^{-1})$, see eq (B.5) :

```
In[*]:= fzetNB[r_, ξ_] [x_] :=  $\frac{1 - x - r \xi^r yr[r, \xi] [x]}{1 - x - r yr[r, \xi] [x]} - (1 + yr[r, \xi] [x])^r$ ;
```

Choose PinfNB = 0.7, 0.8, 0.9 and r = 6 :

```
In[*]:= Plot[{fzetNB[6, 0.7^(1/6)][x], fzetNB[6, 0.8^(1/6)][x], fzetNB[6, 0.9^(1/6)][x]},
{x, 0.5, 1}, PlotRange -> {0, 0.001}]
```



```
In[*]:= Simplify[Solve[y == yr[r, ζ][x], x]]
```

Out[*]=

$$\left\{ \left\{ x \rightarrow \frac{y - y \zeta^r + (-1 + \zeta) \zeta^r}{-1 + \zeta} \right\} \right\}$$

```
In[*]:= substxy = {x -> ζ^r - y \frac{1 - ζ^r}{1 - ζ}};
```

Multiply fzetNB by $(1 - x - r \text{ yr}[r, \zeta][x])/(1 - \zeta)^2$ and substitute x by y (we know $1 - x - r y > 0$ (see below (B.10))):

```
In[*]:= Simplify[
- \frac{1 - x + r x - r x \zeta - \zeta^r - r \zeta^r + x \zeta^r + r \zeta^{1+r}}{-1 + \zeta} - \frac{1 - \zeta^r (1 + r (1 - \zeta)) - x (1 - \zeta^r - r (1 - \zeta))}{1 - \zeta^r}]
```

Out[*]=

0

```
In[*]:= Simplify[(1 - x - r yr[r, ζ][x]) /. substxy]
```

Out[*]=

$$1 - \zeta^r + \frac{y (-1 + r - r \zeta + \zeta^r)}{-1 + \zeta}$$

As in (B.7), we define

```
In[*]:= gzetNB[r_, ζ_] [y_] := fzetNB[r, ζ] \left[ \zeta^r - y \frac{(1 - \zeta^r)}{1 - \zeta} \right] \frac{(1 - \zeta) (1 - \zeta^r) + y (1 - \zeta^r - r (1 - \zeta))}{(1 - \zeta)^3}
```

Then

```
In[*]:= gzetNBA[r_, ζ_] [y_] := \frac{r y ((1 + y)^r - \zeta^r) (1 - \zeta) - ((1 + y)^r - 1) (1 - \zeta^r) (1 - \zeta + y)}{(1 - \zeta)^3}
```

and

```
In[*]:= Simplify[gzetNB[r, ξ][y] - gzetNBA[r, ξ][y]]
Out[*]=
0
```

The following is (B.8) :

```
In[*]:= gzetNBAA[r_, ξ_][y_] :=
  1 / (1 - ξ)^3 (y ((1 + y)^r - 1) (r (1 - ξ) - (1 - ξ^r)) - (1 - ξ) (1 - ξ^r) (((1 + y)^r - 1) - r y))
```

```
In[*]:= Simplify[gzetNB[r, ξ][y] - gzetNBAA[r, ξ][y]]
Out[*]=
0
```

Now use binomial expansion of $(1 + y)^r - 1$:

```
In[*]:= gzetNBB[r_, ξ_][y_] :=
  y^2 1 / (1 - ξ)^3 (Sum[Binoimial[r, j + 1] y^j, {j, 0, r - 1}] (r (1 - ξ) - (1 - ξ^r)) -
  (1 - ξ^r) (1 - ξ) Sum[Binoimial[r, j + 2] y^j, {j, 0, r - 2}])
```

```
In[*]:= Simplify[gzetNB[r, ξ][y] - gzetNBB[r, ξ][y]]
Out[*]=
0
```

The following is (B.9a) :

```
In[*]:= gzetNBC[r_, ξ_][y_] := y^2 1 / (1 - ξ)^2 (Sum[
  y^j (Binoimial[r, j + 1] (r - 1 - ξ^r / (1 - ξ)) - Binoimial[r, j + 2] (1 - ξ^r)), {j, 0, r - 1}])
```

```
In[*]:= Simplify[gzetNB[5, ξ][y] - gzetNBB[5, ξ][y]]
Out[*]=
0
```

And here is (B.9b) :

```
In[*]:= gzetNBC[r_, ξ_][y_] := y^2 1 / (1 - ξ)^2
  (Sum[y^j Binoimial[r, j + 1] ((r - 1 - ξ^r / (1 - ξ)) - (r - j - 1) / (j + 2) (1 - ξ^r)), {j, 0, r - 1}])
```

```
In[*]:= Simplify[gzetNBC[5, ξ][y] - gzetNB[5, ξ][y]]
Out[*]=
0
```

The following is $cg(r, \varphi, \zeta)$ used in (B.10) and defined (B.11) :

```
In[*]:= coeffgzetNBfinal[r_, j_, ξ_] := 
$$\frac{1}{2(j+2)} \left( \text{Sum}[\xi^k (k+1) (2r(1+j) - (2+j)k - 2), \{k, 0, r-2\}] + \frac{\xi^{(r-1)}}{(1-\xi)} r(r+1)j \right);$$

```

```
In[*]:= FullSimplify[
$$\frac{1}{(1-\xi)^2} \left( \left( r - \frac{1-\xi^r}{1-\xi} \right) - \frac{r-j-1}{j+2} (1-\xi^r) \right) - \text{coeffgzetNBfinal}[r, j, \xi]$$
]
```

```
Out[*]=
```

0

The following is (B.10) :

```
In[*]:= gzetNBfinal[r_, ξ_] [y_] :=  
y^2 (Sum[y^j Binomial[r, j+1] coeffgzetNBfinal[r, j, ξ], {j, 0, r-1}])
```

```
In[*]:= FullSimplify[gzetNB[r, ξ] [y] - gzetNBfinal[r, ξ] [y]]
```

```
Out[*]=
```

0

We check that coeffgzetNBfinal > 0 :

```
In[*]:= Simplify[Reduce[(2r(1+j) - (2+j)k - 2) ≥ 2], Assumptions → 0 ≤ j ≤ r-1 && 0 ≤ k ≤ r-2]
```

```
Out[*]=
```

True

This finishes the proof (if y>0).

4.3. Proof of $\varphi_{\text{NBFL}} \leq \varphi_{\text{NB}}$ for $0 \leq x \leq 1$ for $n = 2, 3, 4, 5, 6$

We use the transformation $y \rightarrow u - (1 - \zeta)$, where $1 - \zeta > u > 0$, i.e., $-(1 - \zeta) < y < 0$ or, equivalently, $\zeta^r < x < 1$ (cf. (B.2)).

We show that $\frac{1-\zeta}{y^2} \text{gzetNB}[r, \zeta][u - (1 - \zeta)] > 0$.

```
In[*]:= Simplify[Series[(1 - ξ) gzetNB[2, ξ] [y] / y^2 /. y → u - (1 - ξ), {u, 0, 2}]]
```

```
Out[*]=
```

$u + O[u]^3$

```
In[*]:= Simplify[Series[(1 - ξ) gzetNB[3, ξ] [y] / y^2 /. y → u - (1 - ξ), {u, 0, 3}]]
```

```
Out[*]=
```

$(1 + 4\xi + \xi^2)u + (2 + \xi)u^2 + O[u]^4$

```
In[*]:= Simplify[Series[(1 - ξ) gzetNB[4, ξ] [y] / y^2 /. y → u - (1 - ξ), {u, 0, 4}]]
```

```
Out[*]=
```

$(1 + 4\xi + 10\xi^2 + 4\xi^3 + \xi^4)u + 2(1 + 5\xi + 3\xi^2 + \xi^3)u^2 + (3 + 2\xi + \xi^2)u^3 + O[u]^5$

```
In[*]:= Simplify[Series[(1 - ξ) gzetNB[5, ξ] [y] / y^2 /. y → u - (1 - ξ), {u, 0, 5}]]
```

```
Out[*]=
```

$(1 + 4\xi + 10\xi^2 + 20\xi^3 + 10\xi^4 + 4\xi^5 + \xi^6)u + (2 + 10\xi + 30\xi^2 + 20\xi^3 + 10\xi^4 + 3\xi^5)u^2 + (3 + 18\xi + 13\xi^2 + 8\xi^3 + 3\xi^4)u^3 + (4 + 3\xi + 2\xi^2 + \xi^3)u^4 + O[u]^6$

```
In[*]:= Simplify[Series[(1 - ζ) gzetNB[6, ζ][y] / y^2 /. y → u - (1 - ζ), {u, 0, 6}]]
```

```
Out[*]=
```

$$\begin{aligned} & (1 + 4\zeta + 10\zeta^2 + 20\zeta^3 + 35\zeta^4 + 20\zeta^5 + 10\zeta^6 + 4\zeta^7 + \zeta^8) u + \\ & 2(1 + 5\zeta + 15\zeta^2 + 35\zeta^3 + 25\zeta^4 + 15\zeta^5 + 7\zeta^6 + 2\zeta^7) u^2 + \\ & 3(1 + 6\zeta + 21\zeta^2 + 16\zeta^3 + 11\zeta^4 + 6\zeta^5 + 2\zeta^6) u^3 + \\ & 2(2 + 14\zeta + 11\zeta^2 + 8\zeta^3 + 5\zeta^4 + 2\zeta^5) u^4 + (5 + 4\zeta + 3\zeta^2 + 2\zeta^3 + \zeta^4) u^5 + O[u]^7 \end{aligned}$$

5. Distributions with at most three offspring

We recall that the basic definitions are given in Section 1

5.1. Important properties of the function $f_{F3}(x)$ and verification of Remarks 4.8 and 4.9

The following is (4.34) :

```
In[*]:= ff3[x_] := φF3[p0, p2, p3][x] - φF3FL[p0, p2, p3][x]
```

Simple checks:

```
In[*]:= Simplify[ff3[PinfF3[p0, p2, p3]]]
```

```
Out[*]=
```

$$0$$

```
In[*]:= Simplify[D[ff3[x], x] /. x → PinfF3[p0, p2, p3]]
```

```
Out[*]=
```

$$0$$

Assumption: admissible region R in (4.33):

```
In[*]:= assumeR = p0 > 0 && p2 ≥ 0 && p3 > 0 && p0 + p2 + p3 ≤ 1 && p0 < p2 + 2 p3;
```

Define p0plus, p0r, p0gamma in (4.36), (4.37), (4.38):

```
In[*]:=
```

$$\begin{aligned} p0plus &= \frac{p3 - (p2 + p3)^2}{4 p3}; \\ p0r &= \frac{1}{2} - \frac{(p2 + p3) \left(p2 + p3 + \sqrt{8 p3 + (p2 + p3)^2} \right)}{8 p3}; \\ p0gam &= \frac{1}{2} - \frac{2 (p2 + p3)^2 + (p2 + 3 p3) \sqrt{8 p3 + (p2 + 3 p3)^2} - (p2 + 3 p3)^2}{8 p3}; \end{aligned}$$

Remark 4.8

(a)

In[]:= Simplify[D[fF3[x], {x, 2}] /. x → PinfF3[p0, p2, p3]]

Out[]:=
$$\frac{(-p_3 + 4 p_0 p_3 + (p_2 + p_3)^2) (p_2 + 3 p_3 - \sqrt{4 p_0 p_3 + (p_2 + p_3)^2})}{p_3}$$

The following confirms (8.13):

In[]:= Simplify[Reduce[$\frac{(-p_3 + 4 p_0 p_3 + (p_2 + p_3)^2) (p_2 + 3 p_3 - \sqrt{4 p_0 p_3 + (p_2 + p_3)^2})}{p_3} > 0 \&\&$
 $p_0 > 0 \&\& p_2 \geq 0 \&\& p_3 > 0$], Assumptions → assumeR && p0 > p0plus]

Out[]:= True

In[]:= Simplify[Reduce[$\frac{(-p_3 + 4 p_0 p_3 + (p_2 + p_3)^2) (p_2 + 3 p_3 - \sqrt{4 p_0 p_3 + (p_2 + p_3)^2})}{p_3} > 0 \&\&$
 $p_0 > 0 \&\& p_2 \geq 0 \&\& p_3 > 0$], Assumptions → assumeR && p0 ≤ p0plus]

Out[]:= False

(b)

In[]:= r /. substprF3[p0, p2, p3]

Out[]:=
$$\frac{2 p_0 \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}{p_2 + p_3 + (1 + 2 p_0) \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}$$

In[]:= Simplify[Solve[p0 == $\frac{2 p_0 \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}{p_2 + p_3 + (1 + 2 p_0) \sqrt{4 p_0 p_3 + (p_2 + p_3)^2}}$, p0]]

... Solve: There may be values of the parameters for which some or all solutions are not valid.

Out[]:=
$$\left\{ \left\{ p_0 \rightarrow -\frac{p_2^2 - 4 p_3 + 2 p_2 p_3 + p_3^2 + (p_2 + p_3) \sqrt{p_2^2 + 2 p_2 p_3 + p_3 (8 + p_3)}}{8 p_3} \right\}, \right.$$

$$\left. \left\{ p_0 \rightarrow -\frac{p_2^2 - 4 p_3 + 2 p_2 p_3 + p_3^2 - (p_2 + p_3) \sqrt{p_2^2 + 2 p_2 p_3 + p_3 (8 + p_3)}}{8 p_3} \right\}, \{ p_0 \rightarrow 0 \} \right\}$$

In[]:= p0r

Out[]:=
$$\frac{1}{2} - \frac{(p_2 + p_3) (p_2 + p_3 + \sqrt{8 p_3 + (p_2 + p_3)^2})}{8 p_3}$$

In[]:= Simplify[p0r - $\left(-\frac{p_2^2 - 4 p_3 + 2 p_2 p_3 + p_3^2 + (p_2 + p_3) \sqrt{p_2^2 + 2 p_2 p_3 + p_3 (8 + p_3)}}{8 p_3} \right)$]

Out[]:= 0

The other solution is not admissible (p0 = 0 isn't either)

```
In[*]:= Simplify[Reduce[1 > - $\frac{p^2 - 4 p^3 + 2 p^2 p^3 + p^3^2 - (p^2 + p^3) \sqrt{p^2 + 2 p^2 p^3 + p^3 (8 + p^3)}}{8 p^3}$  > 0 &&
- $\frac{p^2 - 4 p^3 + 2 p^2 p^3 + p^3^2 - (p^2 + p^3) \sqrt{p^2 + 2 p^2 p^3 + p^3 (8 + p^3)}}{8 p^3}$  < p2 + 2 p3], Assumptions →
assumer && - $\frac{p^2 - 4 p^3 + 2 p^2 p^3 + p^3^2 - (p^2 + p^3) \sqrt{p^2 + 2 p^2 p^3 + p^3 (8 + p^3)}}{8 p^3}$  + p2 + p3 ≤ 1]
```

Out[*]=

False

(c)

```
In[*]:= Simplify[Solve[gammaF3[p0, p2, p3] × meanF3[p0, p2, p3] == 1, p0]]
```

... Solve: There may be values of the parameters for which some or all solutions are not valid.

Out[*]=

$$\left\{ \left\{ p_0 \rightarrow p_2 + 2 p_3 \right\}, \left\{ p_0 \rightarrow -\frac{p^2 - 4 p^3 - 2 p^2 p^3 - 7 p^3^2 + (p^2 + 3 p^3) \sqrt{p^2 + 6 p^2 p^3 + p^3 (8 + 9 p^3)}}{8 p^3} \right\}, \right. \\ \left. \left\{ p_0 \rightarrow \frac{-p^2 + 4 p^3 + 2 p^2 p^3 + 7 p^3^2 + (p^2 + 3 p^3) \sqrt{p^2 + 6 p^2 p^3 + p^3 (8 + 9 p^3)}}{8 p^3} \right\} \right\}$$

```
In[*]:= Simplify[p0gam -  $\left( -\frac{p^2 - 4 p^3 - 2 p^2 p^3 - 7 p^3^2 + (p^2 + 3 p^3) \sqrt{p^2 + 6 p^2 p^3 + p^3 (8 + 9 p^3)}}{8 p^3} \right)$ ]
```

Out[*]=

0

The other two solutions are not admissible:

```
In[*]:= Simplify[Reduce[ $\frac{-p^2 + 4 p^3 + 2 p^2 p^3 + 7 p^3^2 + (p^2 + 3 p^3) \sqrt{p^2 + 6 p^2 p^3 + p^3 (8 + 9 p^3)}}{8 p^3}$  > 0 &&
 $\frac{-p^2 + 4 p^3 + 2 p^2 p^3 + 7 p^3^2 + (p^2 + 3 p^3) \sqrt{p^2 + 6 p^2 p^3 + p^3 (8 + 9 p^3)}}{8 p^3}$  <
p2 + 2 p3], Assumptions → assumer]
```

Out[*]=

False

Remark 4.9

(a) is obvious

(b)

```
In[*]:= Simplify[Reduce[p0r > 0, p2], Assumptions → assumer]
```

Out[*]=

$p_2 + p_3 < \sqrt{p_3}$

```
In[*]:= Simplify[Reduce[p0r > p0plus, p2], Assumptions → assumer]
```

Out[*]=

$p_2 + p_3 < \sqrt{p_3}$

```
In[*]:= Simplify[Reduce[p0r + p2 + p3 < 1 && 0 ≤ p2 < 1 && p2 + p3 < 1], Assumptions → assumeR]
```

```
Out[*]=  
True
```

```
In[*]:= Simplify[Reduce[p0r < p2 + 2 p3 && 0 ≤ p2 < 1 && p2 + p3 < 1], Assumptions → assumeR]
```

```
Out[*]=  
 $6 p_3 > 1 \mid \mid \sqrt{p_3 (4 + p_3)} < 2 p_2 + 5 p_3$   
(c)
```

```
In[*]:= Simplify[Reduce[0 < p0gam, p2], Assumptions → assumeR]
```

```
Out[*]=  
 $\sqrt{p_3 (4 + p_3)} > 2 p_2 + 3 p_3$ 
```

```
In[*]:= Simplify[Reduce[ $\sqrt{p_3 (4 + p_3)} - 3 p_3 > 0$ ], Assumptions → assumeR]
```

```
Out[*]=  
 $2 p_3 < 1$ 
```

```
In[*]:= Simplify[Reduce[p0gam < p0r], Assumptions → assumeR]
```

```
Out[*]=  
True  
(d)
```

```
In[*]:= Simplify[Reduce[p0gam < p0plus && p2 ≥ 0], Assumptions → assumeR]
```

```
Out[*]=  
 $9 p_3 > 1 \mid \mid p_2 + 3 p_3 > \sqrt{p_3}$ 
```

```
In[*]:= Simplify[Reduce[p0plus < p2 + 2 p3 && p2 ≥ 0], Assumptions → assumeR]
```

```
Out[*]=  
 $9 p_3 > 1 \mid \mid p_2 + 3 p_3 > \sqrt{p_3}$ 
```

```
In[*]:= Simplify[Reduce[p0gam < p2 + 2 p3 && p2 ≥ 0], Assumptions → assumeR]
```

```
Out[*]=  
 $9 p_3 > 1 \mid \mid p_2 + 3 p_3 > \sqrt{p_3}$ 
```

(e)

```
In[*]:= Simplify[Reduce[ $\sqrt{p_3} - 3 p_3 < \frac{1}{2} (\sqrt{p_3 (4 + p_3)} - 5 p_3) < \frac{1}{2} (\sqrt{p_3 (4 + p_3)} - 3 p_3) < \sqrt{p_3} - p_3$ ],  
Assumptions → assumeR]
```

```
Out[*]=  
True
```

5.2. Elements for the proof of Lemma 4.10

5.2.1 We show (C.2)

```
In[*]:= FullSimplify[Solve[ $\sqrt{4 p_0 p_3 + (p_2 + p_3)^2} = z$ , p0]]
```

⋯ Solve: There may be values of the parameters for which some or all solutions are not valid.

```
Out[*]=  
 $\left\{ \left\{ p_0 \rightarrow \frac{-(p_2 + p_3)^2 + z^2}{4 p_3} \right\} \right\}$ 
```

```
In[*]:= Factor[
  FullSimplify[ $\left( \text{gammaF3}[p0, p2, p3] \times \text{meanF3}[p0, p2, p3] - 1 /. p0 \rightarrow -\frac{p2^2 + 2 p2 p3 + p3^2 - z^2}{4 p3} \right)$ ,
  Assumptions  $\rightarrow z > 0$ ]
```

```
Out[*]=
```

$$-\frac{(p2 + 3 p3 - z)^2 (-2 p3 + p2 z + 3 p3 z + z^2)}{8 p3^2}$$

```
In[*]:= FullSimplify[ $-\frac{(p2 + 3 p3 - z)^2 (-2 p3 + p2 z + 3 p3 z + z^2)}{8 p3^2} /. z \rightarrow \sqrt{4 p0 p3 + (p2 + p3)^2}$ ]
```

```
Out[*]=
```

$$-\frac{1}{8 p3^2} \left(p2 + 3 p3 - \sqrt{4 p0 p3 + (p2 + p3)^2} \right)^2$$

$$\left(-2 p3 + 4 p0 p3 + (p2 + p3)^2 + p2 \sqrt{4 p0 p3 + (p2 + p3)^2} + 3 p3 \sqrt{4 p0 p3 + (p2 + p3)^2} \right)$$

Define:

```
In[*]:= facgam =  $-\left( -2 p3 + 4 p0 p3 + (p2 + p3)^2 + p2 \sqrt{4 p0 p3 + (p2 + p3)^2} + 3 p3 \sqrt{4 p0 p3 + (p2 + p3)^2} \right)$ ;
```

```
In[*]:= Simplify[Solve[facgam == 0, p0], Assumptions  $\rightarrow p2 \geq 0 \&\& p3 \geq 0$ ]
```

 **Solve:** There may be values of the parameters for which some or all solutions are not valid.

```
Out[*]=
```

$$\left\{ \left\{ p0 \rightarrow -\frac{p2^2 - 4 p3 - 2 p2 p3 - 7 p3^2 + (p2 + 3 p3) \sqrt{p2^2 + 6 p2 p3 + p3 (8 + 9 p3)}}{8 p3} \right\}, \right.$$

$$\left. \left\{ p0 \rightarrow \frac{-p2^2 + 4 p3 + 2 p2 p3 + 7 p3^2 + (p2 + 3 p3) \sqrt{p2^2 + 6 p2 p3 + p3 (8 + 9 p3)}}{8 p3} \right\} \right\}$$

Therefore, $p0r$ is indeed a zero of facgam (the only one in $\mathbb{R}$)

5.2.2 We check (C.3a) and (C.3b)

```
In[*]:= FullSimplify[Reduce[facgam > 0 && p2 ≥ 0 && p3 > 0], Assumptions  $\rightarrow \text{assumer} \&\& p0 < p0gam$ ]
```

```
Out[*]=
```

True

```
In[*]:= FullSimplify[Reduce[facgam > 0 && p0 ≥ p0gam && p2 ≥ 0 && p3 > 0], Assumptions  $\rightarrow \text{assumer}$ ]
```

```
Out[*]=
```

False

```
In[*]:= FullSimplify[Reduce[facgam < 0 && p2 ≥ 0 && p3 > 0], Assumptions  $\rightarrow \text{assumer} \&\& p0 > p0gam$ ]
```

```
Out[*]=
```

True

```
In[*]:= FullSimplify[Reduce[facgam < 0 && p0 ≤ p0gam && p2 ≥ 0 && p3 > 0], Assumptions  $\rightarrow \text{assumer}$ ]
```

```
Out[*]=
```

False

The rest of the proof of Lemma 4.10 is clear from Appendix C.

5.3. Elements for the proof of Theorem 4.11

We check (C.4), which was obtained by the substitution

```
In[ ]:= Simplify[Solve[PinfF3[p0, p2, p3] == PiF3, p0]]
```

Solve: There may be values of the parameters for which some or all solutions are not valid.

```
Out[ ]:= { {p0 -> PiF3 (p2 + p3 + p3 PiF3) } }
```

Define

```
In[ ]:= ff3Pinf[x_] := (1 - x) (PinfF3[p0, p2, p3] - x)^2
  ((-p3 + (p2 + p3 + 2 p3 PinfF3[p0, p2, p3]) (p2 + p3 + p3 PinfF3[p0, p2, p3] + p3 x)) /
  (1 + (p2 + p3 + 2 p3 PinfF3[p0, p2, p3]) (PinfF3[p0, p2, p3] - x)));
```

The following confirms (C.4):

```
In[ ]:= Simplify[ff3[x] - ff3Pinf[x]]
```

```
Out[ ]:= 0
```

Remark C.1 (b): we check ff3''' if p0=p0plus

```
In[ ]:= Simplify[D[ff3Pinf[x], {x, 3}] /. x -> PinfF3[p0, p2, p3] /. p0 -> p0plus]
```

```
Out[ ]:= 3 Sqrt[p3] (p2 - Sqrt[p3] + 3 p3)
```

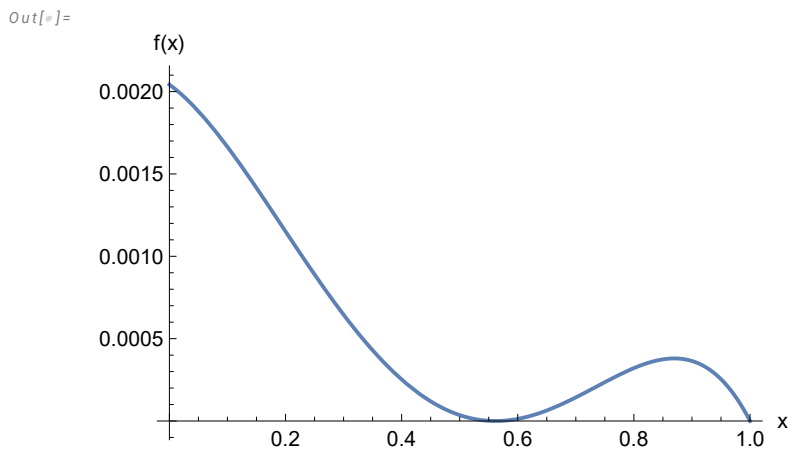
No further tedious calculations are required to follow the proof of Theorem 4.12 in Appendix C.

5.4. Code for figures such as Figure 4.2A

```
In[ ]:= PinfF3[p0, p2, p3] /. {p3 -> p2 / 2, p0 -> p2} /. p2 -> 0.15
```

```
Out[ ]:= 0.561553
```

```
In[ ]:= Plot[ff3[x] /. {p3 -> p2 / 2, p0 -> p2} /. p2 -> 0.15, {x, 0, 1},
  AxesLabel -> {"x", "f(x)"}, LabelStyle -> Directive[Black, FontSize -> 11]]
```



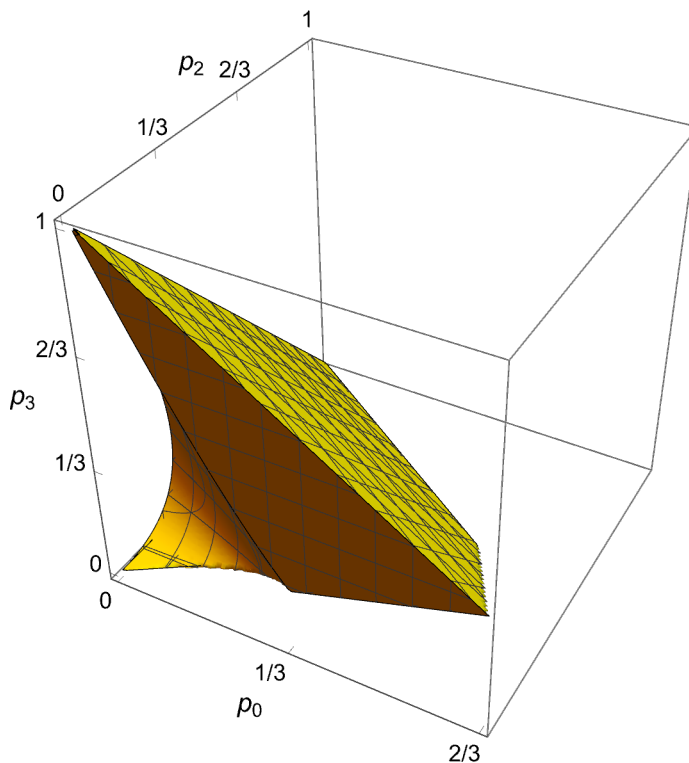
5.5. Code for figures such as Figure 4.3A

```
In[*]:= p0rplot[p2_, p3_] :=  $\frac{1}{2} - \frac{(p2 + p3) (p2 + p3 + \sqrt{8 p3 + (p2 + p3)^2})}{8 p3}$ ;
```

$$p0plusplot[p2_, p3_] := \frac{p3 - (p2 + p3)^2}{4 p3};$$

```
In[*]:= plotR1 = RegionPlot3D[p0rplot[p2, p3] ≤ p0 ≤ p2 + 2 p3 && p0 + p2 + p3 < 1,
  {p0, 0, 2/3}, {p2, 0, 1}, {p3, 0.00001, 1},
  AxesLabel → {Style["p0", FontFamily → "Helvetica", FontSize → 14, Black],
    Style["p2", FontFamily → "Helvetica", FontSize → 14, Black],
    Style["p3 ", FontFamily → "Helvetica", FontSize → 14, Black]},
  Ticks → {{0, {0.333, "1/3"}, {0.667, "2/3"}}, {0, {0.333, "1/3"}, {0.667, "2/3"}}, 1},
    {0, {0.333, "1/3"}, {0.667, "2/3"}}, 1}},
  TicksStyle → Directive[FontFamily → "Helvetica", FontSize → 12, Black],
  LabelStyle → Directive[FontFamily → "Helvetica", FontSize → 12, Black],
  PlotPoints → 100, PlotStyle → Yellow, Mesh → 10]
```

Out[*]=

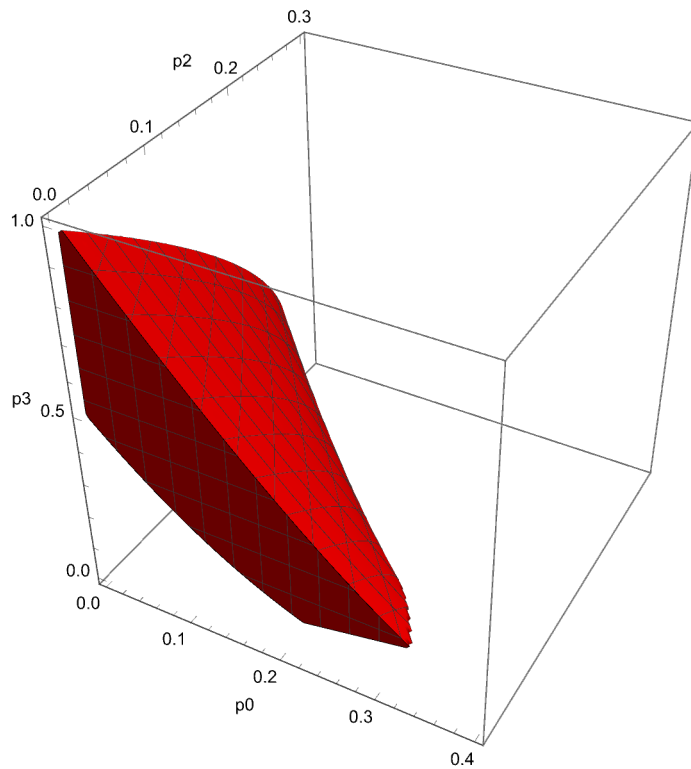


```

In[ ]:= plotR2 = RegionPlot3D[p0gam < p0 ≤ p0rplot[p2, p3] && 0 < p0 < p2 + 2 p3,
  {p0, 0, 0.4}, {p2, 0, 0.3}, {p3, 0.00001, 1},
  AxesLabel → {p0, p2, p3}, PlotPoints → 100, PlotStyle → Red, Mesh → 10]

```

Out[]=

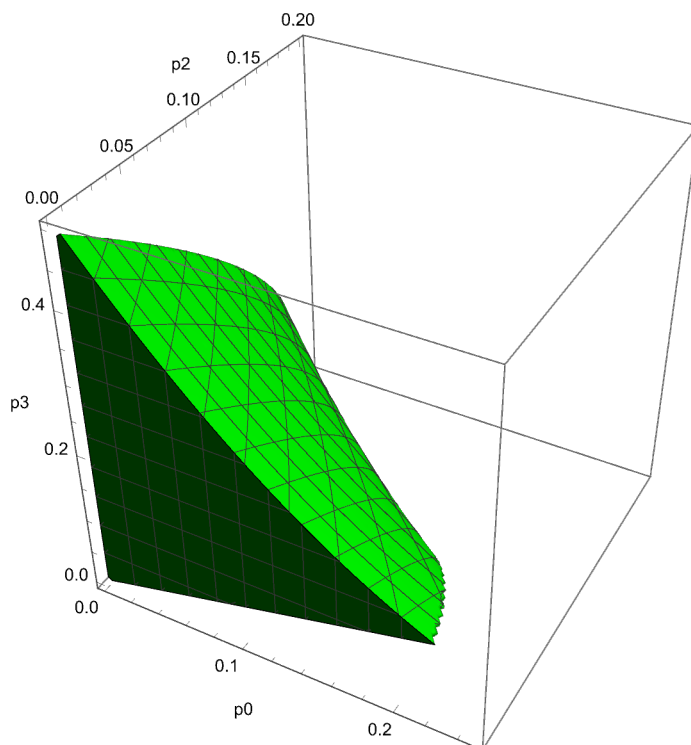


```

In[ ]:= plotR3 = RegionPlot3D[0 < p0 ≤ p0gam && p0 < p2 + 2 p3,
  {p0, 0, 0.25}, {p2, 0, 0.2}, {p3, 0.00001, 0.5},
  AxesLabel → {p0, p2, p3}, PlotPoints → 100, PlotStyle → Green, Mesh → 10]

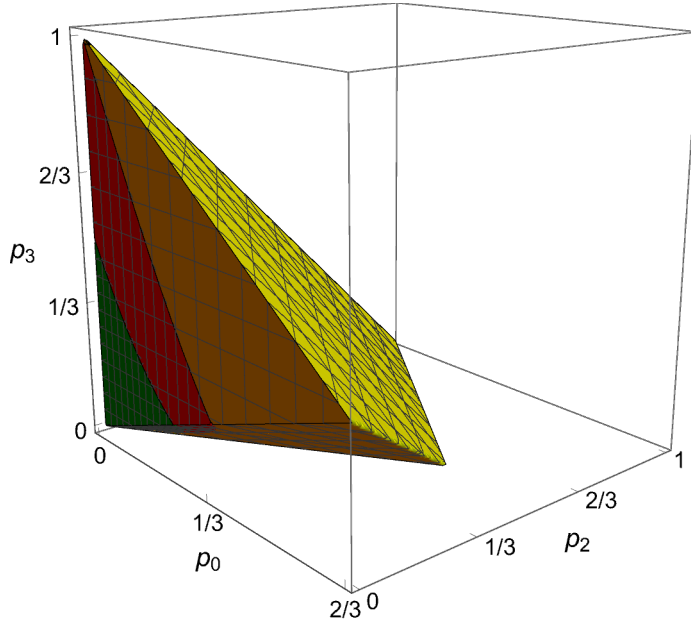
```

Out[]=



```
In[ ]:= Show[plotR1, plotR2, plotR3, ViewPoint -> {2.2, -1.9, 0.8},
  AxesEdge -> {{0, 0}, Automatic, {0, 0}}]
```

```
Out[ ]:=
```



6. Bounds and approximations for the ultimate survival probability S_φ^∞ (Sects. 5.1, 5.2, 5.3, 5.5, 5.6, 5.7 in the ms)

This section contains most of the tedious calculations needed for the results in Section 5 of the ms .

6.1. General definitions of the bounds

We provide the definitions for S_φ^∞ , whereas in the original papers they were defined for P_φ^∞ . The notation is very similar to that in the manuscript and should be self-explanatory

6.1.1 Quine's (1976) lower and upper bounds for S_φ^∞

```
In[ ]:= 
$$\begin{aligned} \phi\text{bound} &= \frac{2(m-1)}{b}; \\ \text{LQbound} &= \frac{2(m-1)}{b} + \left(\frac{2(m-1)}{b}\right)^2 \frac{1}{3b} \varphi^{(3,0)}\left[\frac{4+b-4m}{b}, s\right]; \\ \text{UQbound} &= \phi\text{bound} + \phi\text{bound}^2 \frac{c}{3b} \left(1 - \frac{4c}{3b} \phi\text{bound}\right)^{(-3/2)}; \end{aligned}$$

```

6.1.2 Daley and Narayan's (1980) upper bound for S_φ^∞

```
In[ ]:= 
$$\text{UDNbound} = \frac{3b - \sqrt{9b^2 - 24c(m-1)}}{2c};$$

```

D & N gave this bound in the equivalent form $\frac{4(m-1)/b}{1 + \sqrt{1 - 8(m-1)c/(3b^2)}}$:

```
In[*]:= FullSimplify[UDNbound -  $\frac{4(m-1)/b}{1 + \sqrt{1 - 8(m-1)c/(3b^2)}}$ , Assumptions → b > 0]
```

```
Out[*]=
```

0

Remark: The UDNbound is derived by solving $\text{fLNB}[x] = x$ with $k = 2$ (see From 2007, who refers to Narayan 1981), where

```
In[*]:= fLNB[x_] := 1 - m +  $\frac{b}{k} - \frac{c - (k-2)b}{k(k+1)} +$   

 $x \left( m - \frac{b}{k-1} + \frac{c - (k-2)b}{k(k-1)} \right) + x^k \frac{(k-1)b - c}{k(k-1)} + x^{(k+1)} \frac{c - (k-2)b}{k(k+1)}$ 
```

is a lower bound to the generating function φ (in general, $k = \text{IntegerPart}[1 + b/m]$). Note that in From's eq (1.13) there are typos! Verification:

```
In[*]:= Simplify[Solve[(fLNB[x] /. k → 2) == x, x]]
```

```
Out[*]=
```

```
{ {x → 1}, {x → - $\frac{3b - 2c + \sqrt{9b^2 - 24c(-1+m)}}{2c}$ }, {x →  $\frac{-3b + 2c + \sqrt{9b^2 - 24c(-1+m)}}{2c}$ }} }
```

6.2. Series expansions of S_φ^∞ and γ_φ for general φ

We set $\text{Pinf} = \text{Sum}[\delta[i] (-s)^i, \{i, 0, 4\}]$ with $\delta[0] = 1$

```
In[*]:= Clear[δ]
```

```
In[*]:= δ[0] = 1;  

δ[1] = 0;  

Pinfser = Sum[δ[i] (-s)^i, {i, 0, 4}];
```

We use the following assumptions and substitutions:

```
In[*]:= φ[1, s_] := 1;  

φ(1,0)[1, s_] := 1 + s;  

μ[0, k_] := 0;  

μ[1, 1] = 1;  

μ[1, 2] = 0;  

μ[1, 3] = 0;  

μ[1, 4] = 0;  

substderiv = {φ(k-,1-)[1, 0] → μ[k, 1]};
```

In[*]:= **Simplify**[Series[ϕ [Pinfser, s] - Pinfser, {s, 0, 5}] // . substderiv // Normal

Out[*]=

$$\begin{aligned} & \frac{1}{2} s^2 \theta (-2 + \theta \mu[2, 0]) + s^3 \left(\delta[2] (1 - \theta \mu[2, 0]) - \frac{1}{6} \theta^2 (-3 \mu[2, 1] + \theta \mu[3, 0]) \right) + \\ & s^4 \left(\delta[3] (-1 + \theta \mu[2, 0]) + \frac{1}{24} (12 \delta[2]^2 \mu[2, 0] + \right. \\ & \quad \left. 12 \theta \delta[2] (-2 \mu[2, 1] + \theta \mu[3, 0]) + \theta^2 (6 \mu[2, 2] - 4 \theta \mu[3, 1] + \theta^2 \mu[4, 0]) \right) + \\ & s^5 \left(\delta[4] (1 - \theta \mu[2, 0]) + \frac{1}{120} (60 \delta[2]^2 (\mu[2, 1] - \theta \mu[3, 0]) - 20 \delta[2] (6 \delta[3] \times \mu[2, 0] + \right. \\ & \quad \left. \theta (3 \mu[2, 2] - 3 \theta \mu[3, 1] + \theta^2 \mu[4, 0]) \right) + \theta (60 \delta[3] (2 \mu[2, 1] - \theta \mu[3, 0]) + \\ & \quad \left. \theta (10 \mu[2, 3] - 10 \theta \mu[3, 2] + 5 \theta^2 \mu[4, 1] - \theta^3 \mu[5, 0]) \right) \end{aligned}$$

In[*]:= **coeffs** = **CoefficientList**[% , s]

Out[*]=

$$\begin{aligned} & \left\{ 0, 0, \frac{1}{2} \theta (-2 + \theta \mu[2, 0]), \delta[2] (1 - \theta \mu[2, 0]) - \frac{1}{6} \theta^2 (-3 \mu[2, 1] + \theta \mu[3, 0]), \right. \\ & \delta[3] (-1 + \theta \mu[2, 0]) + \frac{1}{24} (12 \delta[2]^2 \mu[2, 0] + \\ & \quad \left. 12 \theta \delta[2] (-2 \mu[2, 1] + \theta \mu[3, 0]) + \theta^2 (6 \mu[2, 2] - 4 \theta \mu[3, 1] + \theta^2 \mu[4, 0]) \right), \\ & \delta[4] (1 - \theta \mu[2, 0]) + \frac{1}{120} (60 \delta[2]^2 (\mu[2, 1] - \theta \mu[3, 0]) - \\ & \quad \left. 20 \delta[2] (6 \delta[3] \times \mu[2, 0] + \theta (3 \mu[2, 2] - 3 \theta \mu[3, 1] + \theta^2 \mu[4, 0]) \right) + \theta (60 \delta[3] \\ & \quad \left. (2 \mu[2, 1] - \theta \mu[3, 0]) + \theta (10 \mu[2, 3] - 10 \theta \mu[3, 2] + 5 \theta^2 \mu[4, 1] - \theta^3 \mu[5, 0]) \right) \} \end{aligned}$$

The non-trivial (second) solution below gives the desired coefficients; see eqs (5.9) -- (5.11) in the ms:

In[*]:= **Simplify**[**Solve**[**Delete**[coeffs, {{1}, {2}}] == 0, **Table**[$\delta[i]$, {i, 1, 4}]]]

Out[*]=

$$\begin{aligned} & \left\{ \{ \theta \rightarrow 0, \delta[2] \rightarrow 0, \delta[3] \rightarrow 0, \delta[4] \rightarrow 0 \}, \left\{ \theta \rightarrow \frac{2}{\mu[2, 0]}, \right. \right. \\ & \delta[2] \rightarrow \frac{6 \mu[2, 0] \times \mu[2, 1] - 4 \mu[3, 0]}{3 \mu[2, 0]^3}, \delta[3] \rightarrow \frac{1}{9 \mu[2, 0]^5} (-9 \mu[2, 0]^3 \mu[2, 2] + 16 \mu[3, 0]^2 + \\ & \quad \left. 6 \mu[2, 0]^2 (3 \mu[2, 1]^2 + 2 \mu[3, 1]) - 6 \mu[2, 0] (6 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0]) \right), \\ & \delta[4] \rightarrow \frac{1}{135 \mu[2, 0]^7} (45 \mu[2, 0]^5 \mu[2, 3] - 400 \mu[3, 0]^3 - 90 \mu[2, 0]^4 \\ & \quad (3 \mu[2, 1] \times \mu[2, 2] + \mu[3, 2]) + 300 \mu[2, 0] \times \mu[3, 0] (4 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0]) + \\ & \quad \left. 90 \mu[2, 0]^3 (3 \mu[2, 1]^3 + 3 \mu[2, 2] \times \mu[3, 0] + 6 \mu[2, 1] \times \mu[3, 1] + \mu[4, 1]) - 12 \mu[2, 0]^2 \right. \\ & \quad \left. (90 \mu[2, 1]^2 \mu[3, 0] + 40 \mu[3, 0] \times \mu[3, 1] + 30 \mu[2, 1] \times \mu[4, 0] + 3 \mu[5, 0]) \right) \} \} \end{aligned}$$

In[*]:= **Expand**[$-9 \mu[2, 0]^3 \mu[2, 2] + 16 \mu[3, 0]^2 +$
 $6 \mu[2, 0]^2 (3 \mu[2, 1]^2 + 2 \mu[3, 1]) - 6 \mu[2, 0] (6 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0])$]

Out[*]=

$$\begin{aligned} & 18 \mu[2, 0]^2 \mu[2, 1]^2 - 9 \mu[2, 0]^3 \mu[2, 2] - 36 \mu[2, 0] \times \mu[2, 1] \times \mu[3, 0] + \\ & 16 \mu[3, 0]^2 + 12 \mu[2, 0]^2 \mu[3, 1] - 6 \mu[2, 0] \times \mu[4, 0] \end{aligned}$$

```
In[*]:= 
$$\delta\text{list} = \left\{ \delta[2] \rightarrow \frac{6 \mu[2, 0] \times \mu[2, 1] - 4 \mu[3, 0]}{3 \mu[2, 0]^3}, \delta[3] \rightarrow \frac{1}{9 \mu[2, 0]^5} \left( 18 \mu[2, 0]^2 \mu[2, 1]^2 - 9 \mu[2, 0]^3 \mu[2, 2] - 36 \mu[2, 0] \times \mu[2, 1] \times \mu[3, 0] + \right. \right.$$


$$16 \mu[3, 0]^2 + 12 \mu[2, 0]^2 \mu[3, 1] - 6 \mu[2, 0] \times \mu[4, 0] \left. \right), \delta[4] \rightarrow \frac{1}{135 \mu[2, 0]^7}$$


$$\left( 45 \mu[2, 0]^5 \mu[2, 3] - 400 \mu[3, 0]^3 - 90 \mu[2, 0]^4 (3 \mu[2, 1] \times \mu[2, 2] + \mu[3, 2]) + \right.$$


$$300 \mu[2, 0] \times \mu[3, 0] (4 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0]) + 90 \mu[2, 0]^3$$


$$(3 \mu[2, 1]^3 + 3 \mu[2, 2] \times \mu[3, 0] + 6 \mu[2, 1] \times \mu[3, 1] + \mu[4, 1]) - 12 \mu[2, 0]^2$$


$$\left. \left( 90 \mu[2, 1]^2 \mu[3, 0] + 40 \mu[3, 0] \times \mu[3, 1] + 30 \mu[2, 1] \times \mu[4, 0] + 3 \mu[5, 0] \right) \right\};$$

```

Now we compute the series expansion of gamma in (5.12) and (5.13):

```
In[*]:= Simplify[Series[D[φ[x, s], x] /. x → 1 - θ s + δ[2] s^2 - δ[3] s^3 + δ[4] s^4, {s, 0, 4}] /.
substderiv /. δlist /. θ → 2 / μ[2, 0]]

Out[*]= 1 - s + 
$$\frac{2 \mu[3, 0] s^2}{3 \mu[2, 0]^2} +$$


$$\frac{(8 \mu[3, 0]^2 + 6 \mu[2, 0]^2 \mu[3, 1] - 6 \mu[2, 0] (2 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0])) s^3}{9 \mu[2, 0]^4} + \frac{1}{135 \mu[2, 0]^6}$$


$$(200 \mu[3, 0]^3 + 45 \mu[2, 0]^4 \mu[3, 2] - 240 \mu[2, 0] \times \mu[3, 0] (2 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0]) -$$


$$90 \mu[2, 0]^3 (\mu[2, 2] \times \mu[3, 0] + 2 \mu[2, 1] \times \mu[3, 1] + \mu[4, 1]) + 6 \mu[2, 0]^2$$


$$(45 \mu[2, 1]^2 \mu[3, 0] + 40 \mu[3, 0] \times \mu[3, 1] + 45 \mu[2, 1] \times \mu[4, 0] + 9 \mu[5, 0])) s^4 + O[s]^5$$


In[*]:= Expand[-
$$\frac{1}{2} (8 \mu[3, 0]^2 + 6 \mu[2, 0]^2 \mu[3, 1] - 6 \mu[2, 0] (2 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0]))]$$


Out[*]= 6 μ[2, 0] × μ[2, 1] × μ[3, 0] - 4 μ[3, 0]^2 - 3 μ[2, 0]^2 μ[3, 1] + 3 μ[2, 0] × μ[4, 0]
```

We define

```
In[*]:= γ[0] = 1;
γ[1] = 1;
γlist = {γ[2] → 
$$\frac{2 \mu[3, 0]}{3 \mu[2, 0]^2}, \gamma[3] \rightarrow \frac{2}{9 \mu[2, 0]^4} (6 \mu[2, 0] \times \mu[2, 1] \times \mu[3, 0] - 4 \mu[3, 0]^2 -$$


$$3 \mu[2, 0]^2 \mu[3, 1] + 3 \mu[2, 0] \times \mu[4, 0]) , \gamma[4] \rightarrow \frac{1}{135 \mu[2, 0]^6} (200 \mu[3, 0]^3 +$$


$$45 \mu[2, 0]^4 \mu[3, 2] - 240 \mu[2, 0] \times \mu[3, 0] (2 \mu[2, 1] \times \mu[3, 0] + \mu[4, 0]) -$$


$$90 \mu[2, 0]^3 (\mu[2, 2] \times \mu[3, 0] + 2 \mu[2, 1] \times \mu[3, 1] + \mu[4, 1]) + 6 \mu[2, 0]^2$$


$$(45 \mu[2, 1]^2 \mu[3, 0] + 40 \mu[3, 0] \times \mu[3, 1] + 45 \mu[2, 1] \times \mu[4, 0] + 9 \mu[5, 0])) \};$$


In[*]:= Sinfser[s_, n_] := Sum[δ[k] (-1)^(k-1) s^k, {k, 1, n}];
gammaser[s_, n_] := Sum[γ[k] (-s)^k, {k, 0, n}]
```

6.3. Series expansions of Quine's and Daley and Narayana bounds

```

In[*]:= Simplify[
  Series[ϕbound /. {m → 1 + s, b → ϕ(2,0)[1, s], c → ϕ(3,0)[1, s]}, {s, 0, 3}] /. substderiv]
Out[*]=

$$\frac{2s}{\mu[2, 0]} - \frac{2\mu[2, 1]s^2}{\mu[2, 0]^2} + \frac{(2\mu[2, 1]^2 - \mu[2, 0] \times \mu[2, 2])s^3}{\mu[2, 0]^3} + O[s]^4$$


In[*]:= Simplify[Series[UQbound /. {m → 1 + s, b → ϕ(2,0)[1, s], c → ϕ(3,0)[1, s]}, {s, 0, 3}] /.
  substderiv] // Normal
Out[*]=

$$\frac{2s}{\mu[2, 0]} + \frac{s^2(-6\mu[2, 0] \times \mu[2, 1] + 4\mu[3, 0])}{3\mu[2, 0]^3} +$$


$$\frac{1}{3\mu[2, 0]^5} s^3(-3\mu[2, 0]^3 \mu[2, 2] - 12\mu[2, 0] \times \mu[2, 1] \times \mu[3, 0] +$$


$$16\mu[3, 0]^2 + \mu[2, 0]^2(6\mu[2, 1]^2 + 4\mu[3, 1]))$$


In[*]:= Simplify[Series[% - Sinfser[s, 3], {s, 0, 3}] /. δlist /. θ → 2 / μ[2, 0]]
Out[*]=

$$\frac{(32\mu[3, 0]^2 + 6\mu[2, 0] \times \mu[4, 0])s^3}{9\mu[2, 0]^5} + O[s]^4$$


In[*]:= Simplify[Series[LQbound /. {m → 1 + s, b → ϕ(2,0)[1, s], c → ϕ(3,0)[1, s]}, {s, 0, 3}] /.
  substderiv] // Normal
Out[*]=

$$\frac{2s}{\mu[2, 0]} + \frac{s^2(-6\mu[2, 0] \times \mu[2, 1] + 4\mu[3, 0])}{3\mu[2, 0]^3} + \frac{1}{3\mu[2, 0]^4} s^3$$


$$(-3\mu[2, 0]^2 \mu[2, 2] + \mu[2, 0](6\mu[2, 1]^2 + 4\mu[3, 1]) - 4(3\mu[2, 1] \times \mu[3, 0] + 4\mu[4, 0]))$$


In[*]:= Simplify[Series[% - Sinfser[s, 3], {s, 0, 3}] /. δlist /. θ → 2 / μ[2, 0]]
Out[*]=

$$-\frac{2(8\mu[3, 0]^2 + 21\mu[2, 0] \times \mu[4, 0])s^3}{9\mu[2, 0]^5} + O[s]^4$$


In[*]:= Simplify[Series[UDNbound /. {m → 1 + s, b → ϕ(2,0)[1, s], c → ϕ(3,0)[1, s]}, {s, 0, 3}] /.
  substderiv, Assumptions → μ[2, 0] > 0] // Normal
Out[*]=

$$\frac{2s}{\mu[2, 0]} + \frac{2s^2(-3\mu[2, 0] \times \mu[2, 1] + 2\mu[3, 0])}{3\mu[2, 0]^3} +$$


$$\frac{1}{9\mu[2, 0]^5} s^3(-9\mu[2, 0]^3 \mu[2, 2] - 36\mu[2, 0] \times \mu[2, 1] \times \mu[3, 0] +$$


$$16\mu[3, 0]^2 + 6\mu[2, 0]^2(3\mu[2, 1]^2 + 2\mu[3, 1]))$$


In[*]:= Simplify[Series[% - Sinfser[s, 3], {s, 0, 3}] /. δlist /. θ → 2 / μ[2, 0]]
Out[*]=

$$\frac{2\mu[4, 0]s^3}{3\mu[2, 0]^4} + O[s]^4$$


```

Therefore, the series expansions of the UQbound, LQbound, and UDNbound have correct terms of

order 1 and 2 in s

6.4. Series expansions for specific distributions (Sects. 5.5 - 5.7)

Poisson distribution (Sect. 5.5)

The Lambert function has the following Taylor expansion about $x=0$: $W(x) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} x^k$.

```
In[*]:= Table[ $\frac{(-k)^{k-1}}{k!} x^k$ , {k, 1, 7}]
```

Out[*]=

$$\left\{x, -x^2, \frac{3x^3}{2}, -\frac{8x^4}{3}, \frac{125x^5}{24}, -\frac{54x^6}{5}, \frac{16807x^7}{720}\right\}$$

Here, we can work with the explicit expression for SinfPoi

```
In[*]:= Simplify[Series[ProductLog[x], {x, 0, 7}], Assumptions -> x > 0]
```

Out[*]=

$$x - x^2 + \frac{3x^3}{2} - \frac{8x^4}{3} + \frac{125x^5}{24} - \frac{54x^6}{5} + \frac{16807x^7}{720} + O[x]^8$$

```
In[*]:= Simplify[Series[SinfPoi[1 + s], {s, 0, 4}], Assumptions -> s > 0]
```

Out[*]=

$$2s - \frac{8s^2}{3} + \frac{28s^3}{9} - \frac{464s^4}{135} + O[s]^5$$

```
In[*]:= Simplify[Series[gammaPoi[1 + s], {s, 0, 4}], Assumptions -> s > 0]
```

Out[*]=

$$1 - s + \frac{2s^2}{3} - \frac{4s^3}{9} + \frac{44s^4}{135} + O[s]^5$$

Binomial distribution (Sect. 5.6)

```
In[*]:= Table[ $\mu[k, 1] \rightarrow D[D[\phi\text{Bin}[n, \frac{1+s}{n}][x], \{x, k\}]$ , {s, 1}] /. {x -> 1, s -> 0},  
{k, 1, 5}, {1, 0, 5 - k}] // Flatten
```

Out[*]=

$$\left\{\begin{aligned} &\mu[1, 0] \rightarrow 1, 1 \rightarrow 1, 0 \rightarrow 0, 0 \rightarrow 0, 0 \rightarrow 0, \mu[2, 0] \rightarrow \frac{-1+n}{n}, \mu[2, 1] \rightarrow \frac{2(-1+n)}{n}, \\ &\mu[2, 2] \rightarrow \frac{2(-1+n)}{n}, \mu[2, 3] \rightarrow 0, \mu[3, 0] \rightarrow \frac{(-2+n)(-1+n)}{n^2}, \mu[3, 1] \rightarrow \frac{3(-2+n)(-1+n)}{n^2}, \\ &\mu[3, 2] \rightarrow \frac{6(-2+n)(-1+n)}{n^2}, \mu[4, 0] \rightarrow \frac{(-3+n)(-2+n)(-1+n)}{n^3}, \\ &\mu[4, 1] \rightarrow \frac{4(-3+n)(-2+n)(-1+n)}{n^3}, \mu[5, 0] \rightarrow \frac{(-4+n)(-3+n)(-2+n)(-1+n)}{n^4} \end{aligned}\right\}$$

```
In[*]:= substμBin = { θ →  $\frac{2n}{n-1}$ , μ[1, 0] → 1, 1 → 1, 0 → 0, 0 → 0,
  0 → 0, μ[2, 0] →  $\frac{-1+n}{n}$ , μ[2, 1] →  $\frac{2(-1+n)}{n}$ , μ[2, 2] →  $\frac{2(-1+n)}{n}$ ,
  μ[2, 3] → 0, μ[3, 0] →  $\frac{(-2+n)(-1+n)}{n^2}$ , μ[3, 1] →  $\frac{3(-2+n)(-1+n)}{n^2}$ ,
  μ[3, 2] →  $\frac{6(-2+n)(-1+n)}{n^2}$ , μ[4, 0] →  $\frac{(-3+n)(-2+n)(-1+n)}{n^3}$ ,
  μ[4, 1] →  $\frac{4(-3+n)(-2+n)(-1+n)}{n^3}$ , μ[5, 0] →  $\frac{(-4+n)(-3+n)(-2+n)(-1+n)}{n^4}$  };
```

```
In[*]:= Simplify[Series[Sinfser[s, 4] /. δlist /. substμBin, {s, 0, 4}]]
```

```
Out[*]= 
$$\frac{2ns}{-1+n} - \frac{4(n(-1+2n))s^2}{3(-1+n)^2} + \frac{2n(5-17n+14n^2)s^3}{9(-1+n)^3} - \frac{8(n(-17+78n-117n^2+58n^3))s^4}{135(-1+n)^4} + O[s]^5$$

```

```
In[*]:= SinfBinser[s_, 4] :=
```

```

$$\frac{2n}{n-1}s - \frac{4(n(-1+2n))s^2}{3(-1+n)^2} + \frac{2n(5-17n+14n^2)s^3}{9(-1+n)^3} - \frac{8n(-17+78n-117n^2+58n^3)s^4}{135(-1+n)^4}$$

```

```
In[*]:= FullSimplify[Series[gammaSer[s, 4] /. δlist /. γlist /. substμBin, {s, 0, 4}]]
```

```
Out[*]= 
$$1 - s + \frac{2(-2+n)s^2}{3(-1+n)} - \frac{4(-2+n)^2s^3}{9(-1+n)^2} + \frac{2(-2+n)(79+n(-79+22n))s^4}{135(-1+n)^3} + O[s]^5$$

```

Other version:

```
In[*]:= FullSimplify[
  Series[ $\frac{n p \text{PinfB}}{(1-p+p \text{PinfB})}$  /. p →  $\frac{1+s}{n}$  /. PinfB → 1 - SinfBinser[s, 4], {s, 0, 4}]]
```

```
Out[*]= 
$$1 - s + \frac{2(-2+n)s^2}{3(-1+n)} - \frac{4(-2+n)^2s^3}{9(-1+n)^2} + \frac{2(-2+n)(79+n(-79+22n))s^4}{135(-1+n)^3} + O[s]^5$$

```

Negative binomial distribution (Sect. 5.7)

```
In[*]:= Simplify[Table[μ[k, 1] → D[D[φNB[n,  $\frac{n}{n+1+s}$ ][x], {x, k}], {s, 1}],
  {k, 1, 5}, {1, 0, 5-k}] // Flatten]
```

```
Out[*]= 
$$\left\{ \begin{aligned} &\mu[1, 0] \rightarrow 1, 1 \rightarrow 1, 0 \rightarrow 0, 0 \rightarrow 0, 0 \rightarrow 0, \mu[2, 0] \rightarrow 1 + \frac{1}{n}, \mu[2, 1] \rightarrow 2 + \frac{2}{n}, \\ &\mu[2, 2] \rightarrow 2 + \frac{2}{n}, \mu[2, 3] \rightarrow 0, \mu[3, 0] \rightarrow \frac{(1+n)(2+n)}{n^2}, \mu[3, 1] \rightarrow \frac{3(1+n)(2+n)}{n^2}, \\ &\mu[3, 2] \rightarrow \frac{6(1+n)(2+n)}{n^2}, \mu[4, 0] \rightarrow \frac{(1+n)(2+n)(3+n)}{n^3}, \\ &\mu[4, 1] \rightarrow \frac{4(1+n)(2+n)(3+n)}{n^3}, \mu[5, 0] \rightarrow \frac{(1+n)(2+n)(3+n)(4+n)}{n^4} \end{aligned} \right\}$$

```

```
In[*]:= substμNB = {θ →  $\frac{2n}{1+n}$ , μ[1, 0] → 1, 1 → 1, 0 → 0, 0 → 0, 0 → 0, μ[2, 0] →  $1 + \frac{1}{n}$ ,  

  μ[2, 1] →  $2 + \frac{2}{n}$ , μ[2, 2] →  $2 + \frac{2}{n}$ , μ[2, 3] → 0, μ[3, 0] →  $\frac{(1+n)(2+n)}{n^2}$ ,  

  μ[3, 1] →  $\frac{3(1+n)(2+n)}{n^2}$ , μ[3, 2] →  $\frac{6(1+n)(2+n)}{n^2}$ , μ[4, 0] →  $\frac{(1+n)(2+n)(3+n)}{n^3}$ ,  

  μ[4, 1] →  $\frac{4(1+n)(2+n)(3+n)}{n^3}$ , μ[5, 0] →  $\frac{(1+n)(2+n)(3+n)(4+n)}{n^4}$ };
```

```
In[*]:= SinfNBser[s_, 4] :=  


$$\frac{2n}{n+1} s - \frac{4(n(1+2n))s^2}{3(1+n)^2} + \frac{2n(5+17n+14n^2)s^3}{9(1+n)^3} - \frac{8n(17+78n+117n^2+58n^3)s^4}{135(1+n)^4}$$

```

```
In[*]:= FullSimplify[Series[gammaser[s, 4] /. δlist /. γlist /. substμNB, {s, 0, 4}]]
```

```
Out[*]=  


$$1 - s + \frac{2(2+n)s^2}{3(1+n)} - \frac{4(2+n)^2s^3}{9(1+n)^2} + \frac{2(2+n)(79+n(79+22n))s^4}{135(1+n)^3} + O[s]^5$$

```

7. Generalized Poisson distribution (Sect. 5.8)

Generalized Poisson distribution (Consul and Jain 1973, Tuentner 2006), where $m > 0$ and $0 \leq \lambda < 1$. In Johnson and Kotz (2005, Univariate Discrete Distribution, 7.2 .6), this is called the Lagrangian Poisson distribution

7.1 . Basics

```
In[*]:= GenPoi[μ_, λ_, k_] :=  $\frac{\mu (\mu + k \lambda) ^ (k - 1)}{k !} \text{Exp}[-\mu - k \lambda]$ 
```

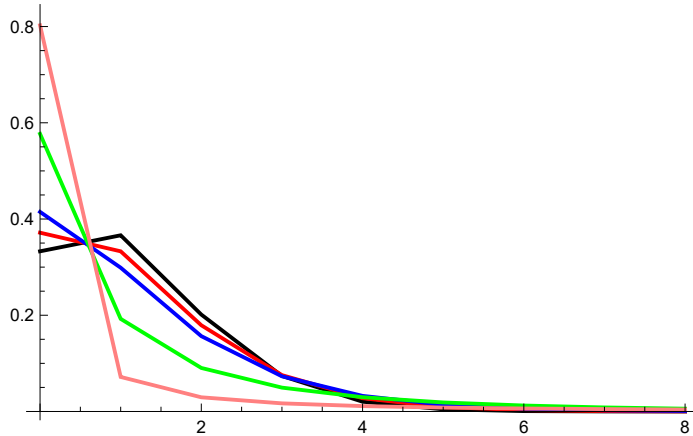
Fix mean =1.1 and change λ ; then $\mu = \text{mean} (1 - \lambda)$:

```

In[ ]:= ListLinePlot[Table[{k, GenPoi[1.1, 0, k]}, {k, 0, 8}],
  Table[{k, GenPoi[1.1 (1 - 0.1), 0.1, k]}, {k, 0, 8}],
  Table[{k, GenPoi[1.1 (1 - 0.2), 0.2, k]}, {k, 0, 8}],
  Table[{k, GenPoi[1.1 (1 - 0.5), 0.5, k]}, {k, 0, 8}],
  Table[{k, GenPoi[1.1 (1 - 0.9) 2, 0.9, k]}, {k, 0, 8}]],
  PlotRange -> All, PlotStyle -> {Black, Red, Blue, Green, Pink}]

```

Out[]:=



```

In[ ]:= Table[varGP[(1 + 0.1) (1 - λ), λ], {λ, {0, 0.1, 0.2, 0.5, 0.9}}]

```

Out[]:=

```
{1.1, 1.35802, 1.71875, 4.4, 110.}
```

7.2. Series expansion of S_{GP}^{∞} and γ_{GP}

Fix mean $m = 1 + s$. Then $\mu = (1 + s) (1 - \lambda)$

The following takes a while

```

FullSimplify[
  Table[μ[k, 1] -> D[D[φGP[(1 + s) (1 - λ), λ][x], {x, k}], {s, 1}] /. {x -> 1, s -> 0},
    {k, 1, 5}], {1, 0, 5 - k}], Assumptions -> s > 0 && 0 ≤ λ < 1 // Flatten

```

Therefore, we define it from the output:

```

In[ ]:= substμGP = {θ -> 2 (1 - λ)2, μ[1, 0] -> 1, 1 -> 1, 0 -> 0, 0 -> 0, 0 -> 0, μ[2, 0] ->  $\frac{1}{(-1 + \lambda)^2}$ ,
  μ[2, 1] ->  $1 + \frac{1}{(-1 + \lambda)^2}$ , μ[2, 2] -> 2, μ[2, 3] -> 0, μ[3, 0] ->  $\frac{1 + 2\lambda}{(-1 + \lambda)^4}$ ,
  μ[3, 1] ->  $\frac{3 - (-3 + \lambda) (-1 + \lambda) \lambda^2}{(-1 + \lambda)^4}$ , μ[3, 2] ->  $\frac{6}{(-1 + \lambda)^2}$ , μ[4, 0] ->  $\frac{1 + \lambda (6 + 9\lambda - \lambda^3)}{(-1 + \lambda)^6}$ ,
  μ[4, 1] ->  $\frac{4 + \lambda (12 + \lambda (-6 + \lambda (-8 + \lambda (23 + 2 (-6 + \lambda) \lambda)))}{(-1 + \lambda)^6}$ ,
  μ[5, 0] ->  $\frac{1 + \lambda (12 + \lambda (48 + \lambda (64 + 5\lambda (-1 + (-4 + \lambda) \lambda)))}{(-1 + \lambda)^8}$ };

```

```
In[*]:= FullSimplify[Series[Sinfser[s, 4] /. δlist /. substμGP, {s, 0, 4}]]
```

```
Out[*]=
```

$$2(-1+\lambda)^2 s - \frac{2}{3} \left((-1+\lambda)^2 (4+\lambda(-10+3\lambda)) \right) s^2 + \frac{4}{9} (-1+\lambda)^3 (-7+\lambda(31+3(-7+\lambda)\lambda)) s^3 - \frac{2}{135} \left((-1+\lambda)^4 (232+\lambda(-1552+15\lambda(116+3(-12+\lambda)\lambda))) \right) s^4 + O[s]^5$$

```
In[*]:= Expand[(232 + λ (-1552 + 15 λ (116 + 3 (-12 + λ) λ)))]
```

```
Out[*]=
```

$$232 - 1552 \lambda + 1740 \lambda^2 - 540 \lambda^3 + 45 \lambda^4$$

```
In[*]:= SinfGPser[s_, 4] :=
```

$$2(1-\lambda)^2 s - \frac{2}{3} \left((1-\lambda)^2 (4-10\lambda+3\lambda^2) \right) s^2 + \frac{4}{9} (1-\lambda)^3 (7-31\lambda+21\lambda^2-3\lambda^3) s^3 - \frac{2}{135} \left((1-\lambda)^4 (232-1552\lambda+1740\lambda^2-540\lambda^3+45\lambda^4) \right) s^4$$

```
In[*]:= Simplify[Series[gammaSer[s, 4] /. δlist /. γlist /. substμGP, {s, 0, 4}]]
```

```
Out[*]=
```

$$1 - s + \frac{2}{3} (1+2\lambda) s^2 - \frac{4}{9} (1+7\lambda+\lambda^2) s^3 + \frac{4}{135} (11+162\lambda+93\lambda^2+4\lambda^3) s^4 + O[s]^5$$

```
In[*]:= gammaGPser[s_, 4] :=
```

$$1 - s + \frac{2}{3} (1+2\lambda) s^2 - \frac{4}{9} (1+7\lambda+\lambda^2) s^3 + \frac{4}{135} (11+162\lambda+93\lambda^2+4\lambda^3) s^4;$$

7.3 . Study fGP = ϕGP - ϕGPFL with the help of the series expansion

Determine the parameters π and ρ for the bounding fractional linear generating function ϕ_{GPFL} by series expansion

We use (3.2) -- (3.4) in the ms to calculate the fractional linear approximations

```
In[*]:= FullSimplify[ $\left\{\frac{1-a2}{1-a1 a2}, a1 \frac{1-a2}{1-a1 a2}\right\} /. \{a1 \rightarrow 1 - \text{SinfGPser}[s, 4], a2 \rightarrow \text{gammaGPser}[s, 4]\}$ ]
```

```
Out[*]=
```

$$\left\{ \left(s - \frac{2}{3} s^2 (1+2\lambda) + \frac{4}{9} s^3 (1+\lambda(7+\lambda)) - \frac{4}{135} s^4 (11+\lambda(162+\lambda(93+4\lambda))) \right) / \right. \\ \left(1 - \frac{1}{18225} (135+2s(-1+\lambda)^2 \right. \\ \left. (-135+45s(4+\lambda(-10+3\lambda)) - 30s^2(-1+\lambda)(-7+\lambda(31+3(-7+\lambda)\lambda)) + \right. \\ \left. s^3(-1+\lambda)^2(232+\lambda(-1552+15\lambda(116+3(-12+\lambda)\lambda))) \right) \left. (135+s(-135+2s(45+90\lambda-30s(1+\lambda(7+\lambda))+2s^2(11+\lambda(162+\lambda(93+4\lambda)))))) \right), \\ \left(\left(s - \frac{2}{3} s^2 (1+2\lambda) + \frac{4}{9} s^3 (1+\lambda(7+\lambda)) - \frac{4}{135} s^4 (11+\lambda(162+\lambda(93+4\lambda))) \right) \right. \\ \left(1 + \frac{2}{135} s(-1+\lambda)^2 \right. \\ \left. (-135+45s(4+\lambda(-10+3\lambda)) - 30s^2(-1+\lambda)(-7+\lambda(31+3(-7+\lambda)\lambda)) + \right. \\ \left. s^3(-1+\lambda)^2(232+\lambda(-1552+15\lambda(116+3(-12+\lambda)\lambda))) \right) \left. \right) / \\ \left(1 - \frac{1}{18225} (135+2s(-1+\lambda)^2 (-135+45s(4+\lambda(-10+3\lambda)) - 30s^2(-1+\lambda)(-7+\lambda(31+ \right. \\ \left. 3(-7+\lambda)\lambda)) + s^3(-1+\lambda)^2(232+\lambda(-1552+15\lambda(116+3(-12+\lambda)\lambda))) \right) \left. (135+s(-135+2s(45+90\lambda-30s(1+\lambda(7+\lambda))+2s^2(11+\lambda(162+\lambda(93+4\lambda)))))) \right) \right\}$$

We use this result to define the substitution:

```
In[*]:=
```

```
substprGPs4[s_, λ_] :=
```

$$\left\{ p \rightarrow \left(s - \frac{2}{3} s^2 (1+2\lambda) + \frac{4}{9} s^3 (1+\lambda(7+\lambda)) - \frac{4}{135} s^4 (11+\lambda(162+\lambda(93+4\lambda))) \right) / \right. \\ \left(1 - \frac{1}{18225} (135+2s(-1+\lambda)^2 \right. \\ \left. (-135+45s(4+\lambda(-10+3\lambda)) - 30s^2(-1+\lambda)(-7+\lambda(31+3(-7+\lambda)\lambda)) + \right. \\ \left. s^3(-1+\lambda)^2(232+\lambda(-1552+15\lambda(116+3(-12+\lambda)\lambda))) \right) \left. (135+s(-135+ \right. \\ \left. 2s(45+90\lambda-30s(1+\lambda(7+\lambda))+2s^2(11+\lambda(162+\lambda(93+4\lambda)))))) \right), \\ r \rightarrow \left(\left(s - \frac{2}{3} s^2 (1+2\lambda) + \frac{4}{9} s^3 (1+\lambda(7+\lambda)) - \frac{4}{135} s^4 (11+\lambda(162+\lambda(93+4\lambda))) \right) \right. \\ \left(1 + \frac{2}{135} s(-1+\lambda)^2 \right. \\ \left. (-135+45s(4+\lambda(-10+3\lambda)) - 30s^2(-1+\lambda)(-7+\lambda(31+3(-7+\lambda)\lambda)) + \right. \\ \left. s^3(-1+\lambda)^2(232+\lambda(-1552+15\lambda(116+3(-12+\lambda)\lambda))) \right) \left. \right) / \\ \left(1 - \frac{1}{18225} (135+2s(-1+\lambda)^2 (-135+45s(4+\lambda(-10+3\lambda)) - \right. \\ \left. 30s^2(-1+\lambda)(-7+\lambda(31+3(-7+\lambda)\lambda)) + \right. \\ \left. s^3(-1+\lambda)^2(232+\lambda(-1552+15\lambda(116+3(-12+\lambda)\lambda))) \right) \left. (135+s(-135+ \right. \\ \left. 2s(45+90\lambda-30s(1+\lambda(7+\lambda))+2s^2(11+\lambda(162+\lambda(93+4\lambda)))))) \right) \right\};$$

Calculate the series expansion of $fGP = \varphi GP - \varphi GPFL$

`In[]:= FullSimplify[Series[$\varphi GP[(1+s)(1-\lambda), \lambda][x] - \varphi FL[p, r][x] /. \text{substprGPs4}[s, \lambda]$,
{s, 0, 2}], Assumptions $\rightarrow s > 0 \ \&\& \ 0 \leq \lambda < 1 \ \&\& \ 0 \leq x \leq 1]$`

`Out[ ]:=`

$$\left(e^{\frac{(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda}} + \frac{1+x+2x(-2+\lambda)\lambda}{-3+x-2(-2+\lambda)\lambda} \right) +$$

$$\left(-\frac{2(-1+x)(-1+\lambda)^2(4+\lambda(2+3\lambda)+x(2+\lambda(-14+3\lambda)))}{3(-3+x-2(-2+\lambda)\lambda)^2} + \right.$$

$$\left. \frac{e^{\frac{(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda}}(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda} \right) s +$$

$$\frac{1}{18}(-1+\lambda)^2 \left(\frac{1}{(-3+x-2(-2+\lambda)\lambda)^3} 8(-1+x) \right.$$

$$\left(-7+3x^2(1+\lambda(-14+(-5+\lambda)^2\lambda)) + \lambda(-14+\lambda(27+\lambda(22+\lambda(-22-3(-2+\lambda)\lambda)))) + \right.$$

$$\left. x(10+\lambda(8+\lambda(30+\lambda(-160+\lambda(121+3(-10+\lambda)\lambda)))) \right) +$$

$$\left. \frac{9e^{\frac{(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda}}(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])^2}{\lambda^2} \right) s^2 + O[s]^3$$

The above is slow, there we use the result to define

$$\text{fGPser2}[s_, \lambda_][x_]:= \left(e^{\frac{(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda}} + \frac{1+x+2x(-2+\lambda)\lambda}{-3+x-2(-2+\lambda)\lambda} \right) +$$

$$\left(-\frac{2(-1+x)(-1+\lambda)^2(4+\lambda(2+3\lambda)+x(2+\lambda(-14+3\lambda)))}{3(-3+x-2(-2+\lambda)\lambda)^2} + \right.$$

$$\left. \frac{e^{\frac{(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda}}(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda} \right) s +$$

$$\frac{1}{18}(-1+\lambda)^2 \left(\frac{1}{(-3+x-2(-2+\lambda)\lambda)^3} 8(-1+x) \right.$$

$$\lambda(-14+\lambda(27+\lambda(22+\lambda(-22-3(-2+\lambda)\lambda)))) +$$

$$x(10+\lambda(8+\lambda(30+\lambda(-160+\lambda(121+3(-10+\lambda)\lambda)))) \right) +$$

$$\left. \frac{9e^{\frac{(-1+\lambda)(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])}{\lambda}}(\lambda + \text{ProductLog}[-e^{-\lambda} x \lambda])^2}{\lambda^2} \right) s^2$$

Analysis of fGPser2 (eqs. (5.35) - (5.39))

The following gives (5.35):

In[*]:= Simplify[Series[fGPser2[s, λ][0], {s, 0, 2}]]

Out[*]=

$$\left(e^{-1+\lambda} + \frac{1}{-3-2(-2+\lambda)\lambda} \right) + \left(e^{-1+\lambda}(-1+\lambda) + \frac{2(-1+\lambda)^2(4+\lambda(2+3\lambda))}{3(3-4\lambda+2\lambda^2)^2} \right) s + \frac{1}{18}(-1+\lambda)^2 \left(9e^{-1+\lambda} - \frac{8(-7+\lambda(-14+\lambda(27+\lambda(22+\lambda(-22-3(-2+\lambda)\lambda))))}{(-3-2(-2+\lambda)\lambda)^3} \right) s^2 + O[s]^3$$

Now show (5.36):

In[*]:= NSolve[$e^{-1+\lambda} + \frac{1}{-3+4\lambda-2\lambda^2} = 0$, λ, Reals]

Out[*]=

$$\{\{\lambda \rightarrow -1.84267\}, \{\lambda \rightarrow 0.25915\}, \{\lambda \rightarrow 1.\}\}$$

In[*]:= Simplify[Series[$\left(e^{-1+\lambda} + \frac{1}{-3-2(-2+\lambda)\lambda} \right) + \left(e^{-1+\lambda}(-1+\lambda) + \frac{2(-1+\lambda)^2(4+\lambda(2+3\lambda))}{3(3-4\lambda+2\lambda^2)^2} \right) s$ /. λ → λ0 + λ01 s, {s, 0, 1}]]

Out[*]=

$$\left(e^{-1+\lambda_0} + \frac{1}{-3-2(-2+\lambda_0)\lambda_0} \right) + \left(e^{-1+\lambda_0}(-1+\lambda_0) + \frac{2(-1+\lambda_0)^2(4+\lambda_0(2+3\lambda_0))}{3(3-4\lambda_0+2\lambda_0^2)^2} + e^{-1+\lambda_0}\lambda_{01} + \frac{4(-1+\lambda_0)\lambda_{01}}{(3-4\lambda_0+2\lambda_0^2)^2} \right) s + O[s]^2$$

In[*]:= NSolve[$\left(\left(e^{-1+\lambda_0}(-1+\lambda_0) + \frac{2(-1+\lambda_0)^2(4+\lambda_0(2+3\lambda_0))}{3(3-4\lambda_0+2\lambda_0^2)^2} + e^{-1+\lambda_0}\lambda_{01} + \frac{4(-1+\lambda_0)\lambda_{01}}{(3-4\lambda_0+2\lambda_0^2)^2} \right) \right) / . \lambda_0 \rightarrow 0.2591495702201173 = 0$, λ01]

Out[*]=

$$\{\{\lambda_{01} \rightarrow 0.199726\}\}$$

Show (5.37):

In[*]:= Simplify[Series[(1+s) gammaGPser[s, 4], {s, 0, 3}]]

Out[*]=

$$1 + \frac{1}{3}(-1+4\lambda)s^2 - \frac{2}{9}(-1+8\lambda+2\lambda^2)s^3 + O[s]^4$$

In[*]:= Simplify[Solve[$\frac{1}{3}(-1+4\lambda) - \frac{2}{9}(-1+8\lambda+2\lambda^2)s = 0$, λ]]

Out[*]=

$$\left\{ \left\{ \lambda \rightarrow \frac{3-4s-3\sqrt{1-3s+2s^2}}{2s} \right\}, \left\{ \lambda \rightarrow \frac{3-4s+3\sqrt{1-3s+2s^2}}{2s} \right\} \right\}$$

In[*]:= Simplify[Series[$\frac{3-4s-3\sqrt{1-3s+2s^2}}{2s}$, {s, 0, 1}]]

Out[*]=

$$\frac{1}{4} + \frac{3s}{16} + O[s]^2$$

Show (5.38) and (5.39):

```

In[*]:= FullSimplify[Series[D[fGPser2[s, λ][x], {x, 2}] /. x → 1 - SinfGPser[s, 4], {s, 0, 2}],
Assumptions → s > 0 && 0 ≤ λ < 1]

Out[*]:=

$$\frac{(1 - 4\lambda)s}{3(-1 + \lambda)^2} - \frac{(1 + \lambda(4 + \lambda(-74 + 3\lambda(4 + \lambda))))s^2}{9(-1 + \lambda)^2} + O[s]^3$$


In[*]:= Simplify[Series[ $\frac{(1 - 4\lambda)}{3} - \frac{(1 + \lambda(4 + \lambda(-74 + 3\lambda(4 + \lambda))))s}{9}$ , {λ, 1/4, 1}]] // Normal

Out[*]:=

$$\frac{69s}{256} + \left(-\frac{4}{3} + \frac{163s}{48}\right)\left(-\frac{1}{4} + \lambda\right)$$


In[*]:= Solve[% == 0, λ]

Out[*]:=

$$\left\{\left\{\lambda \rightarrow \frac{-256 + 445s}{16(-64 + 163s)}\right\}\right\}$$


In[*]:= N[Normal[Series[ $\frac{-256 + 445s}{16(-64 + 163s)}$ , {s, 0, 1}]]]

Out[*]:=
0.25 + 0.202148 s

```

7.4. The numerically exact version of fGP and Figure 5.1

We define fGP such that it is efficient for calculations required for graphics and tables. We use (3.3) and (3.4) in the ms:

```

In[*]:= fGP[μ_, λ_][x_] := (a1 = PinfGP[μ, λ];
a2 = gammaGP[μ, λ];
φGP[μ, λ][x] - φFL[ $\frac{1 - a2}{1 - a1 a2}$ ,  $\frac{a1(1 - a2)}{1 - a1 a2}$ ][x])

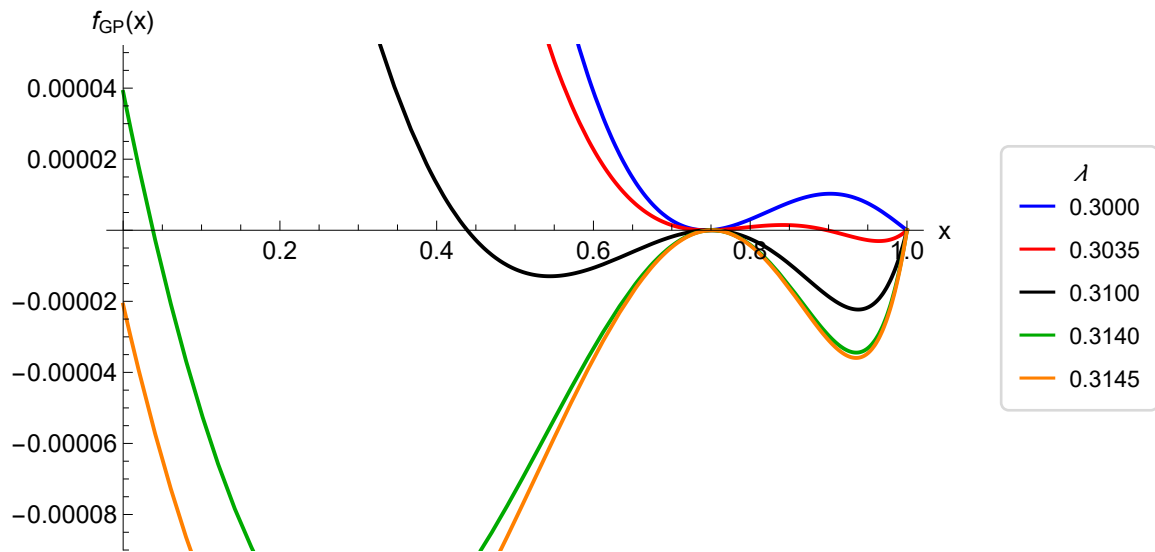
In[*]:= plotfGPfig =
Plot[{fGP[(1 + 0.3)(1 - 0.30), 0.30][x], fGP[(1 + 0.3)(1 - 0.3035), 0.3035][x],
fGP[(1 + 0.3)(1 - 0.31), 0.31][x], fGP[(1 + 0.3)(1 - 0.314), 0.314][x],
fGP[(1 + 0.3)(1 - 0.3145), 0.3145][x]}, {x, 0, 1}, AxesLabel → {"x", "fGP(x)"},
LabelStyle → {FontFamily → "Helvetica", FontSize → 13, Black},
PlotStyle → {Blue, Red, Black, Darker[Green], Orange},
PlotRange → {-0.00009, 0.000052}, ImageSize → 450];

In[*]:= plotlegend = LineLegend[{Blue, Red, Black, Darker[Green], Orange},
{"0.3000", "0.3035", "0.3100", "0.3140", "0.3145"}, LegendLabel → " λ",
LabelStyle → Directive[FontFamily → "Helvetica", FontSize → 12], LegendFunction →
(Framed[#, RoundingRadius → 4, FrameStyle → LightGray, ContentPadding → False] &)];

```

In[]:= Show[Legended[plotfGPfig, Placed[plotlegend, {1.05, 0.5}]], ImageSize → 600]

Out[]:=

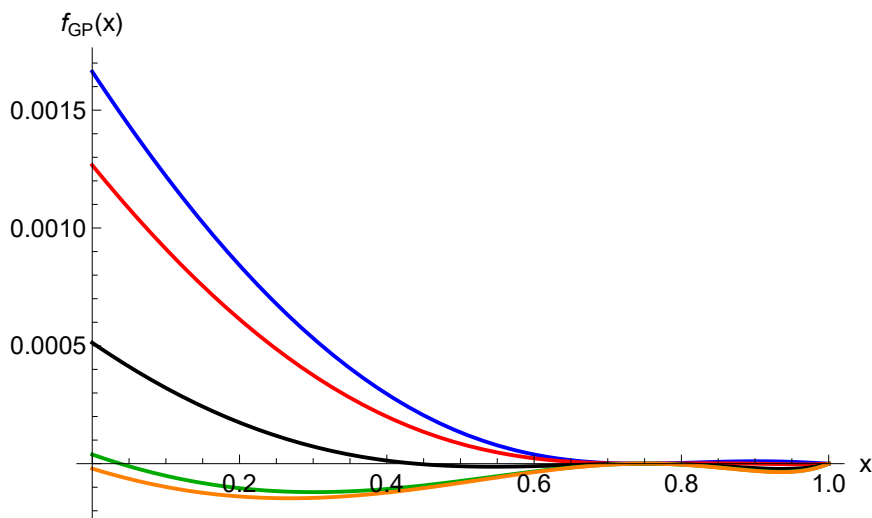


In[]:= plotfGPfigall =

```
Plot[{fGP[(1 + 0.3) (1 - 0.30), 0.30][x], fGP[(1 + 0.3) (1 - 0.3035), 0.3035][x],
      fGP[(1 + 0.3) (1 - 0.31), 0.31][x], fGP[(1 + 0.3) (1 - 0.314), 0.314][x],
      fGP[(1 + 0.3) (1 - 0.3145), 0.3145][x]}, {x, 0, 1}, AxesLabel → {"x", "fGP(x)"},
      LabelStyle → {FontFamily → "Helvetica", FontSize → 13, Black}, PlotStyle →
      {Blue, Red, Black, Darker[Green], Orange}, PlotRange → All, ImageSize → 450];
```

In[]:= plotfGPfigall

Out[]:=



8. Convergence time $T_\phi(\epsilon)$ and graphs of relative errors of survival probabilities $S_\phi^{(n)}$ (Sects. 6.1, 6.2)

8.1 Calculate $T_{\text{app}}(\epsilon)$ and $T_{\text{ser}}(\epsilon)$

Definitions

The following is the approximation (6.1) obtained from the bound by the fractional linear distribution:

```
In[*]:= TepFLapp[Pinf_, gam_, ε_] := Ceiling[ $\frac{\text{Log}[(1 + 1/\epsilon) \text{Pinf}]}{-\text{Log}[\text{gam}]}$ ]
```

Show (6.2):

```
In[*]:= gammaser[s, 3]
Out[*]= -s + s^2 γ[2] - s^3 γ[3]

In[*]:= Simplify[Series[ $\frac{\text{Log}[(1 + 1/\epsilon) \text{Pinf}]}{-\text{Log}[\text{gam}]}$  /. Pinf → 1 - Sinfser[s, 3] /. gam → gammaser[s, 3] /.
  substderiv, {s, 0, 1}], Assumptions → s > 0]
Out[*]= 
$$\frac{\text{Log}\left[1 + \frac{1}{\epsilon}\right]}{s} + \left(-\theta + \text{Log}\left[1 + \frac{1}{\epsilon}\right] \left(-\frac{1}{2} + \gamma[2]\right)\right) +$$


$$\left(\text{Log}\left[1 + \frac{1}{\epsilon}\right] \left(-\frac{1}{12} + \gamma[2]^2 - \gamma[3]\right) + \frac{1}{2} (\theta - \theta^2 - 2\theta\gamma[2] + 2\delta[2])\right) s + O[s]^2$$


In[*]:= TepSer[s_, ε_] := Ceiling[ $\left(\frac{1}{s} - \frac{1}{2} + \gamma[2]\right) \text{Log}\left[1 + \frac{1}{\epsilon}\right] - \theta$ ]
```

Calculate values in Table 6.1

The following computes TepFLapp for the generating functions introduced above

```
In[*]:= TepFLPoi[m_, ε_] := TepFLapp[PinfPoi[m], gammaPoi[m], ε] // Ceiling;
TepFLGP[μ_, λ_, ε_] := TepFLapp[PinfGP[μ, λ], gammaGP[μ, λ], ε] // Ceiling;
TepFLBin[n_, p_, ε_] := TepFLapp[PinfBin[n, p], gammaBin[n, p], ε] // Ceiling;
TepFLNB[n_, p_, ε_] := TepFLapp[PinfNB[n, p], gammaNB[n, p], ε] // Ceiling;
TepFLF3[p0_, p2_, p3_, ε_] :=
  TepFLapp[PinfF3[p0, p2, p3], gammaF3[p0, p2, p3], ε] // Ceiling;
TepFLFL[p_, r_, ε_] := TepFLapp[PinfFL[p, r], gammaFL[p, r], ε] // Ceiling;
```

The following yields the numerically exact value by iteration of the generating function:

```

In[*]:= TepPoi[m_, ε_] :=
  (NestWhileList[φPoi[m], 0, 1 - #1 > (1 + ε) (1 - PinfPoi[m]) &] // Length) - 1;
TepListPoi[m_, ε_] := NestWhileList[φPoi[m], 0, 1 - #1 > (1 + ε) (1 - PinfPoi[m]) &];
TepGP[μ_, λ_, ε_] :=
  (NestWhileList[φGP[μ, λ], 0, 1 - #1 > (1 + ε) (1 - PinfGP[μ, λ]) &] // Length) - 1;
TepBin[n_, p_, ε_] :=
  (NestWhileList[φBin[n, p], 0, 1 - #1 > (1 + ε) (1 - PinfBin[n, p]) &] // Length) - 1;
TepNB[n_, p_, ε_] :=
  (NestWhileList[φNB[n, p], 0, 1 - #1 > (1 + ε) (1 - PinfNB[n, p]) &] // Length) - 1;
TepF3[p0_, p2_, p3_, ε_] := (NestWhileList[φF3[p0, p2, p3],
  0, 1 - #1 > (1 + ε) (1 - PinfF3[p0, p2, p3]) &] // Length) - 1;
TepFL[p_, r_, ε_] :=
  (NestWhileList[φFL[p, r], 0, 1 - #1 > (1 + ε) (1 - PinfFL[p, r]) &] // Length) - 1;

```

Some examples :

s = 0.01

```

In[*]:= Table[{"ε" → ε, Table[TepGP[(1 + 0.01) (1 - λ), λ, ε], {λ, {0, 0.1, 0.259, 0.5, 0.9}}]},
  {ε, {0.01}}]
Out[*]=
{{ε → 0.01, {459, 461, 463, 467, 474}}}

In[*]:= Table[{"ε" → ε, Table[TepFLGP[(1 + 0.01) (1 - λ), λ, ε], {λ, {0, 0.1, 0.259, 0.5, 0.9}}]},
  {ε, {0.01}}]
Out[*]=
{{ε → 0.01, {461, 462, 463, 465, 468}}}

In[*]:= Table[
  {"ε" → ε, Table[TepSer[0.01, ε] /. γlist /. substμGP, {λ, {0, 0.1, 0.259, 0.5, 0.9}}]},
  {ε, {0.01}}]
Out[*]=
{{ε → 0.01, {461, 462, 463, 465, 468}}}

```

s = 0.1

```

In[*]:= Table[{"ε" → ε, Table[TepGP[(1 + 0.1) (1 - λ), λ, ε], {λ, {0, 0.1, 0.276, 0.5, 0.9}}]},
  {ε, {0.1, 0.01, 0.0001}}]
Out[*]=
{{ε → 0.1, {22, 23, 25, 27, 31}},
 {ε → 0.01, {45, 46, 48, 51, 56}}, {ε → 0.0001, {91, 93, 96, 100, 108}}}

In[*]:= Table[{"ε" → ε, Table[TepFLGP[(1 + 0.1) (1 - λ), λ, ε], {λ, {0, 0.1, 0.276, 0.5, 0.9}}]},
  {ε, {0.1, 0.01, 0.0001}}]
Out[*]=
{{ε → 0.1, {23, 24, 25, 26, 28}},
 {ε → 0.01, {45, 46, 48, 50, 53}}, {ε → 0.0001, {92, 94, 96, 99, 105}}}

In[*]:= Table[
  {"ε" → ε, Table[TepSer[0.1, ε] /. γlist /. substμGP, {λ, {0, 0.1, 0.276, 0.5, 0.9}}]},
  {ε, {0.1, 0.01, 0.0001}}]
Out[*]=
{{ε → 0.1, {23, 24, 25, 26, 28}},
 {ε → 0.01, {45, 46, 48, 50, 53}}, {ε → 0.0001, {92, 94, 96, 100, 105}}}

```

```

In[*]:= Table[{{"ε" → ε, TepNB[5,  $\frac{5}{5+1.1}$ , ε]}, {ε, {0.1, 0.01, 0.0001}}}]
Out[*]=
{{ε → 0.1, 23}, {ε → 0.01, 46}, {ε → 0.0001, 93}}

In[*]:= Table[{{"ε" → ε, TepFLNB[5,  $\frac{5}{5+1.1}$ , ε]}, {ε, {0.1, 0.01, 0.0001}}}]
Out[*]=
{{ε → 0.1, 23}, {ε → 0.01, 46}, {ε → 0.0001, 93}}

In[*]:= Table[{"ε" → ε, TepSer[0.1, ε] /.  $\gamma$ list /. subst $\mu$ NB /. n → 5}, {ε, {0.1, 0.01, 0.0001}}]
Out[*]=
{{ε → 0.1, 23}, {ε → 0.01, 46}, {ε → 0.0001, 93}}

s = 0.3

In[*]:= Table[{"ε" → ε, Table[TepGP[(1 + 0.3) (1 - λ), λ, ε], {λ, {0, 0.2, 0.312, 0.5, 0.9}}]},
{ε, {0.01}}]
Out[*]=
{{ε → 0.01, {14, 16, 17, 19, 24}}}}

In[*]:= Table[{"ε" → ε, Table[TepFLGP[(1 + 0.3) (1 - λ), λ, ε], {λ, {0, 0.2, 0.312, 0.5, 0.9}}]},
{ε, {0.01}}]
Out[*]=
{{ε → 0.01, {15, 16, 17, 19, 22}}}}

In[*]:= Table[
{"ε" → ε, Table[TepSer[0.3, ε] /.  $\gamma$ list /. subst $\mu$ GP, {λ, {0, 0.2, 0.312, 0.5, 0.9}}]},
{ε, {0.01}}]
Out[*]=
{{ε → 0.01, {15, 17, 18, 19, 22}}}}

```

8.2 Relative errors of survival probabilities $S_{\varphi}^{(n)}$ and Figure 6.1

Define iterates 1,2,3,...,t of generating functions

```

In[*]:=  $\varphi$ itPoi[m_, t_][x_] := Delete[NestList[ $\varphi$ Poi[m], x, t], 1];
 $\varphi$ itFL[p_, r_, t_][x_] := Delete[NestList[ $\varphi$ FL[p, r], x, t], 1];
 $\varphi$ itBin[n_, p_, t_][x_] := Delete[NestList[ $\varphi$ Bin[n, p], x, t], 1];
 $\varphi$ itNB[n_, p_, t_][x_] := Delete[NestList[ $\varphi$ NB[n, p], x, t], 1];
 $\varphi$ itF3[p0_, p2_, p3_, t_][x_] := Delete[NestList[ $\varphi$ F3[p0, p2, p3], x, t], 1];
 $\varphi$ itGP[μ_, λ_, t_][x_] := Delete[NestList[ $\varphi$ GP[μ, λ], x, t], 1];

In[*]:=  $\varphi$ itPoi[1.1, 5][0.3]
Out[*]=
{0.463013, 0.553947, 0.612223, 0.652754, 0.682515}

```

The following are the iterates of the fractional linear approximation to the given generating function

```

In[*]:=  $\phi\text{itPoiFL}[m\_ , t\_ ][x\_ ] := (\phi\text{itFL}[p, r, t] /. \text{substprPoi}[m]) [x];$ 
 $\phi\text{itBinFL}[n\_ , pp\_ , t\_ ][x\_ ] := (\phi\text{itFL}[p, r, t] /. \text{substprBin}[n, pp]) [x];$ 
 $\phi\text{itNBFL}[n\_ , pp\_ , t\_ ][x\_ ] := (\phi\text{itFL}[p, r, t] /. \text{substprNB}[n, pp]) [x];$ 
 $\phi\text{itF3FL}[p0\_ , p2\_ , p3\_ , t\_ ][x\_ ] := (\phi\text{itFL}[p, r, t] /. \text{substprF3}[p0, p2, p3]) [x];$ 
 $\phi\text{itGPFL}[\mu\_ , \lambda\_ , t\_ ][x\_ ] := (\phi\text{itFL}[p, r, t] /. \text{substprGP}[\mu, \lambda]) [x];$ 

```

The following gives yields extinction probabilities from generation 1 -- 15 :

```

In[*]:=  $\phi\text{itBin}[10, 0.12, 15] [0]$ 
Out[*]= {0.278501, 0.404302, 0.476316, 0.52255, 0.554323, 0.577154, 0.59408,
0.606915, 0.616812, 0.624544, 0.630645, 0.635497, 0.639379, 0.6425, 0.64502}

```

Plot relative errors of extinction probabilities,
 $(\phi\text{itGP}[\text{pars}][0] - \phi\text{itGPFL}[\text{pars}][0]) / (1 - \phi\text{itGP}[\text{pars}][0])$, for the generalized
Poisson

Generalized Poisson for $s = 0.1$, λ given in the legend

```

In[*]:= plotSnGP = Show[{ListPlot[( $\phi\text{itGP}[1.1, 0, 40] [0] - \phi\text{itGPFL}[1.1, 0, 40] [0]$ ) /  

(1 -  $\phi\text{itGP}[1.1, 0, 40] [0]$ ), PlotStyle → Red, PlotRange → All],  

ListPlot[( $\phi\text{itGP}[1.1 (1 - 0.1), 0.1, 40] [0] - \phi\text{itGPFL}[1.1 (1 - 0.1), 0.1, 40] [0]$ ) /  

(1 -  $\phi\text{itGP}[1.1 (1 - 0.1), 0.1, 40] [0]$ ), PlotStyle → Blue, PlotRange → All],  

ListPlot[  

( $\phi\text{itGP}[1.1 (1 - 0.276), 0.276, 40] [0] - \phi\text{itGPFL}[1.1 (1 - 0.276), 0.276, 40] [0]$ ) /  

(1 -  $\phi\text{itGP}[1.1 (1 - 0.276), 0.276, 40] [0]$ ), PlotStyle → Orange, PlotRange → All],  

ListPlot[( $\phi\text{itGP}[1.1 (1 - 0.4), 0.4, 40] [0] - \phi\text{itGPFL}[1.1 (1 - 0.4), 0.4, 40] [0]$ ) /  

(1 -  $\phi\text{itGP}[1.1 (1 - 0.4), 0.4, 40] [0]$ ), PlotStyle → Black, PlotRange → All],  

ListPlot[( $\phi\text{itGP}[1.1 (1 - 0.5), 0.5, 40] [0] - \phi\text{itGPFL}[1.1 (1 - 0.5), 0.5, 40] [0]$ ) /  

(1 -  $\phi\text{itGP}[1.1 (1 - 0.5), 0.5, 40] [0]$ ), PlotStyle → Darker[Green], PlotRange → All]},  

PlotRange → All, AxesOrigin → {1, 0}, ImageSize → 450];

In[*]:= plotlegendSnGP = PointLegend[{Red, Blue, Orange, Black, Darker[Green]},  

{"0", "0.1", "0.276", "0.4", "0.5"}, LegendLabel → " $\lambda$  ",  

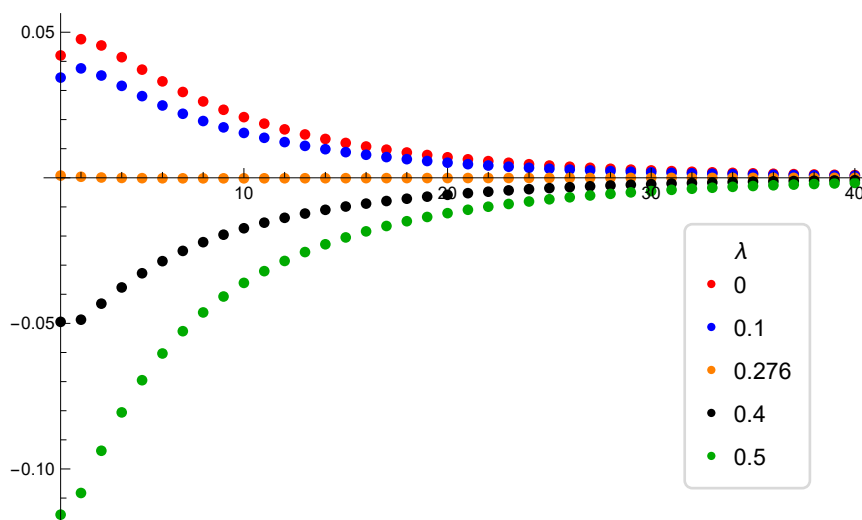
LabelStyle → Directive[FontFamily → "Helvetica", FontSize → 12], LegendFunction →  

(Framed[#, RoundingRadius → 4, FrameStyle → LightGray, ContentPadding → False] &)];

```

```
In[ ]:= Show[Legended[plotSnGP, Placed[plotlegendSnGP, {0.85, 0.33}]]]
```

```
Out[ ]:=
```



8.3 Series approximation of $S_{GP}^{(n)}$

Approximate $S_{GP}^{(n)}$ by using SinfGPser and gammaGPser

We define approximations up to s^3 :

```
In[ ]:= SinfGPser3[s_, λ_] :=
  2 (1 - λ)^2 s - 2/3 ((1 - λ)^2 (4 - 10 λ + 3 λ^2)) s^2 + 4/9 (1 - λ)^3 (7 - 31 λ + 21 λ^2 - 3 λ^3) s^3;

gammaGPser3[s_, λ_] := 1 - s + 2/3 (1 + 2 λ) s^2 - 4/9 (1 + 7 λ + λ^2) s^3
```

The following is the approx to S_n using the approx:

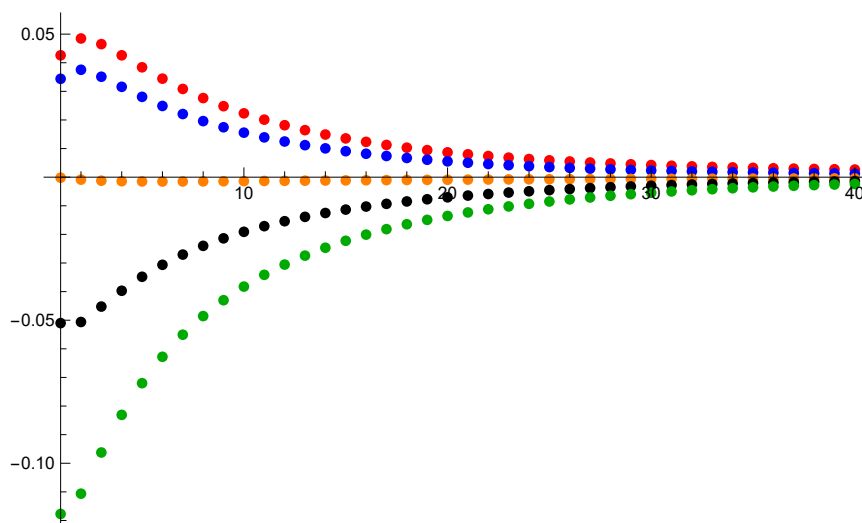
```
In[ ]:= SnGPser3[s_, λ_, n_] := SinfGPser3[s, λ] / (1 - gammaGPser3[s, λ]^n (1 - SinfGPser3[s, λ]));
```

The following shows the relative errors of this approx (for the same parameters as above):

```
In[*]:= plotSnGPser3 =
```

```
Show[{ListPlot[( $\phi$ itGP[1.1, 0, 40][0] - (1 - Table[SnGPser3[0.1, 0, n], {n, 1, 40}])) /
  (1 -  $\phi$ itGP[1.1, 0, 40][0]), PlotStyle → Red, PlotRange → All], ListPlot[
  ( $\phi$ itGP[1.1 (1 - 0.1), 0.1, 40][0] - (1 - Table[SnGPser3[0.1, 0.1, n], {n, 1, 40}])) /
  (1 -  $\phi$ itGP[1.1 (1 - 0.1), 0.1, 40][0]), PlotStyle → Blue, PlotRange → All],
  ListPlot[( $\phi$ itGP[1.1 (1 - 0.276), 0.276, 40][0] -
    (1 - Table[SnGPser3[0.1, 0.276, n], {n, 1, 40}])) /
    (1 -  $\phi$ itGP[1.1 (1 - 0.276), 0.276, 40][0]),
  PlotStyle → Orange, PlotRange → All], ListPlot[
  ( $\phi$ itGP[1.1 (1 - 0.4), 0.4, 40][0] - (1 - Table[SnGPser3[0.1, 0.4, n], {n, 1, 40}])) /
  (1 -  $\phi$ itGP[1.1 (1 - 0.4), 0.4, 40][0]), PlotStyle → Black, PlotRange → All], ListPlot[
  ( $\phi$ itGP[1.1 (1 - 0.5), 0.5, 40][0] - (1 - Table[SnGPser3[0.1, 0.5, n], {n, 1, 40}])) /
  (1 -  $\phi$ itGP[1.1 (1 - 0.5), 0.5, 40][0]), PlotStyle → Darker[Green], PlotRange → All],
  PlotRange → All, AxesOrigin → {1, 0}, ImageSize → 450]
```

```
Out[*]=
```



Because on this scale of resolution there is little visible difference to the above figure, below I show the relative errors of the precisely determined fractional linear approx and of the approx obtained by series expansion to order s^3 of Sinf . This is shown on a longer time scale. It illustrates the fact that the series approximation of Sinf has an error of $O(s^3)$.

```

In[ ]:= Show[ {ListPlot[ (φitGPFL[1.1, 0, 100][0] - (1 - Table[SnGPser3[0.1, 0, n], {n, 1, 100}])) /
  (1 - φitGPFL[1.1, 0, 100][0]), PlotStyle → Red, PlotRange → All], ListPlot[
  (φitGPFL[1.1 (1 - 0.1), 0.1, 100][0] - (1 - Table[SnGPser3[0.1, 0.1, n], {n, 1, 100}])) /
  (1 - φitGPFL[1.1 (1 - 0.1), 0.1, 100][0]), PlotStyle → Blue, PlotRange → All],
  ListPlot[ (φitGPFL[1.1 (1 - 0.276), 0.276, 100][0] -
    (1 - Table[SnGPser3[0.1, 0.276, n], {n, 1, 100}])) /
    (1 - φitGPFL[1.1 (1 - 0.276), 0.276, 100][0]),
  PlotStyle → Orange, PlotRange → All], ListPlot[
  (φitGPFL[1.1 (1 - 0.4), 0.4, 100][0] - (1 - Table[SnGPser3[0.1, 0.4, n], {n, 1, 100}])) /
  (1 - φitGPFL[1.1 (1 - 0.4), 0.4, 100][0]),
  PlotStyle → Black, PlotRange → All], ListPlot[
  (φitGPFL[1.1 (1 - 0.5), 0.5, 100][0] - (1 - Table[SnGPser3[0.1, 0.5, n], {n, 1, 100}])) /
  (1 - φitGPFL[1.1 (1 - 0.5), 0.5, 100][0]),
  PlotStyle → Darker[Green], PlotRange → All]},
  PlotRange → All, AxesOrigin → {1, 0}, ImageSize → 450]

```

Out[]=

