Supplement: Heuristic Solutions for Nonlinear Dynamic Pricing in the Presence of Multiunit Demand and Two-dimensional Customer Heterogeneity

S.1 Proof of Lemma 1

We will show the first part of the lemma in detail. The second part follows by a similar argumentation. For every $i \neq k$ with $i, k > j$ and $r_i - r_j \neq 0 \neq r_k - r_j$, it holds that $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \Leftrightarrow \frac{r_i - r_j}{r_k - r_j} = \frac{\sum_{m=j}^{i-1} l^m}{\sum_{m=j}^{k-1} l^m}$. W.l.o.g., we assume that $i > k$ (otherwise we could focus on $\frac{r_k - r_j}{r_i - r_j} = \frac{\sum_{m=j}^{k-1} l^m}{\sum_{m=j}^{i-1} l^m}$). We define the function $f(l) = \frac{\sum_{m=j}^{i-1} l^m}{\sum_{m=j}^{k-1} l^m}$ and note that $f(l)$ is continuously differentiable. By differentiating $f(l)$, we show that this function is monotonically increasing on the whole interval $(0, 1)$:

Applying the quotient rule, we note that

$$\frac{d}{dl} f(l) > 0 \Leftrightarrow \sum_{m=j}^{k-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{i-1} l^n - \sum_{m=j}^{i-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{k-1} l^n > 0 \, .$$

After some rearrangements of the sums, we get

$$\begin{aligned}
&\sum_{m=j}^{k-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{i-1} l^n - \sum_{m=j}^{i-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{k-1} l^n = \\
&\qquad = \sum_{m=1}^{k-j} \sum_{n=1}^{m} (i-n) l^{k+i-2-m} + \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (i+k-j-n-m) l^{k+i-2-m} \\
&\qquad + \sum_{m=i-j+1}^{k+i-1-2j} \sum_{n=m}^{k+i-1-2j} (i+k-1-j-n) l^{k+i-2-m} - \sum_{m=1}^{k-j} \sum_{n=1}^{m} (k-n) l^{k+i-2-m} \\
&\qquad - \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (k-n) l^{k+i-2-m} - \sum_{m=i-j+1}^{k+i-1-2j} \sum_{n=m}^{k+i-1-2j} (i+k-1-j-n) l^{k+i-2-m} \\
&\qquad = \sum_{m=1}^{k-j} \sum_{n=1}^{m} (i-k) l^{k+i-2-m} + \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (i-j-m) l^{k+i-2-m} > 0
\end{aligned}$$

for all $l \in (0, 1)$. Together with the independence of $\frac{r_i - r_j}{r_k - r_j}$ from l, it follows that there is at most one $l \in (0,1)$ such that $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$.

To show the second part of this lemma, we use a similar argumentation defining $g(l) = \frac{\sum_{m=k}^{j-1} l^m}{\sum_{m=i}^{j-1} l^m} = 1 - \frac{\sum_{m=i}^{k-1} l^m}{\sum_{m=i}^{j-1} l^m}$. □

S.2 Corollary S1

The proofs of Lemma 2, Lemma 4, and Proposition 3 rely on a specific property regarding the interplay of the curves $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}$, $i > j$. This property is stated in the following corollary.

Corollary S1 *For $j > k > i$, one of the following three cases applies:*

1. $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} < \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} < \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$
2. $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} = \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} = \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$
3. $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} > \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} > \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$

Proof: As $\frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} = \frac{\sum_{m=i}^{k-1} l^m}{\sum_{m=i}^{j-1} l^m} \cdot \frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} + \frac{\sum_{m=k}^{j-1} l^m}{\sum_{m=i}^{j-1} l^m} \cdot \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$ is a convex combination of $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m}$ and $\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$, the statement above immediately follows. □

S.3 Proof of Lemma 2

For any $i > j$, $\left[\underline{l}_{ij}^{min}(\boldsymbol{r}), \overline{l}_{ij}^{min}(\boldsymbol{r})\right]$ is defined such that $\max_{0 \le k \le j-1} \left\{\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}\right\} \le \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \min_{j+1 \le k \le c} \left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}\right\} \le 1$ for every $l \in \left[\underline{l}_{ij}^{min}(\boldsymbol{r}), \overline{l}_{ij}^{min}(\boldsymbol{r})\right]$ and $\left[\underline{l}_{ij}^{max}(\boldsymbol{r}), \overline{l}_{ij}^{max}(\boldsymbol{r})\right]$ such that $\max_{0 \le k \le i-1} \left\{\frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m}\right\} = \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \le \min\left\{\min_{i+1 \le k \le c} \left\{\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m}\right\}, 1\right\}$ for every $l \in \left[\underline{l}_{ij}^{max}(\boldsymbol{r}), \overline{l}_{ij}^{max}(\boldsymbol{r})\right]$. To show the equality of both intervals, we note that

a) $\max_{0 \le k \le j-1} \left\{\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}\right\} \le \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \Leftrightarrow \max_{0 \le k \le j-1} \left\{\frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m}\right\} \le \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}$

and

b) $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \min\left\{\min_{j+1 \le k \le c} \left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}\right\}, 1\right\}$

$$\Leftrightarrow \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \le \min_{j+1 \le k \le i-1} \left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}\right\}\right) \wedge \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \le \min\left\{\min_{i+1 \le k \le c} \left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}\right\}, 1\right\}\right)$$

$$\Leftrightarrow \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \geq \max_{j+1\leq k\leq i-1}\left\{\frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m}\right\}\right) \wedge \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \leq \min\left\{\min_{i+1\leq k\leq c}\left\{\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m}\right\}, 1\right\}\right).$$

Both a) and b) follow directly from Corollary S1 and, together, conclude the proof. □

S.4 Proof of Lemma 3

In this proof, we examine each of the four criteria outlined in the manuscript to demonstrate that these prices are indeed irrelevant, in the sense that we do not need them to find the optimal solution.

1. Let $r_j > r_{j+1}$. It follows that $X_{j+1} - X_j = \omega \cdot \lambda^j \geq 0 > r_{j+1} - r_j$. Consequently, all customers derive higher utility from purchasing $j+1$ units than from purchasing j units, leading to $p_j(\boldsymbol{r}) = 0$. However, the same result can be achieved by setting $r_j = r_{j+1}$. Since $\mathbb{P}(X_{j+1} - X_j > 0) = 1$ (based on the continuous distribution of λ and ω), it follows that $\mathbb{P}(X_{j+1} - X_j > r_{j+1} - r_j) = 1$ for $r_j = r_{j+1}$, which also results in $p_j(\boldsymbol{r}) = 0$.

2. Given that $X_j = \omega \sum_{k=0}^{j-1} \lambda^k = j$ only holds for $\lambda = 1$ and $\omega = 1$, it follows that $\mathbb{P}(X_j = j) = 0$. This implies that $p_j(\boldsymbol{r}) = 0$ for $r_j \geq j$. Consequently, we can drop $r_j > j$ and still have the possibility to achieve $p_j(\boldsymbol{r}) = 0$ by setting $r_j = j$.

3. The rationale behind dismissing prices r_j for which $r_j - r_{j-1} > 1$ is grounded in the following inequality: $\max_{0\leq k\leq j-1}\left\{\frac{r_j - r_k}{\sum_{i=k}^{j-1}\lambda^i}\right\} \geq \frac{r_j - r_{j-1}}{\lambda^{j-1}} \geq 1$, which emerges from the condition $r_j - r_{j-1} \geq 1$. Consequently this leads to $p_j(\boldsymbol{r}) = \int_0^1 \left(F_\omega\left(\min_{j+1\leq k\leq c}\left\{\frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}\right\}\right) - F_\omega\left(\max_{0\leq k\leq j-1}\left\{\frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i}\right\}\right)\right)^+ f_\lambda(l)\, dl = 0$. Therefore, maintaining r_j such that $r_j - r_{j-1} = 1$ in our action space suffices to nullify demand for j units if desired.

4. To prove the remaining part of Lemma 3, we assume that our price vector does not follow the given condition, i.e. we choose r_j such that $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}} \geq 0$. In this case, it holds that $\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j}$ for every $l \in \left[(r_{j+1} - r_j)^{\frac{1}{j}}, 1\right]$ as:

$$\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j} \Leftrightarrow l > \frac{r_{j+1} - r_j}{r_j - r_{j-1}}.$$

With $\frac{r_{j+1} - r_j}{r_j - r_{j-1}} = (r_{j+1} - r_j)^{\frac{1}{j}} \cdot \left(\frac{(r_{j+1} - r_j)^{\frac{1}{j}}}{(r_j - r_{j-1})^{\frac{1}{j-1}}}\right)^{j-1} < (r_{j+1} - r_j)^{\frac{1}{j}}$, it holds that the condition of the right side is met for every $l \in \left[(r_{j+1} - r_j)^{\frac{1}{j}}, 1\right]$. As $\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j}$, it holds that

$$w > \frac{r_j - r_{j-1}}{l^{j-1}} \Rightarrow w > \frac{r_{j+1} - r_j}{l^j}$$

and, thus,

$$w \cdot l^{j-1} > r_j - r_{j-1} \Rightarrow w \cdot l^j > r_{j+1} - r_j.$$

This finally results in $w \cdot \sum_{k=0}^{j} l^k - r_{j+1} \geq w \cdot \sum_{k=0}^{j-1} l^k - r_j$ for every $w \in [0,1], l \in \left[(r_{j+1} - r_j)^{\frac{1}{j}}, 1\right]$ such that $w \cdot \sum_{k=0}^{j-1} l^k - r_j = \max_{i \leq j}\{w \cdot \sum_{k=0}^{i-1} l^k - r_i\}$.

With a similar argumentation, we can conclude that $w \cdot \sum_{k=0}^{j-2} l^k - r_{j-1} \geq w \cdot \sum_{k=0}^{j-1} l^k - r_j$ for every $w \in [0,1], l \in \left[0, (r_{j+1} - r_j)^{\frac{1}{j}}\right]$ such that $w \cdot \sum_{k=0}^{j-1} l^k - r_j = \max_{i \leq j}\{w \cdot \sum_{k=0}^{i-1} l^k - r_i\}$.

All in all, we can summarize that for every batch size j and a corresponding batch price r_j such that $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}}$, there is no customer who is willing to purchase j units.

To eliminate demand for j units, the firm could also choose r_j such that $(r_j - r_{j-1})^{\frac{1}{j-1}} = (r_{j+1} - r_j)^{\frac{1}{j}}$. This choice is always possible because of the continuity of prices and $r_{j-1} \leq r_{j+1}$. Thereof, there is no need to considers prices r_j with $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}}$. □

S.5 Proof of Lemma 4

We recall that $\mathcal{R}_c = \left\{\boldsymbol{r} \in \mathbb{R}^c : 0 \leq r_1 \leq \cdots \leq r_c \leq c,\ r_j \leq j\ \forall j, (r_j - r_{j-1})^{\frac{1}{j-1}} \leq 1\ for\ j \geq 2, and\ (r_j - r_{j-1})^{\frac{1}{j-1}} \leq (r_{j+1} - r_j)^{\frac{1}{j}}\ for\ 2 \leq j \leq c-1\right\}$ and $\Lambda_j(\boldsymbol{r}) = \left\{l \in [0,1] \mid \max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}\right\} \leq 1 \leq \min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}\right\}\right\} = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r})\right]$.

We start with a special case by observing that for $\boldsymbol{r} \in \mathcal{R}_c$ it holds by definition that $r_{j+1} = r_j \Rightarrow r_j = r_{j-1}$ and $r_j > r_{j-1} \Rightarrow r_{j+1} > r_j$. Thus, either $r_1 < r_2 < \cdots < r_c$ or there is index i with $r_1 = r_2 = \cdots = r_i < r_{i+1} < \cdots < r_c$.

For the letter case, it holds that $\Lambda_1(\boldsymbol{r}) = \Lambda_2(\boldsymbol{r}) = \cdots = \Lambda_{i-1}(\boldsymbol{r}) = \emptyset$. There is no clear choice of $\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r}),\ j < i$. As $\lambda \neq l$ a.s. for any $l \in [0,1]$, we can as well choose $\underline{l}_j(\boldsymbol{r}), = \bar{l}_j(\boldsymbol{r}) = 0$ (for consistency) even though $\left[\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r})\right] \neq \emptyset$.

In the following, we will concentrate on j with $r_j < r_{j+1}$ regardless of whether j starts at 1 or at the aforementioned i.

Based on $l \in \Lambda_j(\boldsymbol{r}) \Rightarrow 1 \leq \min_{j+1 \leq k \leq c} \left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}\right\}$, we are interested in the order the lines $f_{kj}(l) = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$, $k \geq j+1$, are dropping below 1 for given $\boldsymbol{r} \in \mathcal{R}_c$. As $f_{kj}(l)$ is decreasing in $l \in [0,1]$ this happens at most once on $[0,1]$.

First, we note that $\frac{r_{j+1} - r_j}{l^j} = 1 \Leftrightarrow l = (r_{j+1} - r_j)^{\frac{1}{j}}$ as $r_j < r_{j+1}$. Thereof, we can already conclude from $\boldsymbol{r} \in \mathcal{R}_c$ that $f_{k+1,k}(l)$ drops below 1 before (or at the same time as) $f_{k+2,k+1}(l)$ does, for every $k \geq j$. By Corollary S1, we know that $f_{k+2,k}(l)$ is dropping below 1 between $f_{k+1,k}(l)$ and $f_{k+2,k+1}(l)$. Repeatedly applying Corollary S1 shows that $f_{k+3,k}(l)$ is between $f_{k+2,k}(l)$ and $f_{k+3,k+2}(l)$, $f_{k+4,i}(l)$ is between $f_{k+3,k}(l)$ and $f_{k+4,k+3}(l)$, and so on. We can therefore conclude that the correct order of lines $f_{kj}(l)$, $k > j$, dropping below 1 is $f_{j+1,j}(l)$, $f_{j+2,j}(l), \ldots, f_{cj}(l)$. Most importantly, the last $l \in [0,1]$ with $1 = \min_{j+1 \leq k \leq c}\{f_{kj}(l), 1\}$ is the point where $f_{j+1,j}(l) = 1$. At this point $f_{kj}(l) \geq 1$, $k > j$, because of the given order and the fact that every $f_{kj}(l)$ is decreasing in l. This point is given by $l = (r_{j+1} - r_j)^{\frac{1}{j}}$ and we know that the condition $1 \leq \min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}\right\}$ is met for every $l \leq (r_{j+1} - r_j)^{\frac{1}{j}}$.

We now can set $\bar{l}_j(\boldsymbol{r}) = (r_{j+1} - r_j)^{\frac{1}{j}}$. For $j = c$, the condition is always met, and we set $\bar{l}_c(\boldsymbol{r}) = 1$.

We now concentrate on $\max_{0 \leq k \leq j-1}\{f_{jk}(l)\} \leq 1$. For $j = 1$, this boils down to $r_1 \leq 1$ and is always fulfilled for $\boldsymbol{r} \in \mathcal{R}_c$. In this case we set $\underline{l}_1(\boldsymbol{r}) = 0$. If there is an index i with $r_1 = r_2 = \cdots = r_i < r_{i+1} < \cdots < r_c$, then $\max_{0 \leq k \leq i-1}\{f_{ik}(l)\} = 0$. In this case, we set $\underline{l}_i(\boldsymbol{r}) = 0$. For the remaining cases, we again apply Corollary S1. With the same argumentation as above, we can conclude that the correct order of lines $f_{jk}(l)$, $j > k$, dropping below 1 is $f_{j0}(l)$, $f_{j1}(l), \ldots, f_{j,j-1}(l)$. Some of these $f_{jk}(l)$, $k < j$, might be below 1 for every $l \in [0,1]$. However, this does not affect the proof of this Lemma. Particularly, it holds that $f_{j,j-1}(l) \leq 1 \Leftrightarrow f_{jk}(l) \leq 1 \; \forall k < j \Leftrightarrow \max_{0 \leq k \leq j-1}\{f_{jk}(l)\} \leq 1$ because of the given order and the fact that every $f_{jk}(l)$ is decreasing in l. Therefore, the condition $\max_{0 \leq k \leq j-1}\{f_{jk}(l)\} \leq 1$ is met for every $l \geq (r_j - r_{j-1})^{\frac{1}{j-1}}$. We now can set $\underline{l}_j(\boldsymbol{r}) = (r_j - r_{j-1})^{\frac{1}{j-1}}$.

To sum up, we have shown that $\underline{l}_1(\boldsymbol{r}) = 0$, $\bar{l}_c(\boldsymbol{r}) = 1$ and $\bar{l}_j(\boldsymbol{r}) = (r_{j+1} - r_j)^{\frac{1}{j}} = \underline{l}_{j+1}(\boldsymbol{r})$ and, thus, the whole lemma. □

S.6 Proof of Lemma 5

Exemplarily, we will walk one of these cases through and focus on $\boldsymbol{i > j}$. The other two would follow in a similar matter and we decided to omit a detailed derivation.

Partially differentiating (9) leads to

$$
\begin{aligned}
\frac{\partial}{\partial\, r_i} p_j(\boldsymbol{r}) ={}& f_\lambda\left(\underline{l}_{j+1}(\boldsymbol{r})\right) \cdot \frac{\partial}{\partial\, r_i} \underline{l}_{j+1}(\boldsymbol{r}) - f_\lambda\left(\underline{l}_j(\boldsymbol{r})\right) \cdot \frac{\partial}{\partial\, r_i} \underline{l}_j(\boldsymbol{r}) \\
&+ \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{1}{\sum_{m=j}^{i-1} l^m}\, f_\omega\left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}\right) f_\lambda(l)\, dl \\
&+ \sum_{k=j+1}^{c} \left(F_\omega\left(\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m}\right) \cdot f_\lambda\left(\bar{l}_{kj}(\boldsymbol{r})\right) \cdot \frac{\partial}{\partial\, r_i} \bar{l}_{kj}(\boldsymbol{r}) - F_\omega\left(\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\underline{l}_{kj}(\boldsymbol{r})\right)^m}\right) \right. \\
&\left. \cdot f_\lambda\left(\underline{l}_{kj}(\boldsymbol{r})\right) \cdot \frac{\partial}{\partial\, r_i} \underline{l}_{kj}(\boldsymbol{r}) \right) \\
&- \sum_{k=0}^{j-1} \left(F_\omega\left(\frac{r_j - r_k}{\sum_{m=k}^{j-1}\left(\bar{l}_{jk}(\boldsymbol{r})\right)^m}\right) \cdot f_\lambda\left(\bar{l}_{jk}(\boldsymbol{r})\right) \cdot \frac{\partial}{\partial\, r_i} \bar{l}_{jk}(\boldsymbol{r}) - F_\omega\left(\frac{r_j - r_k}{\sum_{m=k}^{j-1}\left(\underline{l}_{jk}(\boldsymbol{r})\right)^m}\right) \right. \\
&\left. \cdot f_\lambda\left(\underline{l}_{jk}(\boldsymbol{r})\right) \cdot \frac{\partial}{\partial\, r_i} \underline{l}_{jk}(\boldsymbol{r}) \right) = \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{1}{\sum_{m=j}^{i-1} l^m}\, f_\omega\left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}\right) f_\lambda(l)\, dl.
\end{aligned}
$$

The last equation holds because there are many terms that neutralize each other. This can be observed by:

- In Remark 1, we established that $\bar{l}_j^{end}(\boldsymbol{r})$ is the maximum of $\bar{l}_{kj}(\boldsymbol{r})$ for $k > j$ as well as the maximum of $\bar{l}_{jk}(\boldsymbol{r})$ for $k < j$. This can be expressed as $\bar{l}_j^{end}(\boldsymbol{r}) = \max_{k>j}\{\bar{l}_{kj}(\boldsymbol{r})\} = \max_{k<j}\{\bar{l}_{jk}(\boldsymbol{r})\}$. For now, we define $k^{max} = \arg\max_{k>j}\{\bar{l}_{kj}(\boldsymbol{r})\}$ and $k_{max} = \arg\max_{k<j}\{\bar{l}_{jk}(\boldsymbol{r})\}$. Using the same argumentation as outlined in Remark 2, we conclude that $\frac{r_{k^{max}} - r_j}{\sum_{m=j}^{k^{max}-1}\left(\bar{l}_{k^{max}j}(\boldsymbol{r})\right)^m} = \frac{r_j - r_{k_{max}}}{\sum_{m=k_{max}}^{j-1}\left(\bar{l}_{jk_{max}}(\boldsymbol{r})\right)^m}$ and $\frac{\partial}{\partial\, r_i} \bar{l}_{k^{max}j}(\boldsymbol{r}) = \frac{\partial}{\partial\, r_i} \bar{l}_{jk_{max}}(\boldsymbol{r})$.
- Moreover, we know from Remark 2 that for every $\bar{l}_{kj}(\boldsymbol{r}) \neq \bar{l}_j^{end}(\boldsymbol{r})$ there is $\underline{l}_{hj}(\boldsymbol{r})$, $h > j$, such that $\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m} = \frac{r_h - r_j}{\sum_{m=j}^{h-1}\left(\underline{l}_{hj}(\boldsymbol{r})\right)^m}$ and $\frac{\partial}{\partial\, r_i} \bar{l}_{kj}(\boldsymbol{r}) = \frac{\partial}{\partial\, r_i} \underline{l}_{hj}(\boldsymbol{r})$.
- The same conclusion holds for $\bar{l}_{jk}(\boldsymbol{r}) \neq \bar{l}_j^{end}(\boldsymbol{r})$ and its corresponding $\underline{l}_{jh}(\boldsymbol{r})$, $h < j$.
- Remark 2 further reveals that $\frac{\partial}{\partial\, r_i} \underline{l}_{j+1}(\boldsymbol{r}) = \frac{\partial}{\partial\, r_i} \underline{l}_{j+1,j}(\boldsymbol{r})$ and $\frac{r_{j+1} - r_j}{\left(\underline{l}_{j+1,j}(\boldsymbol{r})\right)^j} = 1$. Analogously, it holds that $\frac{\partial}{\partial\, r_i} \underline{l}_j(\boldsymbol{r}) = \frac{\partial}{\partial\, r_i} \underline{l}_{j,j-1}(\boldsymbol{r})$ and $\frac{r_j - r_{j-1}}{\left(\underline{l}_{j,j-1}(\boldsymbol{r})\right)^{j-1}} = 1$.

Using all of these equations, we get $\frac{\partial}{\partial r_i} p_j(\boldsymbol{r}) = \int_{\underline{l}_{ij}(r)}^{\bar{l}_{ij}(r)} \frac{1}{\sum_{m=j}^{i-1} l^m} f_\omega \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \right) f_\lambda(l)\, dl$. With similar steps, we can build the partial derivatives for cases $i < j$ and $i = j$. □

S.7 Proof of Proposition 1

An optimal solution for (7) is either found on the boundary of feasible region $\mathcal{R}_c$ or when it satisfies the first order condition $\frac{\partial}{\partial r_i} \left(\sum_{k=1}^{c} p_k(\boldsymbol{r}) \cdot \left(r_k - \Delta_k V_{t-1}(c) \right) \right) = 0$ for every i. The condition for optimality in the interior is straightforward; however, exploring optimality on the boundary requires additional consideration: A solution residing on the boundary is marked by at least one batch size j for which $p_j(\boldsymbol{r}) = 0$. For simplicity, let us consider the scenario with only one such batch size j, though the argument can be extended to multiple batch sizes where demand is nullified.

Eliminating the option to purchase j units a priori and reapplying the optimization yields an identical optimality condition, except the summation now omits the index j. This index can be reintegrated into the summation because, by definition $\underline{l}_{jk}(\boldsymbol{r}) = \bar{l}_{jk}(\boldsymbol{r})$ ($\underline{l}_{kj}(\boldsymbol{r}) = \bar{l}_{kj}(\boldsymbol{r})$) for $j > k$ ($j < k$), leading to $\int_{\underline{l}_{jk}(r)}^{\bar{l}_{jk}(r)} \frac{1}{\sum_{m=k}^{j-1} l^m} f_\omega \left(\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \right) f_\lambda(l)\, dl = 0$ ($\int_{\underline{l}_{kj}(r)}^{\bar{l}_{kj}(r)} \frac{1}{\sum_{m=j}^{k-1} l^m} f_\omega \left(\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \right) f_\lambda(l)\, dl = 0$). Particularly, it holds for every j with $p_j(\boldsymbol{r}) = 0$ that $\frac{\partial}{\partial r_j} \left(\sum_{k=1}^{c} p_k(\boldsymbol{r}) \cdot \left(r_k - \Delta_k V_{t-1}(c) \right) \right) = 0$.

In summary, whether on the boundary or in the interior, the optimal solution must satisfy the first order condition $\frac{\partial}{\partial r_i} \left(\sum_{k=1}^{c} p_k(\boldsymbol{r}) \cdot \left(r_k - \Delta_k V_{t-1}(c) \right) \right) = 0$ for every i, reinforcing the comprehensive nature of the optimality condition across the entire decision space. □

S.8 Proof of Proposition 2

To prove that the optimal solution $\boldsymbol{r}^*$ is an interior point, we show $p_k(\boldsymbol{r}^*) \neq 0$ for every k by contradiction. To achieve this, we leverage the observation that $p_k(\boldsymbol{r}) \neq 0$ is equivalent to $\underline{l}_k(\boldsymbol{r}) < \underline{l}_{k+1}(\boldsymbol{r})$, for $\omega, \lambda \sim U[0, 1]$.

We prove Proposition 2 by contradiction, assuming that for the optimal solution $\boldsymbol{r}^* \in \mathcal{R}_c$ there exists a batch size k with $p_k(\boldsymbol{r}^*) = 0$. In the following, we focus on a single k. However, the technique we use in this proof could be repeatedly applied (with small adjustments) to contradict optimal solutions with several batch sizes k such that $p_k(\boldsymbol{r}^*) = 0$.

In a first step, we modify the optimal solution by slightly decreasing r_k^* to $r_k^* - \epsilon$ with $\epsilon > 0$ small enough. We denote this modified version by $\boldsymbol{r}^{*,\epsilon}$. With $\boldsymbol{r}^{*,\epsilon}$, the probability to sell a batch of size k is positive as $\underline{l}_k(\boldsymbol{r}^{*,\epsilon}) < \underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})$.

$\underline{k > 1}$:

We recall the argumentation used in Lemma 4 to conclude that $p_k(\boldsymbol{r}^{*,\epsilon}) = \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl$ with $\bar{l}(\boldsymbol{r}^{*,\epsilon}) = \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}} = \frac{r_{k+1}^{*} - r_k^{*} + \epsilon}{r_k^{*} - \epsilon - r_{k-1}^{*}}$. This $\bar{l}(\boldsymbol{r}^{*,\epsilon})$ exists for every ϵ that is small enough.

Plucking $\boldsymbol{r}^{*,\epsilon}$ into the first partial derivative leads to:

$$\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Bigg|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} = 2 \cdot p_k(\boldsymbol{r}^{*,\epsilon}) - \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right)$$

$$= 2 \cdot \left(\left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int\limits_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int\limits_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl\right)$$

$$- \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right).$$

As $1 \geq \frac{\left(r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}\right)^k}{\left(r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}\right)^{k-1}} \geq \frac{1}{2}$ for ϵ small enough, it holds that $\left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl < \frac{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})}{2}$. Consequently, it holds that $\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Big|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} < 0$.

Based on the negativity of the partial derivative, the objective function is strictly decreasing for every $\boldsymbol{r}^{*,\epsilon}$ (with $\epsilon > 0$ small enough). Thus, in contradiction to the optimality of $\boldsymbol{r}^*$, any $\boldsymbol{r}^{*,\epsilon}$ yields higher expected revenues than $\boldsymbol{r}^*$. A similar argumentation works for r_c if we assume $p_c(\boldsymbol{r}^*) = 0$.

$\underline{k = 1:}$

Our assumption of $p_1(\boldsymbol{r}^*) = 0$ implies that $r_1^* = r_2^*$. To show the suboptimality of this condition, we need to examine two cases: $r_2^* > \exp(-0.5)$ and $r_2^* \leq \exp(-0.5)$. First, we focus on $r_2^* > \exp(-0.5)$.

By definition of $p_1(\boldsymbol{r}^{*,\epsilon})$, $p_1(\boldsymbol{r}^{*,\epsilon}) = (r_2^{*,\epsilon} - r_1^{*,\epsilon}) + \int_{r_2^{*,\epsilon} - r_1^{*,\epsilon}}^{\frac{r_2^{*,\epsilon} - r_1^{*,\epsilon}}{r_1^{*,\epsilon}}} \frac{r_2^{*,\epsilon} - r_1^{*,\epsilon}}{l}\, dl - \int_0^{\frac{r_2^{*,\epsilon} - r_1^{*,\epsilon}}{r_1^{*,\epsilon}}} r_1^{*,\epsilon}\, dl$. Solving the integrals leads to $p_1(\boldsymbol{r}^{*,\epsilon}) = -\log(r_1^{*,\epsilon}) \cdot (r_2^{*,\epsilon} - r_1^{*,\epsilon})$. It holds that $\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Big|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} < 0$ for $r_1^{*,\epsilon} \in (\exp(-0.5), r_2^{*,\epsilon})$. From here on, the same argumentation as above applies.

Next, we focus on $r_2^{*,\epsilon} \leq \exp(-0.5)$. As every customer with $w \cdot (1 + l) \geq r_2^{*,\epsilon}$ is purchasing at least 2 units, we can conclude that $\sum_{j=1}^{c} p_j(\boldsymbol{r}^{*,\epsilon}) \geq 1 - \int_0^1 \frac{r_2^{*,\epsilon}}{1+l} dl = 1 - r_2^{*,\epsilon} \cdot (\log(2) - \log(1)) \geq 1 -$

$\exp(-0.5) \cdot \log(2) > 0.5$. Since the optimal solution must fulfill the condition $\sum_{j=1}^{c} p_j(\boldsymbol{r}) = 0.5$ (cf. Remark 3), $r_2^{*,\epsilon} \leq \exp(-0.5)$ cannot occur and has no relevance for our proof.

All in all, we have shown that $\boldsymbol{r}$ with r_k such that $p_k(\boldsymbol{r}) = 0$ cannot be an optimal solution. This ultimately leads to the statement that the optimal solution is an interior point of $\mathcal{R}_c$ and has to fulfill the first order condition. □

S.9 Calculation of $p_j^m(\boldsymbol{r})$ for $\boldsymbol{\omega}, \boldsymbol{\lambda} \sim \boldsymbol{U}[\mathbf{0}, \mathbf{1}]$

We now only have two possibilities for $\bar{l}_j^{end}(\boldsymbol{r})$ (cf. Remark 1):

1. $\frac{r_{j+1}-r_j}{r_j-r_{j-1}}$

   or

2. l such that $\frac{r_{j+1}-r_j}{l^j} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$.

The lower of these two possible values is $\bar{l}_j^{end}(\boldsymbol{r})$. If the first case applies, it holds that $\Lambda_{j0}^{max}(\boldsymbol{r}) = \emptyset$, $\Lambda_{j,j-1}^{max}(\boldsymbol{r}) = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$, and $\Lambda_{j+1,j}^{min}(\boldsymbol{r}) = \left[\underline{l}_{j+1}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$. If the second case applies, it holds that $\Lambda_{j0}^{max}(\boldsymbol{r}) = \left[\bar{l}_{j-1}^{\mathrm{end}}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$, $\Lambda_{j,j-1}^{max}(\boldsymbol{r}) = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_{j-1}^{end}(\boldsymbol{r})\right]$, and $\Lambda_{j+1,j}^{min}(\boldsymbol{r}) = \left[\underline{l}_{j+1}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$. To express the demand function for both cases with one formulation, we bring back the notation $\underline{l}_{ji}^{max}(\boldsymbol{r})$ and $\bar{l}_{ji}^{max}(\boldsymbol{r})$. In the first case, we choose $\underline{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_j(\boldsymbol{r})$ and $\bar{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_j^{end}(\boldsymbol{r})$. In the second case, we choose $\underline{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_j(\boldsymbol{r})$, $\bar{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_{j0}^{max}(\boldsymbol{r}) = \bar{\mathrm{l}}_{j-1}^{\mathrm{end}}(\boldsymbol{r})$, and $\bar{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_j^{end}(\boldsymbol{r})$.

Now, we can write $p_1^m(\boldsymbol{r}) = -(r_2 - r_1) \cdot \log(r_1)$, $p_2^m(\boldsymbol{r}) = -(r_3 - r_2) \cdot \left(\frac{1}{\bar{l}_2^{\mathrm{end}}(\boldsymbol{r})} - \frac{1}{\underline{l}_3(\boldsymbol{r})}\right) + \underline{l}_3(\boldsymbol{r}) - \underline{l}_2(\boldsymbol{r}) - r_2 \cdot \left(\log\left(1 + \bar{l}_2^{end}(\boldsymbol{r})\right) - \log\left(1 + \underline{l}_{20}^{max}(\boldsymbol{r})\right)\right) + (r_2 - r_1) \cdot \log(r_1)$, and $p_j^m(\boldsymbol{r}) = -\frac{r_{j+1}-r_j}{j-1} \cdot \left(\frac{1}{\left(\bar{l}_j^{\mathrm{end}}(\boldsymbol{r})\right)^{j-1}} - \frac{1}{\left(\underline{l}_{j+1}(\boldsymbol{r})\right)^{j-1}}\right) + \underline{l}_{j+1}(\boldsymbol{r}) - \underline{l}_j(\boldsymbol{r}) - \int_{\underline{l}_{j0}^{max}(\boldsymbol{r})}^{\bar{l}_j^{\mathrm{end}}(\boldsymbol{r})} \frac{r_j}{\sum_{m=0}^{j-1} l^m} dl + \frac{r_j - r_{j-1}}{j-2} \cdot \left(\frac{1}{\left(\bar{l}_{j,j-1}^{max}(\boldsymbol{r})\right)^{j-2}} - \frac{1}{\left(\underline{l}_j(\boldsymbol{r})\right)^{j-2}}\right)$.

S.10 Proof of Proposition 3

In the following proof, we assume that the order condition $\overline{l}_j^{end}(\boldsymbol{r}) \leq \overline{l}_{j+1}^{end}(\boldsymbol{r})$ for every j, with $1 \leq j < c$, is satisfied. In the following lines, we want to show that these conditions lead to a scenario, where only the events of selling $0, j-1$, and $j+1$ units limit the probability of selling j units. We recall that for any given $l \in [0,1]$, selling j units only occur iff

$$\max_{0\leq k\leq j-1}\left\{\frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i}\right\} \leq w \leq \min_{j+1\leq k\leq c}\left\{\frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}\right\}.$$

In our first step, we show that the expression of the upper bound $\min_{j+1\leq k\leq c}\left\{\frac{r_k-r_j}{\sum_{i=j}^{k-1} l^i}\right\}$ can be reduced to $\frac{r_{j+1}-r_j}{l^j}$ for $l \leq \overline{l}_j^{end}(\boldsymbol{r})$. Thereby, we prove that in the customer choice model the options of purchasing more than $j+1$ units are in no direct competition with the option of purchasing j units. Analogously, in the second step, we determine that the lower bound reduces to $\max\left\{\frac{r_j-r_{j-1}}{l^{j-1}}, \frac{r_j}{\sum_{m=0}^{j-1} l^m}\right\}$ for $l \leq \overline{l}_j^{end}(\boldsymbol{r})$, leaving only the events of selling 0 or $j-1$ units as relevant options for customers, who may purchase j units.

1. By definition of $\overline{l}_j^{end}(\boldsymbol{r})$, we know that $\frac{r_{j+1}-r_j}{l^j} \geq \frac{r_j-r_{j-1}}{l^{j-1}}$ for every $l \leq \overline{l}_j^{end}(\boldsymbol{r})$. Additionally, the order condition ensures that $\frac{r_{i+1}-r_i}{l^i} \geq \frac{r_i-r_{i-1}}{l^{i-1}}$ for every $i \geq j$ and $l \leq \overline{l}_j^{end}(\boldsymbol{r})$. Consequently, it follows that $\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m} \geq \frac{r_{j+1}-r_j}{l^j}$ for every $l \leq \overline{l}_j^{end}(\boldsymbol{r})$, which ultimately yields $\min_{j+1\leq k\leq c}\left\{\frac{r_k-r_j}{\sum_{i=j}^{k-1} l^i}\right\} = \frac{r_{j+1}-r_j}{l^j}$ for every $l \leq \overline{l}_j^{end}(\boldsymbol{r})$.

2. To prove that $\max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{i=k}^{j-1} l^i}\right\} = \max\left\{\frac{r_j-r_{j-1}}{l^{j-1}}, \frac{r_j}{\sum_{m=0}^{j-1} l^m}\right\}$ for $l \leq \overline{l}_j^{end}(\boldsymbol{r})$, we exclude the special case where $\overline{l}_j^{end}(\boldsymbol{r}) = 1$ for every j, with $1 \leq j < c$. This case applies for pricing schemes with increasing marginal prices. Since we use the term "increasing" in a weak sense, linear pricing schemes also belong to this special case. In this case, it is straightforward to verify that $\max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{i=k}^{j-1} l^i}\right\} = \frac{r_j-r_{j-1}}{l^{j-1}}$ for all l. Therefore, we omit this case and focus on the other cases, where at least $\overline{l}_1^{end}(\boldsymbol{r}) < 1$. Here, we prove $\max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}\right\} = \frac{r_j-r_{j-1}}{l^{j-1}}$ for $l \leq \overline{l}_{j-1}(\boldsymbol{r})$ and $\max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}\right\} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$ for $l \in \left(\overline{l}_{j-1}^{end}(\boldsymbol{r}), \overline{l}_j^{end}(\boldsymbol{r})\right]$. Similar to the first step, we know from the definition of $\overline{l}_{j-1}^{end}(\boldsymbol{r})$ that $\frac{r_j-r_{j-1}}{l^{j-1}} > \max_{0\leq i\leq j-2}\left\{\frac{r_{j-1}-r_i}{\sum_{m=i}^{j-2} l^m}\right\}$ for $l < \overline{l}_{j-1}^{end}(\boldsymbol{r})$. From Corollary S1, this order

carries over to $\frac{r_j-r_{j-1}}{l^{j-1}} > \max_{0\le i\le j-2}\left\{\frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}\right\}$ for $l < \overline{l}_{j-1}^{end}(\boldsymbol{r})$. Consequently, by continuity, it holds that $\frac{r_j-r_{j-1}}{l^{j-1}} \ge \max_{0\le i\le j-1}\left\{\frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}\right\}$ for $l \le \overline{l}_{j-1}^{end}(\boldsymbol{r})$.

Finally, we prove the last part, namely $\max_{0\le k\le j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}\right\} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$ for $l \in \left(\overline{l}_{j-1}^{end}(\boldsymbol{r}), \overline{l}_j^{end}(\boldsymbol{r})\right]$, by induction over j. We focus on scenarios where $\overline{l}_{j-1}^{end}(\boldsymbol{r}) < 1$, as the statement is obviously true for $\left(\overline{l}_{k-1}^{end}(\boldsymbol{r}), \overline{l}_k^{end}(\boldsymbol{r})\right] = \emptyset$ for all $k \ge j$. As induction hypothesis, we assume that for all $i < j$, it holds that $\max_{0\le k\le i-1}\left\{\frac{r_i-r_k}{\sum_{m=k}^{i-1} l^m}\right\} = \frac{r_i}{\sum_{m=0}^{i-1} l^m}$ for $l \in \left(\overline{l}_{i-1}^{end}(\boldsymbol{r}), \overline{l}_i^{end}(\boldsymbol{r})\right]$. The implication of this assumption is that $\frac{r_{i-1}}{\sum_{m=0}^{i-2} l^m} > \frac{r_i}{\sum_{m=0}^{i-1} l^m}$ for $l > \overline{l}_{i-1}^{end}(\boldsymbol{r})$, which follows from the induction hypothesis in combination with the definition of $\overline{l}_{i-1}^{end}(\boldsymbol{r})$, stating that $\frac{r_{i-1}}{\sum_{m=0}^{i-2} l^m} > \frac{r_i-r_{i-1}}{l^{i-1}}$ for $l > \overline{l}_{i-1}^{end}(\boldsymbol{r})$, and from Corollary S1.

Obviously, the base case holds with $i = 1$, as $\max_{0\le k\le i-1}\left\{\frac{r_i-r_k}{\sum_{m=k}^{i-1} l^m}\right\} = r_1$. For the induction step, we prove that $\max_{0\le k\le j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}\right\} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$ for $l > \overline{l}_{j-1}^{end}(\boldsymbol{r})$, under the assumption that this holds for $i < j$. As above, we conclude $\frac{r_{j-1}}{\sum_{m=0}^{j-2} l^m} > \frac{r_j}{\sum_{m=0}^{j-1} l^m}$ for $l > \overline{l}_{j-1}^{end}(\boldsymbol{r})$. The induction hypothesis, combined with $\overline{l}_i^{end}(\boldsymbol{r}) \le \overline{l}_{i+1}^{end}(\boldsymbol{r})$ for every $i < c$, implies that $\frac{r_i}{\sum_{m=0}^{i-1} l^m} > \frac{r_j}{\sum_{m=0}^{j-1} l^m}$ for every $i < j$ and $l > \overline{l}_{j-1}^{end}(\boldsymbol{r})$. By Corollary S1, we conclude that $\frac{r_j}{\sum_{m=0}^{j-1} l^m} > \frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}$ for every $i < j$ and $l > \overline{l}_{j-1}^{end}(\boldsymbol{r})$. Finally, this leads to $\max_{0\le k\le j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}\right\} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$ for $l > \overline{l}_{j-1}^{end}(\boldsymbol{r})$.

With these two steps, we have shown that batch prices that fulfill the stated condition lead to a certain structure of the probability function. Thereby, the probability of selling $j < c$ units is only limited by three events: selling $0, j-1$, and $j+1$ units. □

S.11 Trade-off between accuracy and efficiency for $E(\boldsymbol{\omega})$

In this section, we analyze the impact of the sample size of realizations of ω on revenues and runtimes. While a sample size of 100 was used in the base case, we examine the effects of varying sample sizes $s \in \{10, 20, 100, 1000\}$. For each sample size, we compute the corresponding policy, measure the runtimes for the computation, and simulate the resulting revenues.

As shown in Table S1, revenues increase with the sample size. However, the marginal gains are decreasing with no noteworthy improvement when increasing the sample size from 100 to 1000. On the other hand, runtimes (as presented in Table S2) increase approximately linearly with the sample size.

Considering the trade-off between revenues and runtimes, sample sizes of 20 and 100 seem to be good choices. However, none of the tested sample size breaks the dominance of mechanism D over $E(\omega)$. Across all sample sizes, mechanism $E(\omega)$ yields lower revenues while requiring more time to compute the corresponding policy.

Table S1: Revenues from mechanism $\boldsymbol{E(\omega)}$ for $\boldsymbol{C \leq 120, T = 40}$, and various sample sizes $\boldsymbol{s}$

$\boldsymbol{T = 40}$	$s = 1000$	$s = 100$	$s = 20$	$s = 10$
$\boldsymbol{C = 1}$	0.91	0.91	0.91	0.89
$\boldsymbol{C = 30}$	16.23	16.23	16.22	16.21
$\boldsymbol{C = 60}$	24.06	24.06	24.05	24.01
$\boldsymbol{C = 90}$	29.02	29.02	29.01	29.00
$\boldsymbol{C = 120}$	32.50	32.50	32.49	32.45

Table S2: Runtimes [$\boldsymbol{s}$] of mechanism $\boldsymbol{E(\omega)}$ for $\boldsymbol{C \leq 120, T = 40}$, and various sample sizes $\boldsymbol{s}$

$\boldsymbol{T = 40}$	$s = 1000$	$s = 100$	$s = 20$	$s = 10$
$\boldsymbol{C = 30}$	773	71	17	14
$\boldsymbol{C = 60}$	3,826	345	73	51
$\boldsymbol{C = 90}$	9,399	848	177	113
$\boldsymbol{C = 120}$	17,073	1,546	318	197

S.12 Fluid approximation

Dynamic pricing problems generally suffer from the infamous curse of dimensionality, where the number of states in which an optimal solution must be found is given by $C \cdot T$. In large-scale scenarios, this makes solving each state computationally burdensome, especially when the problems are complex, thus paving the way for the development of approximation methods. One of these methods is the fluid approximation, which replaces stochastic demand with (deterministic) expected demand rates.

In fluid approximations, capacity is assumed to be continuously divisible, and demand is no longer stochastic. Stochasticity is eliminated by replacing probabilities for stochastic events with fractional weights of events, which technically represent the expected values of the probabilities. In this sense, a customer no longer purchases j units with probability p_j but a fraction p_j of each batch j.[1] Accordingly, the fluid model is given by:

$$\max_{\boldsymbol{r}_t \in \mathcal{R}_c \forall t} \left\{ \sum_{t=1}^{T} \sum_{j=1}^{C} r_{jt} \cdot p_j \ (\boldsymbol{r}_t) : \ \sum_{t=1}^{T} \sum_{j=1}^{C} j \cdot p_j \ (\boldsymbol{r}_t) \leq C \ and \ \boldsymbol{r}_t \geq 0 \right\}.$$

With time-homogeneous customer choice, one can easily verify that the optimization problem above is equivalent to:

$$\max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ T \cdot \sum_{j=1}^{C} r_j \cdot p_j \ (\boldsymbol{r}) : \ \sum_{j=1}^{C} j \cdot p_j \ (\boldsymbol{r}) \leq \frac{C}{T} \right\}.$$

With this equivalence, this optimization problem simplifies to a single-stage problem, resulting in a static pricing policy that proves optimal for the fluid model. Notably, the fluid model can be brought into the form of the multiproduct fluid model examined by Maglaras und Meissner (2006). Similarly, their proof regarding asymptotic optimality of the fluid approximation's optimal solution also holds for the static pricing policy defined by the optimal solution of our fluid model. Applying this static policy in our dynamic setting is therefore asymptotically optimal.

[1] Note that this is slight abuse of notation, as we should technically write $E[p_j]$ to indicate that we are using the expected value of p_j in the context of the fluid approximation. While this would result in the same mathematical formulation, it highlights the different interpretation and allows for predicting capacity consumption. As our results in Section 4 hold for both the dynamic program and the single-stage problem from its fluid approximation, we have decided to use the same notation in both cases.

The most important observation, however, is that the objective function of the fluid model is essentially the same as that in the final period of the dynamic program, where $\Delta_j V_{t-1}(c) = 0$. Therefore, all analytical results from Sections 4.3 and 4.4 can be applied to the fluid approximation.

The optimality conditions described in Propositions 1 and 2 (see Section 4.3) can be utilized to find the optimal solution to our fluid model, which only needs to be solved once, rather than for every possible state as in the dynamic program, significantly reducing computational complexity.

Moreover, the approach to modify our formulation (see Section 4.4) also applies to the fluid approximation. Consequently, all results, such as Remark 4 and Proposition 3, carry over. We tested the conditions described in Proposition 3 in our numerical study and found them validated in each scenario we investigated.

S.13 Confidence intervals

The following table presents the 95% confidence intervals for the mean revenues of all mechanisms across five scenarios, as described in Section 6.1. Since the mean revenues are based on a large number of simulation runs (10,000), we can apply the central limit theorem to approximate the (1-α) confidence interval as follows:

$$\left[\bar{x} - z_{\left(1-\frac{\alpha}{2}\right)}\frac{s}{\sqrt{n}}, \bar{x} + z_{\left(1-\frac{\alpha}{2}\right)}\frac{s}{\sqrt{n}}\right],$$

where $\bar{x}$ is the sample mean revenue, $z_{\left(1-\frac{\alpha}{2}\right)}$ is the $\left(1-\frac{\alpha}{2}\right)$ quantile of the standard normal distribution, s is the sample standard deviation, and n is the sample size.

Table S3: 95% confidence intervals for mean revenues for $C \leq 120, T = 40$

$\boldsymbol{T = 40}$	$E(\lambda)$	$E(\omega)$	D	FA	Opt	S	L	PL
$\boldsymbol{C = 1}$	[0.91, 0.92]	[0.91, 0.92]	[0.91, 0.92]	[0.79, 0.81]	[0.91, 0.92]	[0.91, 0.92]	[0.91, 0.92]	[0.91, 0.92]
$\boldsymbol{C = 30}$	[16.13, 16.23]	[16.19, 16.27]	[16.22, 16.32]	[14.81, 14.97]	[16.24, 16.34]	[15.09, 15.15]	[15.69, 15.79]	[15.89, 15.99]
$\boldsymbol{C = 60}$	[23.71, 23.92]	[23.98, 24.15]	[24.02, 24.22]	[22.70, 22.94]	[24.07, 24.26]	[22.03, 22.30]	[22.76, 22.97]	[23.43, 23.62]
$\boldsymbol{C = 90}$	[28.42, 28.73]	[28.90, 29.16]	[28.93, 29.23]	[27.66, 27.94]	[29.03, 29.32]	[25.02, 25.46]	[27.16, 27.48]	[28.24, 28.53]
$\boldsymbol{C = 120}$	[31.67, 32.07]	[32.34, 32.68]	[32.34, 32.73]	[30.83, 31.13]	[32.47, 32.85]	[26.73, 27.31]	[30.22, 30.65]	[31.66, 32.03]

In every scenario, the confidence intervals for all mechanisms are tight and of comparable size. This suggests that the number of simulation runs is sufficiently large for the mean revenues to serve as a reliable approximation of the expected revenues.

References

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