Appendix 0. Proof of equation (16)

The mean square error of the Liu linear shrinkage estimator is as follows:

$$\begin{aligned}
MSE(\widehat{\boldsymbol{\beta}}^{LLS}) &= E\Big[(\widehat{\boldsymbol{\beta}}^{LLS}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{LLS}-\boldsymbol{\beta})'\Big]\\
&= E\Big[(\delta\,\widehat{\boldsymbol{\beta}}^{RL}+(1-\delta)\,\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\delta\,\widehat{\boldsymbol{\beta}}^{RL}+(1-\delta)\,\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})'\Big]\\
&= E\Big[[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})-\delta\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})]\,[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})-\delta\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})]'\Big]\\
&= E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})'\Big]+\delta^2\,E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]\\
&\quad -2\,\delta\,E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]
\end{aligned}$$

We now derive from above equation with respect to δ as follows:

$$\frac{\partial\, MSE(\widehat{\boldsymbol{\beta}}^{LLS})}{\partial\,\delta} = 2\,\delta\,E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]-2\,E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]=0$$

Therefore, the optimum value of δ is as

$$\delta_{optimal} = \frac{\overbrace{E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]}^{q_1}}{\underbrace{E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]}_{q_2}} \tag{25}$$

Since $\widehat{\boldsymbol{\beta}}^{UR} = \mathbf{A}\,\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\beta}}^{RL} = \mathbf{A}\,\widehat{\boldsymbol{\beta}}_{RMLE} = \mathbf{A}\,\big[\widehat{\boldsymbol{\beta}}-\mathbf{B}\,(\mathbf{H}\,\widehat{\boldsymbol{\beta}}-\mathbf{h})\big]$, where $\mathbf{B} = (\mathbf{X}'\widehat{\mathbf{W}}\mathbf{X}+\mathbf{I}_p)^{-1}(\mathbf{X}'\widehat{\mathbf{W}}\mathbf{X}+d\,\mathbf{I}_p)$ and $\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{I}}^{-1}\mathbf{H}'\Big(\mathbf{H}\boldsymbol{\mathcal{I}}^{-1}\mathbf{H}'\Big)^{-1}$. Therefore, q_1 can be written as:

$$\begin{aligned}
q_1 &= E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]\\
&= E\Big[(\mathbf{B}\,\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})\,(\mathbf{B}\,\widehat{\boldsymbol{\beta}}-\mathbf{B}\,\widehat{\boldsymbol{\beta}}_{RMLE})'\Big]\\
&= E\Big[(\mathbf{B}\,\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})\,(\mathbf{B}\,\widehat{\boldsymbol{\beta}}-\mathbf{B}\,[\widehat{\boldsymbol{\beta}}-\boldsymbol{\mathcal{Z}}\,(\mathbf{H}\,\widehat{\boldsymbol{\beta}}-\mathbf{h})])'\Big]\\
&= E\Big[(\mathbf{B}\,\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})\,(\mathbf{A}\,\boldsymbol{\mathcal{Z}}\,(\mathbf{H}\,\widehat{\boldsymbol{\beta}}-\mathbf{h}))'\Big]\\
&= \mathbf{B}\,E\Big[\widehat{\boldsymbol{\beta}}\,\widehat{\boldsymbol{\beta}}'\Big]\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}'-\mathbf{B}\,E\Big[\widehat{\boldsymbol{\beta}}\Big]\,\mathbf{h}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}'\\
&\quad -\boldsymbol{\beta}\,E\Big[\widehat{\boldsymbol{\beta}}'\Big]\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}'+\boldsymbol{\beta}\,\mathbf{h}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}'
\end{aligned}$$

and q_2 is

$$\begin{aligned}
q_2 &= E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})\,(\widehat{\boldsymbol{\beta}}^{UR}-\widehat{\boldsymbol{\beta}}^{RL})'\Big]\\
&= E\Big[[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})-(\widehat{\boldsymbol{\beta}}^{RL}-\boldsymbol{\beta})]\,[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})-(\widehat{\boldsymbol{\beta}}^{RL}-\boldsymbol{\beta})]'\Big]\\
&= E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})'\Big]-2\,E\Big[(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{RL}-\boldsymbol{\beta})'\Big]\\
&\quad +E\Big[(\widehat{\boldsymbol{\beta}}^{RL}-\boldsymbol{\beta})\,(\widehat{\boldsymbol{\beta}}^{RL}-\boldsymbol{\beta})'\Big]\\
&= \underbrace{\boldsymbol{\Sigma}(\widehat{\boldsymbol{\beta}}^{UR})}_{q_3}-2\,\underbrace{\boldsymbol{\Sigma}(\widehat{\boldsymbol{\beta}}^{UR},\widehat{\boldsymbol{\beta}}^{RL})}_{q_4}+\underbrace{\boldsymbol{\Sigma}(\widehat{\boldsymbol{\beta}}^{RL})}_{q_5}
\end{aligned}$$

where $\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}}^{UR})$ and $\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}}^{RL})$ are variances of $\widehat{\boldsymbol{\beta}}^{UR}$ and $\widehat{\boldsymbol{\beta}}^{RL}$, respectively, and $\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}}^{UR}, \widehat{\boldsymbol{\beta}}^{RL})$ is the variance-covariance matrix of $\widehat{\boldsymbol{\beta}}^{UR}$ and $\widehat{\boldsymbol{\beta}}^{RL}$.

Now we can write q_3 as:

$$q_3 = \mathbf{\Sigma}(\widehat{\boldsymbol{\beta}}^{UR}) = \mathbf{\Sigma}(\mathbf{B}\,\widehat{\boldsymbol{\beta}}) = \mathbf{B}\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\,\mathbf{B}'$$

and q_4 is

$$\begin{aligned} q_4 &= \mathbf{\Sigma}\Big(\widehat{\boldsymbol{\beta}}^{UR}, \widehat{\boldsymbol{\beta}}^{RL}\Big) \\ &= \mathbf{\Sigma}\Big(\mathbf{B}\,\widehat{\boldsymbol{\beta}}, \mathbf{B}\,\widehat{\boldsymbol{\beta}}_{RMLE}\Big) \\ &= \mathbf{\Sigma}\Big(\mathbf{B}\,\widehat{\boldsymbol{\beta}}, \mathbf{B}\,[\widehat{\boldsymbol{\beta}} - \boldsymbol{\mathcal{Z}}\,(\mathbf{H}\,\widehat{\boldsymbol{\beta}} - \mathbf{h})]\Big) \\ &= \mathbf{\Sigma}\Big(\mathbf{B}\,\widehat{\boldsymbol{\beta}}, \mathbf{B}\widehat{\boldsymbol{\beta}}\Big) - \mathbf{\Sigma}\Big(\mathbf{B}\,\widehat{\boldsymbol{\beta}}, \mathbf{B}\,\boldsymbol{\mathcal{Z}}\,(\mathbf{H}\,\widehat{\boldsymbol{\beta}} - \mathbf{h})\Big) \\ &= \mathbf{B}^2\,\mathbf{\Sigma}\Big(\widehat{\boldsymbol{\beta}}\Big)\,(\mathbf{B}^2)' - \mathbf{B}^2\,\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{\Sigma}\Big(\widehat{\boldsymbol{\beta}}\Big)\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\,(\mathbf{B}^2)' \\ &= \mathbf{B}^2\,\mathbf{\Sigma}\Big(\widehat{\boldsymbol{\beta}}\Big)\,\Big[\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\Big]\,(\mathbf{B}^2)' \end{aligned}$$

and q_5 is as follows:

$$\begin{aligned} q_5 &= \mathbf{\Sigma}\Big(\widehat{\boldsymbol{\beta}}^{RL}\Big) \\ &= \mathbf{\Sigma}\Big(\mathbf{B}\,\widehat{\boldsymbol{\beta}}_{RMLE}\Big) \\ &= \mathbf{B}\,\mathbf{\Sigma}\Big(\widehat{\boldsymbol{\beta}} - \boldsymbol{\mathcal{Z}}\,(\mathbf{H}\,\widehat{\boldsymbol{\beta}} - \mathbf{h})\Big)\,\mathbf{B}' \\ &= \mathbf{B}\,\Big(\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}}) - \mathbf{\Sigma}(\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\widehat{\boldsymbol{\beta}})\Big)\,\mathbf{B}' \\ &= \mathbf{B}\,\Big(\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}}) - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\Big)\,\mathbf{B}' \\ &= \mathbf{B}\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\,\Big(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\Big)\,\mathbf{B}' \end{aligned}$$

So, q_2 becomes

$$\begin{aligned} q_2 = \mathbf{B}\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\,\mathbf{B}' &+ \mathbf{B}^2\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\,\Big[\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\Big]\,(\mathbf{B}^2)' \\ &- 2\,\mathbf{B}\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\,\Big[\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\Big]\,\mathbf{B} \\ = \mathbf{B}\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\,\mathbf{B}'&\Big[\mathbf{I}_p - (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}')\,(\mathbf{B}\,\mathbf{B}' - 2\,\mathbf{I}_p)\Big] \end{aligned}$$

Hence, Equation (25) becomes

$$\delta_{optimal} = \frac{\mathbf{B}\,E\Big[\widehat{\boldsymbol{\beta}}\,\widehat{\boldsymbol{\beta}}'\Big]\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}' - \mathbf{B}\,E\Big[\widehat{\boldsymbol{\beta}}\Big]\,\mathbf{h}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}' - \boldsymbol{\beta}\,E\Big[\widehat{\boldsymbol{\beta}}'\Big]\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}' + \boldsymbol{\beta}\,\mathbf{h}'\,\boldsymbol{\mathcal{Z}}'\,\mathbf{B}'}{\mathbf{B}\,\mathbf{\Sigma}(\widehat{\boldsymbol{\beta}})\,\mathbf{B}'\Big[\mathbf{I}_p - (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{H}'\,\boldsymbol{\mathcal{Z}}')\,(\mathbf{B}\,\mathbf{B}' - 2\,\mathbf{I}_p)\Big]}$$

Appendix 1. Proof of Lemma 3.1

We present the following lemma which are useful for the proof of the Theorems 3.2 and 3.3.

Lemma .1. *Let $\boldsymbol{y}$ be a p_2-dimensional random vector distributed as $\mathcal{N}_{p_2}(\boldsymbol{\mu_y}, \boldsymbol{\Sigma_y})$. Then, for any measurable function φ, we have*

$$E[\boldsymbol{y}\,\varphi(\boldsymbol{y}'\boldsymbol{y})] = \boldsymbol{\mu_y}\,E[\varphi(\chi^2_{p_2+2}(\Delta^*))], \tag{26}$$

$$E[\boldsymbol{y}'\boldsymbol{y}\,\varphi(\boldsymbol{y}'\boldsymbol{y})] = \boldsymbol{\Sigma_y}\,E[\varphi(\chi^2_{p_2+2}(\Delta^*))] + \boldsymbol{\mu}'_{\boldsymbol{y}}\,\boldsymbol{\mu_y}E[\varphi(\chi^2_{p_2+4}(\Delta^*))], \tag{27}$$

where Δ^* is the non-centrality parameter.

Proof: See Judge and Bock (1987).

Under local alternatives given in Equation (21) we show $\boldsymbol{\kappa}_1^n$ is asymptotically normally distributed with mean and variance as follows

$$\begin{aligned}\boldsymbol{\kappa}_1^n &= \sqrt{n}(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta})\\ &= \sqrt{n}(\mathbf{B}\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\end{aligned}$$

As $n \to \infty$, $\widehat{\boldsymbol{\beta}} \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{\mathcal{I}}^{-1})$. Therefore, $\boldsymbol{\kappa}_1^n \xrightarrow{D} \boldsymbol{\kappa}_1 \sim \mathcal{N}_p\Big(E(\boldsymbol{\kappa}_1), Var(\boldsymbol{\kappa}_1)\Big)$ where

$$\begin{aligned}E(\boldsymbol{\kappa}_1) &= E[\mathbf{B}\,\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}]\\ &= \mathbf{B}\,\boldsymbol{\beta} - \boldsymbol{\beta}\\ &= (\mathbf{B}\ - \mathbf{I}_p)\boldsymbol{\beta}\end{aligned}$$

$$\begin{aligned}Var(\boldsymbol{\kappa}_1) &= Var[\mathbf{B}\,\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}]\\ &= \mathbf{B}\,Var(\widehat{\boldsymbol{\beta}})\,\mathbf{B}'\\ &= \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\end{aligned}$$

We show $\boldsymbol{\kappa}_2^n$ is asymptotically normally distributed with mean and variance as follow

$$\begin{aligned}\boldsymbol{\kappa}_2^n &= \sqrt{n}(\mathbf{H}\widehat{\boldsymbol{\beta}}^{UR} - \mathbf{h})\\ &= \sqrt{n}(\mathbf{H}\,\widehat{\boldsymbol{\beta}}^{UR} - \mathbf{H}\,\boldsymbol{\beta} + \mathbf{H}\,\boldsymbol{\beta} - \mathbf{h})\\ &= \mathbf{H}\,\sqrt{n}(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta}) + \underbrace{\sqrt{n}\,(\mathbf{H}\,\boldsymbol{\beta} - \mathbf{h})}_{\boldsymbol{\vartheta}}\\ &= \mathbf{H}\,\boldsymbol{\kappa}_1^n + \boldsymbol{\vartheta}\end{aligned}$$

as $n \to \infty$, $\boldsymbol{\kappa}_1^n \xrightarrow{D} \boldsymbol{\kappa}_1 \sim \mathcal{N}_p\Big((\mathbf{B}\ - \mathbf{I}_p)\boldsymbol{\beta}, \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\Big)$. So, $\boldsymbol{\kappa}_2^n \xrightarrow{D} \boldsymbol{\kappa}_2 \sim \mathcal{N}_p\Big(E(\boldsymbol{\kappa}_2), Var(\boldsymbol{\kappa}_2)\Big)$ where

$$\begin{aligned}E(\boldsymbol{\kappa}_2) &= \mathbf{H}\,E(\boldsymbol{\kappa}_1) + \boldsymbol{\vartheta}\\ &= \mathbf{H}\,(\mathbf{B}\ - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\vartheta}\end{aligned}$$

$$\begin{aligned} Var(\boldsymbol{\kappa}_2) &= \mathbf{H}\, Var(\boldsymbol{\kappa}_1)\, \mathbf{H}' \\ &= \mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\mathbf{H}' \end{aligned}$$

Then

$$\begin{aligned} \boldsymbol{\kappa}_3 &= \sqrt{n}(\widehat{\boldsymbol{\beta}}^{RL} - \boldsymbol{\beta}) \\ &= \sqrt{n}(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\mathcal{Z}}(\mathbf{H}\widehat{\boldsymbol{\beta}}^{UR} - \mathbf{h}) - \boldsymbol{\beta}) \\ &= \sqrt{n}(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta}) - \boldsymbol{\mathcal{Z}}\sqrt{n}(\mathbf{H}\widehat{\boldsymbol{\beta}}^{UR} - \mathbf{h}) \\ &= \boldsymbol{\kappa}_1^n - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\kappa}_2^n \\ &= \boldsymbol{\kappa}_1^n - \boldsymbol{\mathcal{Z}}\,(\mathbf{H}\,\boldsymbol{\kappa}_1^n + \boldsymbol{\vartheta}) \\ &= (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\boldsymbol{\kappa}_1^n - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}, \end{aligned}$$

As $n \to \infty$, $\boldsymbol{\kappa}_1^n \xrightarrow{D} \boldsymbol{\kappa}_1 \sim \mathcal{N}_p\Big((\mathbf{B} - \mathbf{I}_p)\boldsymbol{\beta}, \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\Big)$. Thus, $\boldsymbol{\kappa}_3^n \xrightarrow{D} \boldsymbol{\kappa}_3 \sim \mathcal{N}_p\Big(E(\boldsymbol{\kappa}_3), Var(\boldsymbol{\kappa}_3)\Big)$ where

$$\begin{aligned} E(\boldsymbol{\kappa}_3) &= E\big[(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\boldsymbol{\kappa}_1 - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big] \\ &= (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,E[\boldsymbol{\kappa}_1] - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \\ &= (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \end{aligned}$$

$$\begin{aligned} Var(\boldsymbol{\kappa}_3) &= Var\big[(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\boldsymbol{\kappa}_1 - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big] \\ &= Var\big[(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\boldsymbol{\kappa}_1\big] \\ &= (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})' \\ &= \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' \end{aligned}$$

Now we write

$$\begin{aligned} \boldsymbol{\kappa}_4^n &= \sqrt{n}(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}) \\ &= \sqrt{n}\big(\widehat{\boldsymbol{\beta}}^{UR} - \big[\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\mathcal{Z}}(\mathbf{H}\widehat{\boldsymbol{\beta}}^{UR} - \mathbf{h})\big]\big) \\ &= \boldsymbol{\mathcal{Z}}\,\boldsymbol{\kappa}_2^n \\ &= \boldsymbol{\mathcal{Z}}\,(\mathbf{H}\,\boldsymbol{\kappa}_1^n + \boldsymbol{\vartheta}) \\ &= \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\boldsymbol{\kappa}_1^n + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \end{aligned}$$

As $n \to \infty$, $\boldsymbol{\kappa}_1^n \xrightarrow{D} \boldsymbol{\kappa}_1 \sim \mathcal{N}_p\Big((\mathbf{B} - \mathbf{I}_p)\boldsymbol{\beta}, \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\Big)$. Thus, $\boldsymbol{\kappa}_4^n \xrightarrow{D} \boldsymbol{\kappa}_4 \sim \mathcal{N}_p\Big(E(\boldsymbol{\kappa}_4), Var(\boldsymbol{\kappa}_4)\Big)$ where

$$
\begin{aligned}
E(\boldsymbol{\kappa}_4) &= E\big[\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\boldsymbol{\kappa}_1 + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big] \\
&= \ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,E[\boldsymbol{\kappa}_1] + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \\
&= \boldsymbol{\mathcal{Z}}\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\boldsymbol{\vartheta}
\end{aligned}
$$

$$
\begin{aligned}
Var(\boldsymbol{\kappa}_4) &= Var[\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\boldsymbol{\kappa}_1 + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}] \\
&= (\boldsymbol{\mathcal{Z}}\,\mathbf{H})\,Var[\boldsymbol{\kappa}_1]\,(\boldsymbol{\mathcal{Z}}\,\mathbf{H})' \\
&= \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,(\boldsymbol{\mathcal{Z}}\,\mathbf{H})' \\
&= \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'.
\end{aligned}
$$

The joint distribution of $\boldsymbol{\kappa}_1^n$ and $\boldsymbol{\kappa}_4^n$ is as follows:

$$
\begin{aligned}
\begin{pmatrix} \boldsymbol{\kappa}_1^n \\ \boldsymbol{\kappa}_4^n \end{pmatrix} &= \begin{pmatrix} \mathbf{I}_p\,\boldsymbol{\kappa}_1^n + \mathbf{0} \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\boldsymbol{\kappa}_1^n + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{I}_p \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H} \end{pmatrix} \boldsymbol{\kappa}_1^n + \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \end{pmatrix},
\end{aligned}
$$

as $n \to \infty$, $\boldsymbol{\kappa}_1^n \xrightarrow{D} \boldsymbol{\kappa}_1 \sim \mathcal{N}_p\Big((\mathbf{B} - \mathbf{I}_p)\boldsymbol{\beta}, \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\Big)$. So, $\begin{pmatrix} \boldsymbol{\kappa}_1^n \\ \boldsymbol{\kappa}_4^n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_4 \end{pmatrix} \sim \mathcal{N}_{2p}\Big(\boldsymbol{\mu}_1, \boldsymbol{\sigma}_1\Big)$
where

$$
\begin{aligned}
\boldsymbol{\mu}_1 &= E\left[\begin{pmatrix} \mathbf{I}_p \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H} \end{pmatrix} \boldsymbol{\kappa}_1 + \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \end{pmatrix} \right] \\
&= \begin{pmatrix} \mathbf{I}_p \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H} \end{pmatrix} E[\boldsymbol{\kappa}_1] + \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{B} - \mathbf{I}_p)\boldsymbol{\beta} \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}(\mathbf{B} - \mathbf{I}_p)\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \end{pmatrix}
\end{aligned}
$$

and

$$\begin{aligned}
\boldsymbol{\sigma}_1 &= Var\Bigg[\begin{pmatrix}\mathbf{I}_p\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\end{pmatrix}\boldsymbol{\kappa}_1+\begin{pmatrix}\mathbf{0}\\ \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\end{pmatrix}\Bigg]\\
&= Var\Bigg[\begin{pmatrix}\mathbf{I}_p\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\end{pmatrix}\boldsymbol{\kappa}_1\Bigg]\\
&= \begin{pmatrix}\mathbf{I}_p\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\end{pmatrix} Var[\,\boldsymbol{\kappa}_1]\,\Big(\mathbf{I}_p\quad (\boldsymbol{\mathcal{Z}}\,\mathbf{H})'\Big)\\
&= \begin{pmatrix}\mathbf{I}_p\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\end{pmatrix} \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,\Big(\mathbf{I}_p\quad (\boldsymbol{\mathcal{Z}}\,\mathbf{H})'\Big)\\
&= \begin{pmatrix}\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' & \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,(\boldsymbol{\mathcal{Z}}\,\mathbf{H})'\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' & \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,(\boldsymbol{\mathcal{Z}}\,\mathbf{H})'\end{pmatrix}\\
&= \begin{pmatrix}\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' & \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' & \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\end{pmatrix}
\end{aligned}$$

In a similar way, we consider

$$\begin{aligned}
\begin{pmatrix}\boldsymbol{\kappa}_3^n\\ \boldsymbol{\kappa}_4^n\end{pmatrix} &= \begin{pmatrix}(\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\boldsymbol{\kappa}_1-\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\boldsymbol{\kappa}_1+\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\end{pmatrix}\\
&= \begin{pmatrix}\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H}\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\end{pmatrix}\boldsymbol{\kappa}_1+\begin{pmatrix}-\mathbf{I}_p\\ \mathbf{I}_p\end{pmatrix}\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}
\end{aligned}$$

Therefore, as $n\to\infty$, $\boldsymbol{\kappa}_1^n \xrightarrow{D} \boldsymbol{\kappa}_1 \sim \mathcal{N}_p\Big((\mathbf{B}-\mathbf{I}_p)\boldsymbol{\beta}, \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\Big)$. So, $\begin{pmatrix}\boldsymbol{\kappa}_3^n\\ \boldsymbol{\kappa}_4^n\end{pmatrix} \xrightarrow{D} \begin{pmatrix}\boldsymbol{\kappa}_3\\ \boldsymbol{\kappa}_4\end{pmatrix} \sim \mathcal{N}_{2p}\Big(\boldsymbol{\mu}_2, \boldsymbol{\sigma}_2\Big)$ where

$$\begin{aligned}
\boldsymbol{\mu}_2 &= E\Bigg[\begin{pmatrix}\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H}\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\end{pmatrix}\boldsymbol{\kappa}_1+\begin{pmatrix}-\mathbf{I}_p\\ \mathbf{I}_p\end{pmatrix}\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\Bigg]\\
&= \begin{pmatrix}\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H}\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\end{pmatrix} E[\boldsymbol{\kappa}_1]+\begin{pmatrix}-\mathbf{I}_p\\ \mathbf{I}_p\end{pmatrix}\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\\
&= \begin{pmatrix}(\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\end{pmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
\boldsymbol{\sigma}_2 &= Var\left[\begin{pmatrix} \mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H} \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H} \end{pmatrix} \boldsymbol{\kappa}_1 + \begin{pmatrix} -\mathbf{I}_p \\ \mathbf{I}_p \end{pmatrix} \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \right] \\
&= \begin{pmatrix} \mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H} \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H} \end{pmatrix} Var[\boldsymbol{\kappa}_1] \begin{pmatrix} (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})' & (\boldsymbol{\mathcal{Z}}\,\mathbf{H})' \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H} \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H} \end{pmatrix} \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' \begin{pmatrix} (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})' & (\boldsymbol{\mathcal{Z}}\,\mathbf{H})' \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})' & (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,(\boldsymbol{\mathcal{Z}}\,\mathbf{H})' \\ \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})' & \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\,(\boldsymbol{\mathcal{Z}}\,\mathbf{H})' \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' \end{pmatrix}.
\end{aligned}
$$

Appendix 2. Proof of Theorem 3.3

Here, we provide the proof of bias expressions. Based on Lemma 3.1 we have

$$\mathcal{B}(\widehat{\boldsymbol{\beta}}^{UR}) = \lim_{n\to\infty} E\left[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta}\right)\right] = E[\boldsymbol{\kappa}_1] = (\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta}$$

$$\begin{aligned}\mathcal{B}(\widehat{\boldsymbol{\beta}}^{RL}) &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{RL} - \boldsymbol{\beta}\right)\right] \\ &= E[\boldsymbol{\kappa}_3] \\ &(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\widehat{\boldsymbol{\beta}}^{LLS}) &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{LLS} - \boldsymbol{\beta}\right)\right] \\ &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\delta\,\widehat{\boldsymbol{\beta}}^{RL} + (1-\delta)\,\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta}\right)\right] \\ &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta}\right) - \delta\,\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}\right)\right] \\ &= E(\boldsymbol{\kappa}_1) - \delta\,E(\boldsymbol{\kappa}_4) \\ &= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{UR}) - \delta\,\boldsymbol{\mathcal{Z}}\,[\mathbf{H}(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\vartheta}]\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\widehat{\boldsymbol{\beta}}^{LSPE}) &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{LSPE} - \boldsymbol{\beta}\right)\right] \\ &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \delta\,(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL})\,I(T_n \le T_{n,\alpha}) - \boldsymbol{\beta}\right)\right] \\ &= \lim_{n\to\infty} E\left[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta}\right) - \delta\,\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}\right) I(T_n \le T_{n,\alpha})\right] \\ &= E[\boldsymbol{\kappa}_1] - \delta\,E\left[\boldsymbol{\kappa}_4\,I(T_n \le T_{n,\alpha})\right],\end{aligned}$$

based on Equation (26) we can write

$$\begin{aligned}\mathcal{B}(\widehat{\boldsymbol{\beta}}^{LSPE}) &= E[\boldsymbol{\kappa}_1] - \delta\,E[\boldsymbol{\kappa}_4]\,P(\chi^2_{p_2+2}(\Delta^*) \le \chi^2_{p_2,\alpha}) \\ &= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{UR}) - \delta\,\boldsymbol{\mathcal{Z}}\,[\mathbf{H}(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\vartheta}]\,P(\chi^2_{p_2+2}(\Delta^*) \le \chi^2_{p_2,\alpha}) \\ &= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{UR}) - \delta\,\boldsymbol{\mathcal{Z}}\,[\mathbf{H}(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\vartheta}]\,\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha}; \Delta^*).\end{aligned}$$

If $\lambda = 1$,

$$\mathcal{B}(\widehat{\boldsymbol{\beta}}^{LPT}) = \mathcal{B}(\widehat{\boldsymbol{\beta}}^{UR}) - \boldsymbol{\mathcal{Z}}\,[\mathbf{H}(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\vartheta}]\,\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha}; \Delta^*).$$

In a similar way, we can obtain

$$
\begin{aligned}
\mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS}) &= \lim_{n\to\infty} E\Big[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{LS} - \boldsymbol{\beta}\right)\Big] \\
&= \lim_{n\to\infty} E\Big[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{RL} + (1-(p_2-2)\,T_n^{-1})\,(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}) - \boldsymbol{\beta}\right)\Big] \\
&= \lim_{n\to\infty} \left\{E\Big[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \boldsymbol{\beta}\right)\Big] - (p_2-2)\,E\Big[T_n^{-1}\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}\right)\Big]\right\} \\
&= E[\boldsymbol{\kappa}_1] - (p_2-2)\,E[T_n^{-1}\,\boldsymbol{\kappa}_4] \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{UR}) - (p_2-2)\,E[\boldsymbol{\kappa}_4]\,E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big] \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{UR}) - (p_2-2)\,\boldsymbol{\mathcal{Z}}\,\left[\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\vartheta}\right]E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big].
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathcal{B}(\widehat{\boldsymbol{\beta}}^{LPS}) &= \lim_{n\to\infty} E\Big[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{LPS} - \boldsymbol{\beta}\right)\Big] \\
&= \lim_{n\to\infty} E\Big[\sqrt{n}\left(\mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS} - \boldsymbol{\beta}\right) - (1-(p_2-2)\,T_n^{-1}) \\
&\qquad\qquad \times\sqrt{n}\left((\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL})\,I(T_n < p_2-2)\right)\Big] \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS}) - \lim_{n\to\infty} E\Big[\sqrt{n}\left(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}\right)(1-(p_2-2)\,T_n^{-1})\,I(T_n<p_2-2)\Big] \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS}) - E[\boldsymbol{\kappa}_4\,(1-(p_2-2)\,T_n^{-1})\,I(T_n<p_2-2)] \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS}) - E[\boldsymbol{\kappa}_4\,I(T_n<p_2-2)] + (p_2-2)\,E[\boldsymbol{\kappa}_4\,T_n^{-1}\,I(T_n<p_2-2)] \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS}) - E[\boldsymbol{\kappa}_4]\,\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) + (p_2-2)\,E[\boldsymbol{\kappa}_4]E\Big[\frac{I(T_n<p_2-2)}{\chi^2_{p_2+2}(\Delta^*)}\Big] \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS}) - E[\boldsymbol{\kappa}_4]\left\{\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) + (p_2-2)\,E\Big[\frac{I(T_n<p_2-2)}{\chi^2_{p_2+2}(\Delta^*)}\Big]\right\} \\
&= \mathcal{B}(\widehat{\boldsymbol{\beta}}^{LS}) - \boldsymbol{\mathcal{Z}}\,[\mathbf{H}(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\vartheta}]\Big\{\boldsymbol{\Psi}_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) \\
&\qquad\qquad + (p_2-2)\,E\Big[\frac{I(\chi^2_{p_2+2}(\Delta^*)<p_2-2)}{\chi^2_{p_2+2}(\Delta^*)}\Big]\Big\}.
\end{aligned}
$$

Appendix 3. Proof of Theorem 3.3

Here, we compute the asymptotic covariance matrices of the estimators

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})\,\sqrt{n}(\widehat{\boldsymbol{\beta}}^{UR}-\boldsymbol{\beta})'\Big)\\
&= \lim_{n\to\infty} E(\boldsymbol{\kappa}_1^n\,\boldsymbol{\kappa}_1^{n\prime})\\
&= E(\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_1')\\
&= Var(\boldsymbol{\kappa}_1)+E(\boldsymbol{\kappa}_1\,E(\boldsymbol{\kappa}_1{}')\\
&= \mathbf{B}\boldsymbol{\mathcal{I}}^{-1}\mathbf{B}'+\Big[(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}\Big]\,\Big[(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}\Big]'
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{RL}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\widehat{\boldsymbol{\beta}}^{RL}-\boldsymbol{\beta})\,\sqrt{n}(\widehat{\boldsymbol{\beta}}^{RL}-\boldsymbol{\beta})'\Big)\\
&= \lim_{n\to\infty} E(\boldsymbol{\kappa}_3^n\,\boldsymbol{\kappa}_3^{n\prime})\\
&= E(\boldsymbol{\kappa}_3\,\boldsymbol{\kappa}_3{}')\\
&= Var(\boldsymbol{\kappa}_3)+E(\boldsymbol{\kappa}_3)\,E(\boldsymbol{\kappa}_3{}')\\
&= \mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'-\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\\
&\quad+\Big[(\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\Big]\,\Big[(\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\Big]'
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LLS}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\widehat{\boldsymbol{\beta}}^{LLS}-\boldsymbol{\beta})\,\sqrt{n}(\widehat{\boldsymbol{\beta}}^{LLS}-\boldsymbol{\delta})'\Big)\\
&= \lim_{n\to\infty} E[(\boldsymbol{\kappa}_1^n-\delta\,\boldsymbol{\kappa}_4^n)\,(\boldsymbol{\kappa}_1^n-\delta\,\boldsymbol{\kappa}_4^n)']\\
&= E[(\boldsymbol{\kappa}_1-\delta\,\boldsymbol{\kappa}_4)\,(\boldsymbol{\kappa}_1-\delta\,\boldsymbol{\kappa}_4)']\\
&= E[\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_1']-2\delta\,E[\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_4']+\delta^2\,E[\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4']\\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR})-2\delta\,\underbrace{E[\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_4']}_{e_1}+\delta^2\,E[\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4']
\end{aligned}
$$

Using the conditional expectation of a multivariate normal distribution, e_1 becomes

$$
\begin{aligned}
e_1 &= E[\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_4']\\
&= E\big(\boldsymbol{\kappa}_4'\,E[\boldsymbol{\kappa}_1\,|\,\boldsymbol{\kappa}_4]\big)\\
&= E\big(\boldsymbol{\kappa}_4'\big\{(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\kappa}_4-\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}-\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big\}\big)\\
&= (\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}\,E(\boldsymbol{\kappa}_4')+E(\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4')-[\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta}+\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]\,E(\boldsymbol{\kappa}_4')\\
&= E(\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4')+E(\boldsymbol{\kappa}_4')\big\{(\mathbf{I}_p-\boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta})-\boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LLS}) &= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) + \delta^2 E[\boldsymbol{\kappa}_4 \boldsymbol{\kappa}_4'] \\
&- 2\delta \left(E(\boldsymbol{\kappa}_4 \boldsymbol{\kappa}_4') + E(\boldsymbol{\kappa}_4')\big\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta}) - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \right) \\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) - 2\delta\big\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta}) - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big\} E(\boldsymbol{\kappa}_4') - \delta\,(2-\delta)\,E[\boldsymbol{\kappa}_4 \boldsymbol{\kappa}_4'] \\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) - 2\delta\,\big\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta}) - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big\}\big\{\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big\}' \\
&- \delta\,(2-\delta)\,\big\{\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}' + [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]\,[\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]'\big\}
\end{aligned}
$$

Next we obtain $\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LSPE})$ as follows

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LSPE}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\widehat{\boldsymbol{\beta}}^{LSPE} - \boldsymbol{\beta})\,\sqrt{n}(\widehat{\boldsymbol{\beta}}^{LSPE} - \boldsymbol{\beta})'\Big) \\
&= \lim_{n\to\infty} E[\,\{\boldsymbol{\kappa}_1^n - \delta\,\boldsymbol{\kappa}_4^n\,I(T_n \le T_{n,\alpha})\}\,\{\boldsymbol{\kappa}_1^n - \delta\,\boldsymbol{\kappa}_4^n\,I(T_n \le T_{n,\alpha})\}'\,] \\
&= \lim_{n\to\infty} E(\boldsymbol{\kappa}_1^n\,\boldsymbol{\kappa}_1^{n\prime}) - 2\delta \lim_{n\to\infty} E(\boldsymbol{\kappa}_1^n\,\boldsymbol{\kappa}_4^{n\prime}\,I(T_n \le T_{n,\alpha})) \\
&\qquad + \delta^2 \lim_{n\to\infty} E(\boldsymbol{\kappa}_4^n\,\boldsymbol{\kappa}_4^{n\prime}\,I(T_n \le T_{n,\alpha})) \\
&= E(\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_1') - 2\delta\,E(\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})) + \delta^2\,E(\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})) \\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) - 2\delta\,\underbrace{E(\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha}))}_{e_2} + \delta^2\,E(\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})),
\end{aligned}
$$

by using conditional expectation, e_2 becomes

$$
\begin{aligned}
e_2 &= E(\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})) \\
&= E\big[E(\boldsymbol{\kappa}_1\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})\,|\,\boldsymbol{\kappa}_4)\big] \\
&= E\big[E(\boldsymbol{\kappa}_1\,|\,\boldsymbol{\kappa}_4)\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})\big] \\
&= E\big[\,\{(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\kappa}_4 - \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})\big] \\
&= (\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta}\,E[\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})] + E[\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})] \\
&- \big(\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big)\,E[\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})] \\
&= \Big\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\Big\}\,E[\boldsymbol{\kappa}_4]\Psi_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) + E[\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4'\,I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})]
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LSPE}) &= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) \\
&\quad - 2\delta \left(\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}\, E[\boldsymbol{\kappa}_4]\Psi_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) \right) \\
&\quad + E[\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4'\, I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})] + \delta^2\, E(\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4'\, I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})) \\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) - 2\delta \left(\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}\, E[\boldsymbol{\kappa}_4]\Psi_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) \right) \\
&\quad - \delta\,(2-\delta)\, E[\boldsymbol{\kappa}_4\,\boldsymbol{\kappa}_4'\, I(\chi^2_{p_2}(\Delta^*) \le \chi^2_{p_2,\alpha})] \\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) - 2\delta \Big(\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\} \\
&\quad \times [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]'\, \Psi_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) \Big) \\
&\quad - \delta\,(2-\delta) \Big(\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\, \Psi_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) + [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}] \\
&\quad \times [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]'\Psi_{p_2+4}(\chi^2_{p_2,\alpha};\Delta^*) \Big).
\end{aligned}$$

For $\delta = 1$, $\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LPT})$ reduces to

$$\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LPT}) &= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) - 2 \Big(\Big\{ (\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} \Big\} \\
&\quad \times [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]'\, \Psi_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) \Big) \\
&\quad - \Big(\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\, \Psi_{p_2+2}(\chi^2_{p_2,\alpha};\Delta^*) + [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}] \\
&\quad \times [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B} - \mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]'\Psi_{p_2+4}(\chi^2_{p_2,\alpha};\Delta^*) \Big).
\end{aligned}$$

Now we obtain $\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LS})$ as follows:

$$\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LS}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\widehat{\boldsymbol{\beta}}^{LS} - \boldsymbol{\beta})\, \sqrt{n}(\widehat{\boldsymbol{\beta}}^{LS} - \boldsymbol{\beta})' \Big) \\
&= \lim_{n\to\infty} E\Big[\sqrt{n}\Big(\widehat{\boldsymbol{\beta}}^{RL} + \{1 - (p_2 - 2\} T_n^{-1})\, (\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}) - \boldsymbol{\beta} \Big) \\
&= \lim_{n\to\infty} E[(\boldsymbol{\kappa}_1^n - (p_2 - 2)\, T_n^{-1}\, \boldsymbol{\kappa}_4^n)\, (\boldsymbol{\kappa}_1^n - (p_2 - 2)\, T_n^{-1}\, \boldsymbol{\kappa}_4^n)'] \\
&= E[(\boldsymbol{\kappa}_1 - (p_2 - 2)\, T_n^{-1}\, \boldsymbol{\kappa}_4)\, (\boldsymbol{\kappa}_1 - (p_2 - 2)\, T_n^{-1}\, \boldsymbol{\kappa}_4'] \\
&= E[(\boldsymbol{\kappa}_1\, \boldsymbol{\kappa}_1'] - 2(p_2 - 2)\, \underbrace{E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_1'\, T_n^{-1}]}_{e_3} + (p_2 - 2)^2\, \underbrace{E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_4'\, \mathbf{T}_n^{-2}]}_{e_4},
\end{aligned}$$

similar to e_1, we can write e_3 as follows:

$$
\begin{aligned}
e_3 &= E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_1'\, T_n^{-1}] \\
&= E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_4'\, T_n^{-1}] + \big[(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big]\, E[\boldsymbol{\kappa}_4\, T_n^{-1}] \\
&= Var[\boldsymbol{\kappa}_4]\, E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big] + E[\boldsymbol{\kappa}_4]\, E[\boldsymbol{\kappa}_1']\, E\Big[\frac{1}{\chi^2_{p_2+4}(\Delta^*)}\Big] \\
&+ \big[(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big]\, E[\boldsymbol{\kappa}_4\, T_n^{-1}],
\end{aligned}
$$

and by using Equation (27), e_4 becomes

$$
\begin{aligned}
e_4 &= E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_4'\, T_n^{-2}] \\
&= Var[\boldsymbol{\kappa}_4]\, E\Big[\frac{1}{(\chi^2_{p_2+2}(\Delta^*))^2}\Big] + E[\boldsymbol{\kappa}_4]\, E[\boldsymbol{\kappa}_1']\, E\Big[\frac{1}{(\chi^2_{p_2+4}(\Delta^*))^2}\Big],
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LS}) &= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{UR}) \\
&- 2\,(p_2-2)\left(\big[(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big]\,\big[\boldsymbol{\mathcal{Z}}\,\mathbf{H}[\mathbf{B}-\mathbf{I}_p]\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\big]\, E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\right) \\
&+ (p_2-2)\,(p_2-4)\,\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\left(E\Big[\frac{1}{(\chi^2_{p_2+2}(\Delta^*))^2}\Big] - E\Big[\frac{1}{\chi^2_{p_2+2}(\Delta^*)}\Big]\right) \\
&+ (p_2-2)\,(p_2-4)\,[\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]\,[\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]' \\
&\times \left(E\Big[\frac{1}{(\chi^2_{p_2+4}(\Delta^*))^2}\Big] - E\Big[\frac{1}{\chi^2_{p_2+4}(\Delta^*)}\Big]\right).
\end{aligned}
$$

Finally, we can write $\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LPS})$ as follows:

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LPS}) &= \lim_{n\to\infty} E\Big(\sqrt{n}(\widehat{\boldsymbol{\beta}}^{LPS} - \boldsymbol{\beta})\,\sqrt{n}(\widehat{\boldsymbol{\beta}}^{LPS} - \boldsymbol{\beta})'\Big) \\
&= \lim_{n\to\infty} E\Big[\sqrt{n}\,\Big(\widehat{\boldsymbol{\beta}}^{LS} - (1-(p_2-2)\,T_n^{-1})\,I(T_n < p_2-2)\,(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}) - \boldsymbol{\beta}\Big) \\
&\times \sqrt{n}\,\Big(\widehat{\boldsymbol{\beta}}^{LS} - (1-(p_2-2)\,T_n^{-1})\,I(T_n < p_2-2)\,(\widehat{\boldsymbol{\beta}}^{UR} - \widehat{\boldsymbol{\beta}}^{RL}) - \boldsymbol{\beta}\Big)'\Big] \\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{LS}) - 2E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_3'\,(1-(p_2-2)\,T_n^{-1})\,I(T_n < p_2-2)] \\
&- 2E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_4'\,(1-(p_2-2)\,T_n^{-1})^2\,I(T_n < p_2-2)] \\
&+ E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_4'\,(1-(p_2-2)\,T_n^{-1})^2\,I(T_n < p_2-2)] \\
&= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{LS}) - 2\,\underbrace{E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_3'\,(1-(p_2-2)\,T_n^{-1})\,I(T_n < p_2-2)]}_{e_5} \\
&- \underbrace{E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_4'\,(1-(p_2-2)\,T_n^{-1})^2\,I(T_n < p_2-2)]}_{e_6},
\end{aligned}
$$

now we obtain e_5

$$
\begin{aligned}
e_5 &= E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_3' \,(1-(p_2-2)\,T_n^{-1})\, I(T_n < p_2-2)] \\
&= E[\boldsymbol{\kappa}_4\, E\{\boldsymbol{\kappa}_3' \,(1-(p_2-2)\,T_n^{-1})\ I(T_n < p_2-2)\,|\,\boldsymbol{\kappa}_4\}] \\
&= E[\boldsymbol{\kappa}_4\, \{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}' \\
&\times (1-(p_2-2)\,T_n^{-1})\ I(T_n < p_2-2)] \\
&= \{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}\, E[\boldsymbol{\kappa}_4\,(1-(p_2-2)\,T_n^{-1})\ I(T_n < p_2-2)] \\
&= \{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}\, E[\boldsymbol{\kappa}_4]\, E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)\, I(\chi^2_{p_2+2}(\Delta^*) < p_2-2)\Big] \\
&= \{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}\, [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}] \\
&\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)\, I(\chi^2_{p_2+2}(\Delta^*) < p_2-2)\Big]
\end{aligned}
$$

and based on Equation (27), e_6 becomes

$$
\begin{aligned}
e_6 &= E[\boldsymbol{\kappa}_4\, \boldsymbol{\kappa}_4' \,(1-(p_2-2)\,T_n^{-1})^2\, I(T_n < p_2-2)] \\
&= Var(\boldsymbol{\kappa}_4)\, E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2\, I(\chi^2_{p_2+2}(\Delta^*) < p_2-2)\Big] \\
&+ E(\boldsymbol{\kappa}_4)\, E(\boldsymbol{\kappa}_4')\, E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2\, I(\chi^2_{p_2+4}(\Delta^*) < p_2-2)\Big] \\
&= \boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\, E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2\, I(\chi^2_{p_2+2}(\Delta^*) < p_2-2)\Big] \\
&+ [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]\, [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]' \\
&\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2\, I(\chi^2_{p_2+4}(\Delta^*) < p_2-2)\Big]
\end{aligned}
$$

Therefore, $\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LPS})$ becomes

$$
\begin{aligned}
\mathcal{V}(\widehat{\boldsymbol{\beta}}^{LPS}) &= \mathcal{V}(\widehat{\boldsymbol{\beta}}^{LS}) \\
&- 2\Big(\{(\mathbf{I}_p - \boldsymbol{\mathcal{Z}}\,\mathbf{H})\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}\}\, [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} - \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}] \\
&\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)\, I(\chi^2_{p_2+2}(\Delta^*) < p_2-2)\Big]\Big) \\
&- \Big(\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,\mathbf{B}\,\boldsymbol{\mathcal{I}}^{-1}\,\mathbf{B}'\, E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+2}(\Delta^*)}\Big)^2\, I(\chi^2_{p_2+2}(\Delta^*) < p_2-2)\Big] \\
&+ [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]\, [\boldsymbol{\mathcal{Z}}\,\mathbf{H}\,(\mathbf{B}-\mathbf{I}_p)\,\boldsymbol{\beta} + \boldsymbol{\mathcal{Z}}\,\boldsymbol{\vartheta}]' \\
&\times E\Big[\Big(1-\frac{p_2-2}{\chi^2_{p_2+4}(\Delta^*)}\Big)^2\, I(\chi^2_{p_2+4}(\Delta^*) < p_2-2)\Big]\Big).
\end{aligned}
$$

Appendix 4. Bootstrap Standard Errors

Table 6: Coefficients and bootstrapped standard errors of the proposed estimators for econometric data

number	Variables	UR	LR	LLS	LPT	LSPE	LS	LPS
				Coefficients				
1	houseval	0.1035	0.1407	0.1221	0.1092	0.1064	0.1105	0.1105
2	education	-3.5113	-3.5892	-3.5503	-3.5237	-3.5175	-3.5259	-3.5259
3	recreation	-2.9546	-2.6228	-2.7887	-2.9087	-2.9317	-2.8939	-2.894
4	social	-3.3167	-3.4323	-3.3745	-3.3343	-3.3255	-3.3382	-3.3381
5	urbanplanning	-3.6089	-3.6965	-3.6527	-3.6219	-3.6154	-3.6251	-3.6251
6	popdens	-0.0119	-0.0117	-0.0118	-0.0124	-0.0121	-0.0119	-0.0119
7	noleft	-0.0181	-0.021	-0.0195	-0.0200	-0.0191	-0.0191	-0.0191
8	minorityleft	-0.0895	-0.037	-0.0633	-0.0806	-0.0851	-0.0794	-0.0794
9	safety	0.3466	-0.024	0.1613	0.2971	0.3219	0.2792	0.2793
10	tot	-0.0029	-7e-04	-0.0018	-0.0027	-0.0028	-0.0026	-0.0026
				Standard Errors				
1	houseval	0.0573	0.0486	0.0515	0.0594	0.0577	0.0558	0.0557
2	education	0.2625	0.2606	0.2589	0.2662	0.2635	0.2618	0.2616
3	recreation	0.5526	0.4194	0.4585	0.5567	0.5463	0.5165	0.5163
4	social	0.1856	0.1475	0.1588	0.1944	0.1877	0.1787	0.1784
5	urbanplanning	0.159	0.1431	0.1465	0.1671	0.1616	0.1569	0.1566
6	popdens	0.0256	0.0055	0.0141	0.022	0.023	0.0203	0.0203
7	noleft	0.0626	0.006	0.0311	0.0555	0.0573	0.0501	0.05
8	minorityleft	0.0589	0.0073	0.0297	0.0583	0.0562	0.0487	0.0485
9	safety	0.445	0.0055	0.223	0.4492	0.433	0.374	0.3737
10	tot	0.0049	0.0013	0.0031	0.0049	0.0048	0.0044	0.0044

Table 7: Coefficients and bootstrapped standard errors of the proposed estimators for educational data

	Variables	UR	RR	RLS	RPT	SPE	RS	RPS
	Coefficients							
1	Father Education	-0.0319	-0.0361	-0.034	-0.0337	-0.0328	-0.0335	-0.0335
2	Age	-0.1026	-0.1087	-0.1056	-0.1058	-0.1042	-0.1051	-0.105
3	Studytime	-0.0291	-0.0252	-0.0271	-0.0269	-0.028	-0.0274	-0.0274
4	Failures	0.0324	0.0267	0.0295	0.0293	0.0308	0.0299	0.03
5	Health	-0.0378	-0.0378	-0.0378	-0.0377	-0.0378	-0.0378	-0.0378
6	First period grade	0.0482	0.0492	0.0487	0.0486	0.0484	0.0485	0.0485
7	second period grade	0.1658	0.166	0.1659	0.166	0.1659	0.166	0.1659
8	Mother education	-0.0044	-4e-04	-0.0024	-0.0028	-0.0036	-0.0029	-0.0029
9	Travel time	-0.0221	-3e-04	-0.0112	-0.0107	-0.0164	-0.0128	-0.0131
10	Absences	-0.0011	-1e-04	-6e-04	-5e-04	-8e-04	-7e-04	-7e-04
11	Weekend Alcohol consumption	-0.0085	-1e-04	-0.0043	-0.0046	-0.0066	-0.0052	-0.0053
12	quality of family relationships	0.0092	-6e-04	0.0043	0.004	0.0066	0.0048	0.0049
13	going out	-0.0202	-3e-04	-0.0102	-0.0096	-0.0149	-0.0117	-0.0119
14	Workday alcohol consumption	4e-04	1e-04	2e-04	5e-04	4e-04	2e-04	2e-04
	Standard Errors							
1	Father Education	0.0199	0.0155	0.0167	0.0178	0.0183	0.0174	0.0173
2	Age	0.0084	0.0055	0.0064	0.0078	0.0077	0.0071	0.0070
3	Studytime	0.0222	0.0211	0.0213	0.0219	0.0218	0.0216	0.0216
4	Failures	0.0307	0.0302	0.0301	0.0308	0.0306	0.0304	0.0303
5	Health	0.0136	0.0132	0.0133	0.0134	0.0134	0.0134	0.0134
6	First period grade	0.0115	0.0111	0.0112	0.0113	0.0113	0.0113	0.0112
7	second period grade	0.0123	0.0116	0.0118	0.0119	0.0120	0.0119	0.0119
8	Mother education	0.0200	0.0004	0.0100	0.0133	0.0152	0.0123	0.0120
9	Travel time	0.0271	0.0005	0.0135	0.0224	0.0224	0.0183	0.0177
10	Absences	0.0021	0.0000	0.0010	0.0015	0.0016	0.0013	0.0012
11	Weekend Alcohol consumption	0.0183	0.0003	0.0091	0.0130	0.0142	0.0116	0.0112
12	quality of family relationships	0.0190	0.0003	0.0095	0.0136	0.0148	0.0122	0.0116
13	going out	0.0176	0.0003	0.0088	0.0166	0.0153	0.0129	0.0123
14	Workday alcohol consumption	0.0245	0.0004	0.0123	0.0163	0.0187	0.0151	0.0147

Appendix 5. Tables of Simulation Results

Table 8: RMSE values for $\rho = 0.6$

	$n = 50$							$n = 100$							$n = 200$						
	Δ	RL	LLS	LPT	LSPE	LS	LPS	Δ	RL	LLS	LPT	LSPE	LS	LPS	Δ	RL	LLS	LPT	LSPE	LS	LPS
$p_2 = 10$	0.00	2.5900	1.9060	1.5150	1.3540	1.7340	1.7950	0.00	1.5590	1.3860	1.4110	1.2920	1.3850	1.4380	0.00	1.4790	1.3320	1.3640	1.2590	1.3370	1.3790
	0.20	1.5660	1.5420	1.1490	1.1500	1.3690	1.3930	0.20	1.2490	1.2910	1.0610	1.0730	1.2200	1.2230	0.20	0.9800	1.1410	0.9730	1.0210	1.1010	1.1040
	0.40	1.2270	1.3850	1.0260	1.0620	1.2310	1.2460	0.40	1.0430	1.1990	0.9900	1.0110	1.1290	1.1300	0.40	0.7700	1.0260	0.9720	0.9940	1.0410	1.0410
	0.60	1.0320	1.2730	0.9860	1.0240	1.1650	1.1690	0.60	0.9200	1.1210	0.9900	1.0010	1.0860	1.0860	0.60	0.6520	0.9430	0.9940	0.9980	1.0140	1.0140
	0.80	0.8960	1.1750	0.9650	1.0050	1.1060	1.1080	0.80	0.8410	1.0650	0.9920	0.9990	1.0570	1.0570	0.80	0.5870	0.8910	0.9980	0.9990	1.0030	1.0030
	1.00	0.8050	1.1100	0.9570	0.9960	1.0720	1.0730	1.00	0.7800	1.0210	0.9970	0.9990	1.0400	1.0400	1.00	0.5360	0.8460	1.0000	1.0000	0.9930	0.9930
	1.20	0.7290	1.0490	0.9690	0.9960	1.0520	1.0520	1.20	0.7320	0.9780	0.9980	0.9990	1.0220	1.0220	1.20	0.5050	0.8110	1.0000	1.0000	0.9860	0.9860
	2.00	0.6040	0.9150	0.9900	0.9960	1.0020	1.0020	2.00	0.6560	0.9040	1.0000	1.0000	0.9970	0.9970	2.00	0.4410	0.7380	1.0000	1.0000	0.9780	0.9780
$p_2 = 15$	0.00	4.8160	2.4820	1.3420	1.2380	2.0840	2.1040	0.00	1.7960	1.4700	1.5250	1.3280	1.5610	1.5860	0.00	1.6030	1.4080	1.4670	1.3250	1.4640	1.5140
	0.20	3.1930	2.1840	1.1680	1.1290	1.7090	1.7170	0.20	1.0950	1.2120	1.0090	1.0430	1.1670	1.1710	0.20	1.1700	1.2660	1.0250	1.0440	1.2170	1.2190
	0.40	2.2610	1.9320	1.1010	1.0870	1.5470	1.5550	0.40	0.8870	1.1100	0.9750	1.0000	1.0880	1.0880	0.40	0.9250	1.1520	0.9940	1.0010	1.1190	1.1190
	0.60	1.8380	1.7620	1.0710	1.0690	1.4390	1.4450	0.60	0.7510	1.0180	0.9890	0.9980	1.0410	1.0410	0.60	0.7800	1.0610	0.9990	1.0000	1.0730	1.0730
	0.80	1.5880	1.6560	1.0360	1.0470	1.3620	1.3650	0.80	0.6770	0.9650	0.9920	0.9980	1.0180	1.0180	0.80	0.6880	0.9950	1.0000	1.0000	1.0450	1.0450
	1.00	1.3720	1.5470	1.0130	1.0230	1.2880	1.2900	1.00	0.6200	0.9160	0.9970	0.9990	0.9980	0.9980	1.00	0.6250	0.9430	1.0000	1.0000	1.0270	1.0270
	1.20	1.2430	1.4550	1.0120	1.0220	1.2440	1.2450	1.20	0.5810	0.8790	1.0000	1.0000	0.9880	0.9880	1.20	0.5810	0.9020	1.0000	1.0000	1.0160	1.0160
	2.00	0.9140	1.2290	0.9920	1.0020	1.1260	1.1260	2.00	0.4830	0.7840	1.0000	1.0000	0.9670	0.9670	2.00	0.4800	0.7960	1.0000	1.0000	0.9900	0.9900
$p_2 = 20$	0.00	3.3870	2.1300	1.6240	1.4060	2.3570	2.3990	0.00	1.7110	1.4720	1.5200	1.3590	1.5880	1.6260	0.00	1.6260	1.4050	1.5100	1.3380	1.5240	1.5550
	0.20	2.2970	1.9430	1.1600	1.1330	1.8150	1.8190	0.20	1.1830	1.2760	1.0280	1.0500	1.2470	1.2480	0.20	0.9820	1.2030	0.9870	1.0110	1.1730	1.1730
	0.40	1.6600	1.7790	1.0320	1.0370	1.5440	1.5450	0.40	0.9440	1.1660	0.9900	1.0030	1.1420	1.1420	0.40	0.7140	1.0480	0.9980	1.0000	1.0810	1.0810
	0.60	1.2790	1.5970	1.0040	1.0100	1.3960	1.3960	0.60	0.8020	1.0800	0.9970	0.9990	1.0870	1.0870	0.60	0.5690	0.9330	0.9990	1.0000	1.0350	1.0350
	0.80	1.0940	1.4850	1.0000	1.0010	1.3060	1.3060	0.80	0.7200	1.0240	1.0000	1.0000	1.0600	1.0600	0.80	0.4960	0.8560	1.0000	1.0000	1.0100	1.0100
	1.00	0.9600	1.3930	0.9990	1.0000	1.2480	1.2480	1.00	0.6630	0.9770	1.0000	1.0000	1.0390	1.0390	1.00	0.4380	0.7940	1.0000	1.0000	0.9960	0.9960
	1.20	0.8560	1.3130	0.9990	1.0000	1.2050	1.2050	1.20	0.6180	0.9370	1.0000	1.0000	1.0240	1.0240	1.20	0.4070	0.7520	1.0000	1.0000	0.9850	0.9850
	2.00	0.6610	1.1050	1.0000	1.0000	1.1110	1.1110	2.00	0.5240	0.8400	1.0000	1.0000	0.9960	0.9960	2.00	0.3350	0.6490	1.0000	1.0000	0.9670	0.9670

Table 9: RMSE values for $\rho = 0.9$

		$n = 50$							$n = 100$							$n = 200$					
	Δ	RL	LLS	LPT	LSPE	LS	LPS	Δ	RL	LLS	LPT	LSPE	LS	LPS	Δ	RL	LLS	LPT	LSPE	LS	LPS
$p_2 = 10$	0.00	6.3380	2.7530	1.6390	1.4210	2.1990	2.2690	0.00	2.3400	1.7220	1.8310	1.4960	1.7850	1.8630	0.00	1.8400	1.5440	1.6010	1.4080	1.5510	1.6410
	0.20	4.8170	2.4920	1.4890	1.3280	1.9750	2.0310	0.20	2.2310	1.6880	1.5670	1.3620	1.6930	1.7180	0.20	1.6700	1.4740	1.3470	1.2610	1.4440	1.4640
	0.40	3.8880	2.3260	1.4990	1.3400	1.9500	1.9760	0.40	2.0180	1.6400	1.4540	1.3180	1.5870	1.6160	0.40	1.5080	1.4140	1.2210	1.1880	1.3630	1.3750
	0.60	3.4110	2.1970	1.4320	1.3030	1.8170	1.8440	0.60	1.8730	1.6150	1.3140	1.2420	1.5350	1.5420	0.60	1.3770	1.3530	1.1200	1.1200	1.2830	1.2870
	0.80	2.9450	2.0640	1.3950	1.2850	1.7430	1.7720	0.80	1.7230	1.5510	1.2620	1.2120	1.4670	1.4740	0.80	1.2900	1.3140	1.0710	1.0840	1.2410	1.2440
	1.00	2.7440	2.0110	1.3270	1.2420	1.6830	1.7080	1.00	1.6010	1.5130	1.1880	1.1660	1.4110	1.4170	1.00	1.2130	1.2760	1.0400	1.0620	1.2100	1.2110
	1.20	2.4800	1.9250	1.2750	1.2090	1.6070	1.6320	1.20	1.5050	1.4690	1.1450	1.1380	1.3720	1.3740	1.20	1.1440	1.2270	1.0190	1.0430	1.1690	1.1700
	2.00	1.9640	1.7180	1.1910	1.1590	1.4640	1.4740	2.00	1.2640	1.3470	1.0600	1.0820	1.2600	1.2610	2.00	1.0090	1.1380	0.9920	1.0100	1.1030	1.1030
$p_2 = 15$	0.00	6.3480	2.7200	1.3580	1.2460	2.2390	2.2580	0.00	4.2770	2.4180	1.7740	1.5030	2.5140	2.5770	0.00	3.2630	1.9770	2.1750	1.6230	2.3600	2.4310
	0.20	4.0210	2.3630	1.3630	1.2560	2.0500	2.0730	0.20	3.3780	2.1520	1.5180	1.3500	2.1730	2.2010	0.20	3.3300	2.0190	1.6420	1.3900	2.1370	2.1490
	0.40	3.3410	2.1960	1.2770	1.2050	1.8610	1.8740	0.40	2.8410	1.9910	1.4350	1.3020	1.9560	1.9730	0.40	2.9920	1.9820	1.4120	1.2730	1.9560	1.9640
	0.60	2.9400	2.0960	1.2130	1.1660	1.7540	1.7720	0.60	2.5670	1.8900	1.3370	1.2400	1.8340	1.8440	0.60	2.7360	1.9410	1.2890	1.2040	1.8310	1.8340
	0.80	2.4520	1.9390	1.2080	1.1660	1.6580	1.6840	0.80	2.3080	1.7970	1.3070	1.2250	1.7480	1.7540	0.80	2.4370	1.8460	1.1960	1.1450	1.7110	1.7120
	1.00	2.3740	1.9190	1.1640	1.1340	1.5980	1.6210	1.00	2.1760	1.7480	1.2610	1.1950	1.6730	1.6780	1.00	2.2670	1.8120	1.1430	1.1100	1.6400	1.6410
	1.20	2.1000	1.8060	1.1710	1.1440	1.5410	1.5620	1.20	1.9850	1.6590	1.2490	1.1890	1.6000	1.6040	1.20	2.1650	1.7860	1.1050	1.0840	1.5720	1.5730
	2.00	1.6260	1.5960	1.0810	1.0890	1.3850	1.3970	2.00	1.6540	1.4910	1.1540	1.1270	1.4390	1.4410	2.00	1.7730	1.6370	1.0320	1.0290	1.4120	1.4120
$p_2 = 20$	0.00	4.1120	2.3070	1.5860	1.3800	2.5330	2.5660	0.00	2.8100	1.9070	2.1040	1.6320	2.3240	2.3990	0.00	2.6780	1.8600	2.2270	1.6850	2.2660	2.3670
	0.20	3.3680	2.1530	1.3830	1.2660	2.1780	2.1890	0.20	2.2590	1.6970	1.5950	1.3790	1.8960	1.9160	0.20	2.1680	1.7440	1.6190	1.4360	1.9100	1.9470
	0.40	2.9880	2.0740	1.2770	1.2030	2.0050	2.0120	0.40	1.9560	1.5920	1.4250	1.2920	1.6970	1.7090	0.40	1.7990	1.6130	1.3760	1.3080	1.6850	1.7070
	0.60	2.7030	1.9930	1.2010	1.1520	1.8810	1.8830	0.60	1.8170	1.5450	1.2830	1.2090	1.5990	1.6030	0.60	1.6130	1.5540	1.2390	1.2280	1.5640	1.5790
	0.80	2.4460	1.9100	1.1600	1.1260	1.7770	1.7790	0.80	1.6410	1.4670	1.2100	1.1650	1.4900	1.4920	0.80	1.4560	1.4760	1.1500	1.1640	1.4590	1.4680
	1.00	2.3430	1.8910	1.1270	1.1020	1.7320	1.7330	1.00	1.5270	1.4140	1.1620	1.1330	1.4240	1.4260	1.00	1.3570	1.4320	1.0980	1.1260	1.4100	1.4140
	1.20	2.1710	1.8180	1.0870	1.0720	1.6470	1.6480	1.20	1.4610	1.3860	1.1170	1.1030	1.3770	1.3780	1.20	1.2760	1.3910	1.0550	1.0870	1.3580	1.3590
	2.00	1.8220	1.6670	1.0460	1.0420	1.4910	1.4910	2.00	1.2360	1.2560	1.0470	1.0530	1.2380	1.2380	2.00	1.0570	1.2490	0.9900	1.0260	1.2160	1.2170