Supplementary Material

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1 Proof of the qualitative features of $f_U(y)$

Consider the random variable U_n of support $(\alpha, 1]$ and density function $f_U(y)$. Assume $1-\alpha > 0.5$. When $\Theta \sim N(\theta_d, \sigma^2/n_d)$,

$$f_U(y) = G(y)\mathbb{I}_{(\alpha,1]}(y) + D(y) \times \mathbb{I}_{(1-\alpha,1]}(y)$$

where:

$$G(y) = \tau \exp\{-\frac{1}{2}(\Psi + \tau(z_y - z_\alpha))^2 + \frac{1}{2}z_y^2\},$$
$$D(y) = \tau \exp\{-\frac{1}{2}(\Psi - \tau(z_y + z_\alpha))^2 + \frac{1}{2}z_y^2\},$$

and where $\Psi = \frac{\theta_0 - \theta_d}{\sigma/\sqrt{n_d}}$ and $\tau = \sqrt{\frac{n_d}{n}}$. Observe that $\tau > 0$. We obtain the qualitative features of $f_U(y)$ by studying its first derivative wrt y.

For $y \in (\alpha, 1-\alpha]$**:**

$$f_U(y)\mathbb{I}_{(\alpha,1-\alpha]}(y) = G(y).$$

Recall that $\frac{d}{dy}z_y = [\phi(z_y)]^{-1}$. It follows that:

$$f'_U(y) = \frac{dG(y)}{dy} = \frac{G(y)}{\phi(z_y)}\{-\Psi - \tau z_y + \tau z_\alpha + z_y\},$$

which means that

$$f'_U(y) > 0 \iff -\Psi - \tau z_y + \tau z_\alpha + z_y > 0,$$

since $G(y), \phi(z_y) > 0 \; \forall y$.

- $\tau = 1$:

$$\begin{aligned} -\Psi - z_y + z_\alpha + z_y > 0 &\iff -\Psi > -z_\alpha \\ &\iff \frac{\theta_0 - \theta_d}{\sigma/\sqrt{n_d}} < z_\alpha \\ &\iff \theta_d > \theta_0 - z_\alpha \frac{\sigma}{\sqrt{n_d}}. \end{aligned}$$

  Observe that for $\theta_d = \theta_0 - z_\alpha \frac{\sigma}{\sqrt{n_d}}$, $f'_U(y) = 0$ and $f_U(y)$ is constant.

- $\tau \neq 1$:

$$-\Psi - \tau z_y + \tau z_\alpha + z_y > 0 \iff -\Psi > \tau z_y - \tau z_\alpha - z_y.$$

  Since z_y is strictly increasing in $(\alpha, 1-\alpha]$, it has a minimum at $y = \alpha$ and a maximum at $y = 1 - \alpha$. Given that:

$$\lim_{y \longrightarrow \alpha} (\tau z_y - \tau z_\alpha - z_y) = -z_\alpha,$$

  and

$$\lim_{y \longrightarrow 1-\alpha} (\tau z_y - \tau z_\alpha - z_y) = -2\tau z_\alpha + z_\alpha = -z_\alpha(2\tau - 1),$$

  then

$$\begin{aligned} -\Psi - \tau z_y + \tau z_\alpha + z_y > 0 &\iff -\Psi > \max\{-z_\alpha; -z_\alpha(2\tau - 1)\} \\ &\iff \theta_d > \theta^\star, \end{aligned}$$

  where $\theta^\star = \theta_0 + \max\{-z_\alpha; -z_\alpha(2\tau - 1)\} \frac{\sigma}{\sqrt{n_d}}$.

For $y \in (1-\alpha, 1]$**:**

$$f_U(y)\mathbb{I}_{(1-\alpha,1]}(y) = G(y) + D(y).$$

It follows that:

$$\begin{aligned} f'_U(y) &= \frac{G(y)}{\phi(z_y)}(-\Psi - \tau z_y + \tau z_\alpha + z_y) \\ &\quad + \frac{D(y)}{\phi(z_y)}(-\Psi + \tau z_y + \tau z_\alpha + z_y), \end{aligned}$$

and since $G(y), D(y), \phi(z_y) > 0 \; \forall y$,

$$f'_U(y) > 0 \iff \begin{cases} -\Psi > h(y, \tau, \alpha) \\ -\Psi > l(y, \tau, \alpha) \end{cases},$$

where $h(y,\tau,\alpha) = \tau z_y - \tau z_\alpha - z_y$ and $l(y,\tau,\alpha) = -\tau z_y - \tau z_\alpha - z_y$.

- $\tau = 1$:

$$\begin{aligned} h(y,1,\alpha) &= -z_\alpha, \\ l(y,1,\alpha) &= -2z_y - z_\alpha. \end{aligned}$$

  Hence:

$$\begin{aligned} f'_U(y) > 0 &\iff -\Psi > \max\{-z_\alpha, -2z_y - z_\alpha\} = -z_\alpha \\ &\ \text{since } z_y > 0 \text{ for } y \in (1-\alpha, 1] \\ &\iff \theta_d > \theta_0 - z_\alpha \frac{\sigma}{\sqrt{n_d}}. \end{aligned}$$

- $\tau \neq 1$:

$$\begin{aligned} f'_U(y) > 0 &\iff \begin{cases} -\Psi > h(y,\tau,\alpha) \\ -\Psi > l(y,\tau,\alpha) \end{cases} \\ &\iff -\Psi > \max\{z_y(\tau-1), z_y(-\tau-1)\} - \tau z_\alpha \\ &\iff -\Psi > z_y(\tau-1) - \tau z_\alpha \\ &\ \text{since } z_y > 0 \text{ for } y \in (1-\alpha, 1] \text{ and since } \tau - 1 > -\tau - 1 \text{ always} \\ &\iff -\Psi > z_y(\tau-1) - \tau z_\alpha. \end{aligned}$$

  ◇ If $\tau > 1$, $z_y(\tau-1) - \tau z_\alpha$ has a maximum at $y = 1$. Moreover:

$$\lim_{y \longrightarrow 1} z_y(\tau-1) - \tau z_\alpha = +\infty,$$

  which implies that $f'_U(y) > 0 \iff -\Psi > +\infty$, ie never.
  ◇ If $\tau < 1$, $z_y(\tau-1) - \tau z_\alpha$ has a maximum at $y = 1-\alpha$. Moreover:

$$\lim_{y \longrightarrow 1-\alpha} z_y(\tau-1) - \tau z_\alpha = (2\tau-1) z_{1-\alpha},$$

  which implies that

$$\begin{aligned} f'_U(y) > 0 &\iff -\Psi > (2\tau-1) z_{1-\alpha} \\ &\iff \theta_d > \theta_0 - z_\alpha (2\tau-1) \frac{\sigma}{\sqrt{n_d}}. \end{aligned}$$

In conclusion, when $\tau \leq 1$,

$$f'_U(y) > 0 \iff \theta_d > \theta^\star.$$

□

2 Proof of the stochastic order of the PrRVs.

From the expressions of the cdfs of the PrRVs, it is straightforward that:

$$\mathbb{F}_U(y) \leq \mathbb{F}_P(y) \leq \mathbb{F}_J(y),$$

and that

$$\mathbb{F}_C(y) \leq \mathbb{F}_P(y).$$

The stochastic order of $\mathbb{F}_C(y)$ and $\mathbb{F}_U(y)$ is as follows:

- for $y \in [0, \alpha)$, $\mathbb{F}_C(y) = \mathbb{F}_U(y)$,
- for $y \in [\alpha, 1-\alpha]$, $\mathbb{F}_C(y) = \frac{\mathbb{F}_P(y)-\pi_0}{\pi_1} \geq \mathbb{F}_P(y) - \pi_0 = \mathbb{F}_U(y)$,
- for $y \in (1-\alpha, 1]$, $\mathbb{F}_C(y) = \frac{\mathbb{F}_P(y)-\pi_0}{\pi_1} \geq \mathbb{F}_P(y) - \mathbb{F}_P(1-y) = \mathbb{F}_U(y)$ since

  $$\min_{y\in(1-\alpha,1]} \mathbb{F}_P(1-y) = \mathbb{F}_P(\alpha) = \pi_0,$$

  implying that $\mathbb{F}_C(y) \geq \mathbb{F}_U(y)$.

□

3 Simulation algorithm of Section 3.2

```
library(truncnorm)

# Function to compute power given theta

eta_n = function(theta){
num = (theta - theta_0) + sqrt(sigmasq/n)*qnorm(alpha)
den = sqrt(sigmasq/n)
pnorm(num/den)
}

# Specify trial and design prior

theta_0 = 0
n = 128
sigmasq = 4*64
n_d = 4
theta_d = 4
alpha = 0.025
pi1 = 1-pnorm(theta_0, theta_d, sqrt(sigmasq/n_d))

theta_sim = rnorm(100000, theta_d, sqrt(sigmasq/n_d ) )
theta_sim_trunc = rtruncnorm(100000, theta_0, Inf, theta_d, sqrt(sigmasq/n_d ) )

# Simulate the distributions of the PrRVs.
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```
Psim = eta_n(theta_sim)
Jsim = eta_n(theta_sim)*(theta_sim>theta_0)
Csim = eta_n(theta_sim_trunc)
Usim = (1-eta_n(theta_sim))*(theta_sim<=theta_0) +  eta_n(theta_sim)*(theta_sim>theta_0)
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