ONLINE RESOURCE

Characteristic function and moment generating function of multivariate folded normal distribution

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Statistical Papers

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This Online Resource presents a particular form of the moment generating function of the multivariate folded normal distribution for dimensions 4 and 5 and lists an algorithm of evaluation of the characteristic function of the sum of multivariate folded normal distribution. Moreover, this Online Resource expands on the characteristic function of specific cases of the sum of multivariate folded normal distribution and includes the derivation of the second mixed moment for the bivariate folded normal distribution using the results from (Murthy, 2015) and the moment generating function derived in the paper. These materials support a deeper understanding and verification of the theoretical findings presented in the main body of our paper.

Recall that we use the following notation. $\Phi_1(\cdot)$ denotes the cumulative distribution function of the standard normal distribution, $\mathsf{N}(0,1)$ and $\Phi_n(\cdot,\ldots,\cdot;\boldsymbol{R})$ for $n>1$, denotes the cumulative distribution function of the n dimensional normal distribution with zero expected value, unit variances and correlation matrix $\boldsymbol{R}$. Furthermore, we denote $\sigma_{s..} = \mathbf{1}^\top \mathbf{\Sigma}_s \mathbf{1}$ and $\mu_{s.} = \mathbf{1}^\top \boldsymbol{\mu}_s$, the sum of all elements in $\mathbf{\Sigma}_s$ and $\boldsymbol{\mu}_s$, respectively and $\sigma_{sk.} = (\mathbf{\Sigma}_s \mathbf{1})_k$ the sum of elements in the k-th row.

1 Moment generating function of folded normal distributions for selected dimensions

In this section we provide Corollaries for the particular dimensions $n = 4$ and $n = 5$ for the formulation of the mgf of the multivariate folded normal distribution.

Corollary 1 ($n = 4$)**.** *Suppose $\boldsymbol{X} \sim \mathsf{N}_4(\boldsymbol{\mu}, \mathbf{\Sigma})$, $\mathbf{\Sigma}$ is positive definite and $\boldsymbol{W} = |\boldsymbol{X}|$. Denote ϱ_{ij} the elements of the correlation matrix of $\boldsymbol{X}$. We consider the set $\mathcal{S}(4)$, mean vector $\boldsymbol{\mu}_s$ and variance matrices $\mathbf{\Sigma}_s$ of the vector $\boldsymbol{X}^s$, following their definition from Theorem 4. We introduce the vector*

$$\boldsymbol{\nu}^s = \left[\tfrac{(\mathbf{\Sigma}_s \boldsymbol{t} + \boldsymbol{\mu}_s)_1}{\sqrt{\sigma_{11}}}, \tfrac{(\mathbf{\Sigma}_s \boldsymbol{t} + \boldsymbol{\mu}_s)_2}{\sqrt{\sigma_{22}}}, \tfrac{(\mathbf{\Sigma}_s \boldsymbol{t} + \boldsymbol{\mu}_s)_3}{\sqrt{\sigma_{33}}}, \tfrac{(\mathbf{\Sigma}_s \boldsymbol{t} + \boldsymbol{\mu}_s)_4}{\sqrt{\sigma_{44}}} \right].$$

Moment generating function of $\boldsymbol{W}$ has the following form

$$\begin{aligned}
\mathrm{mgf}_{\boldsymbol{W}}(\boldsymbol{t}) = \sum_{s \in \mathcal{S}(4)} & \exp\left(\tfrac{1}{2} \boldsymbol{t}^\top \mathbf{\Sigma}_s \boldsymbol{t} + \boldsymbol{t}^\top \boldsymbol{\mu}_s\right) \cdot \Big(\Phi_1(\nu_4^s) - \\
& - \Phi_2(\nu_4^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{2,\mathcal{C}_1}) - \Phi_2(\nu_4^s, -\nu_2^s; \widetilde{\boldsymbol{R}}^s_{2,\mathcal{C}_2}) - \Phi_2(\nu_4^s, -\nu_3^s; \widetilde{\boldsymbol{R}}^s_{2,\mathcal{C}_3}) + \\
& + \Phi_3(\nu_4^s, -\nu_3^s, -\nu_2^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_1}) + \Phi_3(\nu_4^s, -\nu_3^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_2}) + \Phi_3(\nu_4^s, -\nu_2^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_3}) - \\
& - \Phi_4(\nu_4^s, -\nu_3^s, -\nu_2^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{4,\mathcal{C}_1}) \Big)
\end{aligned}$$

where

$$\widetilde{\boldsymbol{R}}_{2,\mathcal{C}_1} = \begin{pmatrix} 1 & -\varrho_{14} \\ -\varrho_{14} & 1 \end{pmatrix}, \quad \widetilde{\boldsymbol{R}}_{2,\mathcal{C}_2} = \begin{pmatrix} 1 & -\varrho_{24} \\ -\varrho_{24} & 1 \end{pmatrix}, \quad \widetilde{\boldsymbol{R}}_{2,\mathcal{C}_3} = \begin{pmatrix} 1 & -\varrho_{34} \\ -\varrho_{34} & 1 \end{pmatrix}$$

$$\widetilde{\boldsymbol{R}}_{3,\mathcal{C}_1} = \begin{pmatrix} 1 & -\varrho_{34} & -\varrho_{24} \\ -\varrho_{34} & 1 & \varrho_{23} \\ -\varrho_{24} & \varrho_{23} & 1 \end{pmatrix}, \quad \widetilde{\boldsymbol{R}}_{3,\mathcal{C}_2} = \begin{pmatrix} 1 & -\varrho_{34} & -\varrho_{14} \\ -\varrho_{34} & 1 & \varrho_{13} \\ -\varrho_{14} & \varrho_{13} & 1 \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}_{3,\mathcal{C}_3} = \begin{pmatrix} 1 & -\varrho_{24} & -\varrho_{14} \\ -\varrho_{24} & 1 & \varrho_{12} \\ -\varrho_{14} & \varrho_{12} & 1 \end{pmatrix}, \widetilde{\boldsymbol{R}}_{4,\mathcal{C}_1} = \begin{pmatrix} 1 & -\varrho_{34} & -\varrho_{24} & -\varrho_{14} \\ -\varrho_{34} & 1 & \varrho_{23} & \varrho_{13} \\ -\varrho_{24} & \varrho_{23} & 1 & \varrho_{12} \\ -\varrho_{14} & \varrho_{13} & \varrho_{12} & 1 \end{pmatrix}.$$

and $\widetilde{\boldsymbol{R}}^s_{n,\mathcal{C}_j} = \mathbf{\Lambda}_s \widetilde{\boldsymbol{R}}_{n,\mathcal{C}_j} \mathbf{\Lambda}_s$.

Corollary 2 ($n = 5$)**.** *Suppose $\boldsymbol{X} \sim \mathsf{N}_5(\boldsymbol{\mu}, \mathbf{\Sigma})$, $\mathbf{\Sigma}$ is positive definite and $\boldsymbol{W} = |\boldsymbol{X}|$. Denote ϱ_{ij} the elements of the correlation matrix of $\boldsymbol{X}$. We consider the set $\mathcal{S}(5)$, mean vector $\boldsymbol{\mu}_s$ and variance matrices $\mathbf{\Sigma}_s$ of the vector $\boldsymbol{X}^s$, following their definition* *from Theorem 4. We introduce the vector*

$$\boldsymbol{\nu}^s = \left[\frac{(\boldsymbol{\Sigma}_s \boldsymbol{t}+\boldsymbol{\mu}_s)_1}{\sqrt{\sigma_{11}}}, \frac{(\boldsymbol{\Sigma}_s \boldsymbol{t}+\boldsymbol{\mu}_s)_2}{\sqrt{\sigma_{22}}}, \frac{(\boldsymbol{\Sigma}_s \boldsymbol{t}+\boldsymbol{\mu}_s)_3}{\sqrt{\sigma_{33}}}, \frac{(\boldsymbol{\Sigma}_s \boldsymbol{t}+\boldsymbol{\mu}_s)_4}{\sqrt{\sigma_{44}}}, \frac{(\boldsymbol{\Sigma}_s \boldsymbol{t}+\boldsymbol{\mu}_s)_5}{\sqrt{\sigma_{55}}} \right].$$

Moment generating function of $\boldsymbol{W}$ has the following form

$$\begin{aligned}
\mathrm{mgf}_{\boldsymbol{W}}(\boldsymbol{t}) = \sum_{s\in\mathcal{S}(5)} & \exp\left(\tfrac{1}{2}\boldsymbol{t}^\top \boldsymbol{\Sigma}_s \boldsymbol{t} + \boldsymbol{t}^\top \boldsymbol{\mu}_s\right) \cdot \Big(\Phi_1(\nu_5^s) - \\
& - \Phi_2(\nu_5^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{2,\mathcal{C}_1}) - \Phi_2(\nu_5^s, -\nu_2^s; \widetilde{\boldsymbol{R}}^s_{2,\mathcal{C}_2}) - \\
& - \Phi_2(\nu_5^s, -\nu_3^s; \widetilde{\boldsymbol{R}}^s_{2,\mathcal{C}_3}) - \Phi_2(\nu_5^s, -\nu_4^s; \widetilde{\boldsymbol{R}}^s_{2,\mathcal{C}_4}) + \\
& + \Phi_3(\nu_5^s, -\nu_4^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_1}) + \Phi_3(\nu_5^s, -\nu_4^s, -\nu_2^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_2}) + \Phi_3(\nu_5^s, -\nu_4^s, -\nu_3^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_3}) + \\
& + \Phi_3(\nu_5^s, -\nu_3^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_4}) + \Phi_3(\nu_5^s, -\nu_3^s, -\nu_2^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_5}) + \Phi_3(\nu_5^s, -\nu_2^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{3,\mathcal{C}_6}) - \\
& - \Phi_4(\nu_5^s, -\nu_4^s, -\nu_3^s, -\nu_2^s; \widetilde{\boldsymbol{R}}^s_{4,\mathcal{C}_1}) - \Phi_4(\nu_5^s, -\nu_4^s, -\nu_3^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{4,\mathcal{C}_2}) - \\
& - \Phi_4(\nu_5^s, -\nu_4^s, -\nu_2^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{4,\mathcal{C}_3}) - \Phi_4(\nu_5^s, -\nu_3^s, -\nu_2^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{4,\mathcal{C}_4}) + \\
& + \Phi_5(\nu_5^s, -\nu_4^s, -\nu_3^s, -\nu_2^s, -\nu_1^s; \widetilde{\boldsymbol{R}}^s_{5,\mathcal{C}_1})\Big)
\end{aligned}$$

where

$$\widetilde{\boldsymbol{R}}_{2,\mathcal{C}_1} = \begin{pmatrix} 1 & -\varrho_{15} \\ -\varrho_{15} & 1 \end{pmatrix}, \quad \widetilde{\boldsymbol{R}}_{2,\mathcal{C}_2} = \begin{pmatrix} 1 & -\varrho_{25} \\ -\varrho_{25} & 1 \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}_{2,\mathcal{C}_3} = \begin{pmatrix} 1 & -\varrho_{35} \\ -\varrho_{35} & 1 \end{pmatrix}, \quad \widetilde{\boldsymbol{R}}_{2,\mathcal{C}_4} = \begin{pmatrix} 1 & -\varrho_{45} \\ -\varrho_{45} & 1 \end{pmatrix}$$

$$\widetilde{\boldsymbol{R}}_{3,\mathcal{C}_1} = \begin{pmatrix} 1 & -\varrho_{45} & -\varrho_{15} \\ -\varrho_{45} & 1 & \varrho_{14} \\ -\varrho_{15} & \varrho_{14} & 1 \end{pmatrix}, \ \widetilde{\boldsymbol{R}}_{3,\mathcal{C}_2} = \begin{pmatrix} 1 & -\varrho_{45} & -\varrho_{25} \\ -\varrho_{45} & 1 & \varrho_{24} \\ -\varrho_{25} & \varrho_{24} & 1 \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}_{3,\mathcal{C}_3} = \begin{pmatrix} 1 & -\varrho_{45} & -\varrho_{35} \\ -\varrho_{45} & 1 & \varrho_{34} \\ -\varrho_{35} & \varrho_{34} & 1 \end{pmatrix}, \ \widetilde{\boldsymbol{R}}_{3,\mathcal{C}_4} = \begin{pmatrix} 1 & -\varrho_{35} & -\varrho_{15} \\ -\varrho_{35} & 1 & \varrho_{13} \\ -\varrho_{15} & \varrho_{13} & 1 \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}_{3,\mathcal{C}_5} = \begin{pmatrix} 1 & -\varrho_{35} & -\varrho_{25} \\ -\varrho_{35} & 1 & \varrho_{23} \\ -\varrho_{25} & \varrho_{23} & 1 \end{pmatrix}, \ \widetilde{\boldsymbol{R}}_{3,\mathcal{C}_6} = \begin{pmatrix} 1 & -\varrho_{25} & -\varrho_{15} \\ -\varrho_{25} & 1 & \varrho_{12} \\ -\varrho_{15} & \varrho_{12} & 1 \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}_{4,\mathcal{C}_1} = \begin{pmatrix} 1 & -\varrho_{45} & -\varrho_{35} & -\varrho_{25} \\ -\varrho_{45} & 1 & \varrho_{34} & \varrho_{24} \\ -\varrho_{35} & \varrho_{34} & 1 & \varrho_{23} \\ -\varrho_{25} & \varrho_{24} & \varrho_{23} & 1 \end{pmatrix}, \ \widetilde{\boldsymbol{R}}_{4,\mathcal{C}_2} = \begin{pmatrix} 1 & -\varrho_{45} & -\varrho_{35} & -\varrho_{15} \\ -\varrho_{45} & 1 & \varrho_{34} & \varrho_{14} \\ -\varrho_{35} & \varrho_{34} & 1 & \varrho_{13} \\ -\varrho_{15} & \varrho_{14} & \varrho_{13} & 1 \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}_{4,\mathcal{C}_3} = \begin{pmatrix} 1 & -\varrho_{45} & -\varrho_{25} & -\varrho_{15} \\ -\varrho_{45} & 1 & \varrho_{24} & \varrho_{14} \\ -\varrho_{25} & \varrho_{24} & 1 & \varrho_{12} \\ -\varrho_{15} & \varrho_{14} & \varrho_{12} & 1 \end{pmatrix}, \ \widetilde{\boldsymbol{R}}_{4,\mathcal{C}_4} = \begin{pmatrix} 1 & -\varrho_{35} & -\varrho_{25} & -\varrho_{15} \\ -\varrho_{35} & 1 & \varrho_{23} & \varrho_{13} \\ -\varrho_{25} & \varrho_{23} & 1 & \varrho_{12} \\ -\varrho_{15} & \varrho_{13} & \varrho_{12} & 1 \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}_{5,\mathcal{C}_5} = \begin{pmatrix} 1 & -\varrho_{45} & -\varrho_{35} & -\varrho_{25} & -\varrho_{15} \\ -\varrho_{45} & 1 & \varrho_{34} & \varrho_{24} & \varrho_{14} \\ -\varrho_{35} & \varrho_{34} & 1 & \varrho_{23} & \varrho_{13} \\ -\varrho_{25} & \varrho_{24} & \varrho_{23} & 1 & \varrho_{12} \\ -\varrho_{15} & \varrho_{14} & \varrho_{13} & \varrho_{12} & 1 \end{pmatrix}$$

and $\widetilde{\boldsymbol{R}}^s_{n,\mathcal{C}_j} = \boldsymbol{\Lambda}_s \widetilde{\boldsymbol{R}}_{n,\mathcal{C}_j} \boldsymbol{\Lambda}_s$.

2 Algorithm for characteristic function of folded normal distributions

Algorithm 1 Computation of characteristic function of sum of elements of multivariate folded normal vector with parameters $\mathbf{\Sigma}$ and μ in point $\mathbf{t}$ [`charf_nD`]

Permute elements in $\mathbf{\Sigma}$ to obtain a block diagonal matrix $\mathbf{\Sigma}^*$ by identifying connected components in graph, where vertices i and j are connected if $\sigma_{ij} \neq 0$
Identify the number of diagonal submatrices in $\mathbf{\Sigma}^*$
if there is only one block in $\mathbf{\Sigma}^*$ **then**
    Set $d \leftarrow$ the dimension of vector $\mathbf{t}$
    Generate $S = \{s; s = (s_1, \dots, s_d);\ s_i = \pm 1\}$
    Identify unique $\mathbf{\Sigma}_s = \mathbf{\Lambda_s \Sigma \Lambda_s}$ and $\mu_s = \mathbf{\Lambda}_s \mu$, where $\mathbf{\Lambda_s} = \mathrm{diag}(s)$ over all $s \in S$
    Set $ic \leftarrow$ the counts of each unique transformation in the previous step
    Initialize $value \leftarrow 0$
    for each unique $\mathbf{\Sigma}_s$ and μ_s **do**
        Compute $\mathbf{d}_s \leftarrow \mathrm{diag}(\mathbf{\Sigma}_s)^{-1/2}(i\mathbf{\Sigma}_s\mathbf{t} + \mu_s)$
        Standardize the variance matrix $\mathbf{\Sigma}_s$ to correlation matrix $\mathbf{R}_s$
        Initialize $value_s \leftarrow \Phi_1(\mathbf{d}_s[d])$
        for $i = 2$ to d **do**
            Compute all combinations of indexes M for dimension $i - 1$
            Set $\mathbf{D} \leftarrow -\mathbf{I}_i + \mathrm{diag}(2, 0, \dots, 0)$
            for each combination $\mathbf{k}$ in M **do**
                Update $value_s$ by adding $\Phi_n(\mathbf{d}_s[d], -\mathbf{d}_s[\mathbf{k}]; \mathbf{D}(\mathbf{R}_s[\mathbf{k}, \mathbf{k}])\mathbf{D})$
            end for
        end for
        Update $value$ by adding $ic \cdot value_s$
    end for
    Update $value$ by multiplication by constant $\exp(-\frac{1}{2}\mathbf{t}^T\mathbf{\Sigma}_s\mathbf{t} + i\mathbf{t}^T\mu_s)$
else
    Set $value \leftarrow 1$
    for each diagonal block matrix $\mathbf{B}^*$ in $\mathbf{\Sigma}^*$ **do**
        Compute subproblem using `charf_nD` for matrix $\mathbf{B}^*$ and corresponding subvectors of μ^* and $\mathbf{t}^*$
        Update $value$ by multiplication with the result of the previous step
    end for
end if
return $value$

3 Characteristic function of sum of folded normal distributions with equicorrelation matrix with equal expected values

The characteristic function in this specific case can be expressed using the result given in Subsection 4.1. In equicorrelation matrix, all off-diagonal elements are equal to ϱ, which after some derivation gives

$$\begin{aligned}
\sigma_{s..} &= n + (n(n-1) + 2n_s(n_s - n))\,\varrho \\
\sigma_{sk.} &= 1 + (n-1)\varrho, \quad s_k = 1 \\
\sigma_{sk.} &= 1 + (2n_s - n - 1)\varrho, \quad s_k = -1
\end{aligned}$$

where n_s stands for number of negative signs in s. If the expected values are equal, then $\mu_{s.} = (n - 2n_s)\mu$.

4 Characteristic function of sum of absolute values of 3 consecutive values in MA(1) under normality

The characteristic function in this specific case can be expressed by using the result given in Subsection 4.1. For a normal random vector of three consecutive values in MA(1) process with zero expected value the correlation matrix is of the form

$$\boldsymbol{R} = \begin{pmatrix} 1 & \varrho & 0 \\ \varrho & 1 & \varrho \\ 0 & \varrho & 1 \end{pmatrix}$$

The characteristic function of $Z = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} |X_i|$ can be expressed as follows

$$\begin{aligned}
\mathrm{cf}_Z(t) = \sum_{s \in \mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right) \cdot \Bigg(& \Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right) - \Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right) \\
& - \Phi_2\left(\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t; \boldsymbol{R}_2\right) + \Phi_3\left(\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t, -\sigma_{s1.}\,\mathrm{i}\,t; \boldsymbol{R}_3\right)\Bigg)
\end{aligned}$$

with off-diagonal elements of correlation matrices $\boldsymbol{R}_2$, $r_{12}^{(2)} = -[\boldsymbol{\Lambda}_s \boldsymbol{R} \boldsymbol{\Lambda}_s]_{23}$, and of $\boldsymbol{R}_3$

$$r_{12}^{(3)} = -[\boldsymbol{\Lambda}_s \boldsymbol{R} \boldsymbol{\Lambda}_s]_{23}, \quad r_{13}^{(3)} = 0, \quad r_{23}^{(3)} = [\boldsymbol{\Lambda}_s \boldsymbol{R} \boldsymbol{\Lambda}_s]_{12}$$

Here, the formula (7) can be applied to express Φ_3.

$$\mathrm{cf}_Z(t) = \sum_{s \in \mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right) \cdot \Bigg(\Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) - \Phi_2\left(\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t; -r_{23}\right) +$$

$$+ \Phi_2\left(\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t, -r_{23}\right)\Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right) + I_3(\sigma_{s1.}, \sigma_{s2.}, \sigma_{s3.} r_{12}, r_{23}, t)\Bigg)$$

where

$$\begin{aligned} & I_3(-\sigma_{s1.}, -\sigma_{s2.}, \sigma_{s3.} r_{12}, -r_{23}, t) = I_3(\sigma_{s1.}, \sigma_{s2.}, \sigma_{s3.} r_{12}, r_{23}, t) = \\ & = \frac{1}{2\pi}\int_0^{\arcsin(r_{12})} \exp\left\{\frac{\sigma_{s1.}^2 - 2\sin(\theta)\sigma_{s1.}\sigma_{s2.} + \sigma_{s2.}^2}{2\cos^2(\theta)} t^2\right\} \cdot \\ & \qquad \cdot\Phi_1\left(\left(\sigma_{s3.} - \frac{r_{23}(\sigma_{s2.} - \sin(\theta)\sigma_{s1.})}{\cos^2(\theta)}\right)\mathrm{i}\,t\right) d\theta \end{aligned} \tag{1}$$

Then

$$\begin{aligned} \mathrm{cf}_Z(t) = \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(&\Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) - \\ & - \Phi_2\left(\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t; \boldsymbol{R}_2\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) + I_3(\sigma_{s1.}, \sigma_{s2.}, \sigma_{s3.} r_{12}, r_{23}, t)\Bigg) \end{aligned}$$

And again, Φ_2 can be expressed as a product of univariate Φ_1s and an integral

$$\begin{aligned} \mathrm{cf}_Z(t) = \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(&\Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) - \\ & - \Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) + \\ & + I_2(\sigma_{s2.}, \sigma_{s3.}, r_{23}, t)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) + I_3(\sigma_{s1.}, \sigma_{s2.}, \sigma_{s3.} r_{12}, r_{23}, t)\Bigg) \end{aligned}$$

where

$$\begin{aligned} & I_2(-\sigma_{s2.}, \sigma_{s3.}, -r_{23}, t) = \\ & = \frac{1}{2\pi}\int_0^{-\arcsin(r_{23})} \exp\left\{\frac{\sigma_{s2.}^2 + 2\sin(\theta)\sigma_{s2.}\sigma_{s3.} + \sigma_{s3.}^2}{2\cos^2(\theta)} t^2\right\} d\theta \\ & = -\frac{1}{2\pi}\int_0^{\arcsin(r_{23})} \exp\left\{\frac{\sigma_{s2.}^2 - 2\sin(\theta)\sigma_{s2.}\sigma_{s3.} + \sigma_{s3.}^2}{2\cos^2(\theta)} t^2\right\} d\theta \\ & = -I_2(\sigma_{s2.}, \sigma_{s3.}, r_{23}, t) \end{aligned} \tag{2}$$

In the end, we can see how the characteristic function of sum of independent folded normal distribution and the sum of absolute values of MA(1) process differ

$$\begin{aligned}\mathrm{cf}_Z(t) = &\sum_{s\in\mathcal{S}(n)} \exp\left(-\tfrac{1}{2}t^2\sigma_{s..}\right)\cdot\Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) + \\ &+\sum_{s\in\mathcal{S}(n)} \exp\left(-\tfrac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(I_2(\sigma_{s2.},\sigma_{s3.},r_{23},t)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) + \\ &+I_3(\sigma_{s1.},\sigma_{s2.},\sigma_{s3.}r_{12},r_{23},t)\Bigg)\end{aligned}$$

5 Characteristic function of sum of absolute values of 4 consecutive values in MA(1) under normality

Similarly to the previous section, in the following section, we express the characteristic function of the sum of absolute values of four consecutive values in MA(1) under normality with zero expected value the correlation matrix of the form

$$\boldsymbol{R} = \begin{pmatrix} 1 & \varrho & 0 & 0 \\ \varrho & 1 & \varrho & 0 \\ 0 & \varrho & 1 & \varrho \\ 0 & 0 & \varrho & 1 \end{pmatrix}$$

To avoid denoting various correlation matrices in the expressions of multivariate cumulative distribution function, we list the of diagonal elements (by rows) of the correlation matrices in the arguments of Φ_2, Φ_3, and Φ_4 after the semicolumn. Then we can write

$$\begin{aligned}\mathrm{cf}_Z(t) = &\sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right) - \\ &-\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right) - \Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right) - \Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t; -r_{34}\right) \\ &+\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_2\left(-\sigma_{s2.}\,\mathrm{i}\,t, -\sigma_{s1.}\,\mathrm{i}\,t; r_{12}\right) + \Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right)\Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t; -r_{34}\right) \\ &+\Phi_3\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t; -r_{34}, 0, r_{23}\right) - \\ &-\Phi_4\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t, -\sigma_{s1.}\,\mathrm{i}\,t; -r_{34}, 0, 0, r_{23}, 0, r_{12}\right)\Bigg)\end{aligned}$$

Again, the formula (7) can be applied to express Φ_4.

$$\mathrm{cf}_Z(t) = \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right) -$$
$$-\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right) - \Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right) - \Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t; -r_{34}\right) +$$
$$+\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_2\left(-\sigma_{s2.}\,\mathrm{i}\,t, -\sigma_{s1.}\,\mathrm{i}\,t; r_{12}\right) + \Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right)\Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t; -r_{34}\right) +$$
$$+\Phi_3\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t; -r_{34}, 0, r_{23}\right) -$$
$$-\Phi_3\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t; -r_{34}, 0, r_{23}\right)\Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right) -$$
$$-I_4(-\sigma_{s1.}, -\sigma_{s2.}, -\sigma_{s3.}, \sigma_{s4.}, r_{12}, r_{23}, -r_{34}, t)\Bigg)$$

where

$$I_4(-\sigma_{s1.}, -\sigma_{s2.}, -\sigma_{s3.}, \sigma_{s4.}, r_{12}, r_{23}, -r_{34}, t)$$
$$= \frac{1}{2\pi}\int_0^{\arcsin(r_{12})} \exp\left\{\frac{\sigma_{s1.}^2 - 2\sin(\theta)\sigma_{s1.}\sigma_{s2.} + \sigma_{s2.}^2}{2\cos^2(\theta)}t^2\right\}\cdot$$
$$\cdot\Phi_2\left(\sigma_{s4.}it, -\left(\sigma_{s3.} - \tfrac{r_{23}(\sigma_{s2.}-\sin(\theta)\sigma_{s1.})}{\cos^2(\theta)}\right)\Big/\sqrt{1-\tfrac{r_{23}^2}{\cos^2(\theta)}}\,\mathrm{i}\,t, -r_{34}\Big/\sqrt{1-\tfrac{r_{23}^2}{\cos^2(\theta)}}\right)d\theta$$

Also other rearrangements can be done what gives

$$\mathrm{cf}_Z(t) = \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) - \Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right) +$$
$$+\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_2\left(-\sigma_{s2.}\,\mathrm{i}\,t, -\sigma_{s1.}\,\mathrm{i}\,t; r_{12}\right) - \Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)\Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t; -r_{34}\right) +$$
$$+\Phi_3\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t, -\sigma_{s2.}\,\mathrm{i}\,t; -r_{34}, 0, r_{23}\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) -$$
$$-I_4(-\sigma_{s1.}, -\sigma_{s2.}, -\sigma_{s3.}, \sigma_{s4.}, r_{12}, r_{23}, -r_{34}, t)\Bigg)$$

Next, the function Φ_3 can be written using integral (1)

$$\mathrm{cf}_Z(t) = \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) -$$
$$-\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right) +$$
$$+\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_2\left(-\sigma_{s2.}\,\mathrm{i}\,t, -\sigma_{s1.}\,\mathrm{i}\,t; r_{12}\right) - \Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)\Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t; -r_{34}\right) +$$
$$+\Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t, -\sigma_{s3.}\,\mathrm{i}\,t; -r_{34}\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right) +$$
$$+\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)I_3(\sigma_{s2.}, \sigma_{s3.}, \sigma_{s4.}, r_{23}, r_{34}, t) -$$
$$-I_4(-\sigma_{s1.}, -\sigma_{s2.}, -\sigma_{s3.}, \sigma_{s4.}, r_{12}, r_{23}, -r_{34}, t)\Bigg)$$

which gives

$$
\begin{aligned}
\mathrm{cf}_Z(t) \;=\; \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(& \Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)-\\
& -\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right)\\
& +\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_2\left(-\sigma_{s2.}\,\mathrm{i}\,t,-\sigma_{s1.}\,\mathrm{i}\,t;r_{12}\right)-\\
& -\Phi_2\left(\sigma_{s4.}\,\mathrm{i}\,t,-\sigma_{s3.}\,\mathrm{i}\,t;-r_{34}\right)\Phi_1\left(\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)+\\
& +\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)I_3(\sigma_{s2.},\sigma_{s3.},\sigma_{s4.},r_{23},r_{34},t)-\\
& -I_4(-\sigma_{s1.},-\sigma_{s2.},-\sigma_{s3.},\sigma_{s4.},r_{12},r_{23},-r_{34},t)\Bigg)
\end{aligned}
$$

Also, the functions Φ_2 can be written using integral (2)

$$
\begin{aligned}
\mathrm{cf}_Z(t) \;=\; & \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)-\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right)\\
& +\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s1.}\,\mathrm{i}\,t\right)+\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)I_2(\sigma_{s1.},\sigma_{s2.},r_{21})\\
& -\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(-\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)+\\
& +I_2\left(\sigma_{s3.},\sigma_{s4.},r_{34}\right)\Phi_1\left(\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)\\
& +\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)I_3(\sigma_{s2.},\sigma_{s3.},\sigma_{s4.};r_{23},r_{34})-\\
& -I_4(-\sigma_{s1.},-\sigma_{s2.},-\sigma_{s3.},\sigma_{s4.},r_{12},r_{23},-r_{34},t)\Bigg)
\end{aligned}
$$

what makes

$$
\begin{aligned}
\mathrm{cf}_Z(t) \;=\; & \sum_{s\in\mathcal{S}(n)} \exp\left(-\frac{1}{2}t^2\sigma_{s..}\right)\cdot\Bigg(\Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s3.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)+\\
& \Phi_1\left(\sigma_{s4.}\,\mathrm{i}\,t\right)I_2(\sigma_{s1.},\sigma_{s2.},r_{21})+I_2\left(\sigma_{s3.},\sigma_{s4.},r_{34}\right)\Phi_1\left(\sigma_{s2.}\,\mathrm{i}\,t\right)\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)\\
& +\Phi_1\left(\sigma_{s1.}\,\mathrm{i}\,t\right)I_3(\sigma_{s2.},\sigma_{s3.},\sigma_{s4.};r_{23},r_{34})-\\
& -I_4(-\sigma_{s1.},-\sigma_{s2.},-\sigma_{s3.},\sigma_{s4.},r_{12},r_{23},-r_{34},t)\Bigg)
\end{aligned}
$$

6 Second mixed moment of bivariate folded central normal distribution

Proof of Proposition 10 using the expression in (Murthy, 2015). The second mixed moment expression in (Murthy, 2015) is

$$\mathsf{E}(W_1W_2) = \tfrac{2}{\pi}\sqrt{\sigma_{11}\sigma_{22}}\Big((1-\varrho_{12}^2)\sqrt{1-\varrho_{12}^2} + \pi\varrho_{12}\left[\int_0^\infty 2x^2\varphi(x)\Phi_1(\alpha x)\,\mathrm{d}x - \tfrac{1}{2}\right]\Big),$$

where $\varrho_{12} = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}, \alpha = \varrho_{12}/\sqrt{1-\varrho_{12}^2}$. Since $\Phi_1(x) = \frac{1}{2}(1+\operatorname{erf}\frac{x}{\sqrt{2}})$, we can rewrite the integral

$$\int_0^\infty 2x^2\varphi(x)\Phi_1(\alpha x)\,\mathrm{d}x = \tfrac{1}{\sqrt{2\pi}}\int_0^\infty x^2\mathrm{e}^{-x^2}\,\mathrm{d}x + \tfrac{1}{\sqrt{2\pi}}\int_0^\infty x^2\mathrm{e}^{-x^2}\operatorname{erf}\tfrac{\alpha x}{\sqrt{2}}\,\mathrm{d}x$$

and use formulas (Korotkov and Korotkov, 2020, (2.1.3.-1.)) and (Korotkov and Korotkov, 2020, (2.7.3.-1.)) to express the integrals analytically. That is how we gain

$$\int_0^\infty 2x^2\varphi(x)\Phi_1(\alpha x)\,\mathrm{d}x = \tfrac{1}{2} + \tfrac{1}{\pi}\left(\arctan(\alpha) + \varrho\sqrt{1-\varrho_{12}^2}\right),$$

which proves the proposition by observing that $\arctan(\varrho_{12}/\sqrt{1-\varrho_{12}^2}) = \arcsin(\varrho_{12})$ (Rektorys, 2000, Section 2.11, Proposition 2, Item 12). □

Proof of Proposition 10 from derivative of mgf. We provide a derivative of mixed moment $\mathsf{E}[W_1W_2]$ from the moment generating function such that

$$\mathsf{E}[W_1W_2] = \tfrac{\partial^2}{\partial t_1 \partial t_2}\operatorname{mgf}_{\boldsymbol{W}}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0}.$$

We denote $\boldsymbol{t} = (t_1, t_2)$. Recall the formula for moment generating function for random variable $\boldsymbol{W} = |\boldsymbol{X}|$; $\boldsymbol{X} \sim \mathsf{N}_2(\mathbf{0}_{2\times 1}, \boldsymbol{\Sigma})$ as a direct consequence of Theorem 4 and Corollary 8 such that

$$\operatorname{mgf}_{\boldsymbol{W}}(\boldsymbol{t}) = \sum_{s\in\mathcal{S}(2)} \exp\left(\tfrac{1}{2}\boldsymbol{t}^\top\boldsymbol{\Sigma}_s\boldsymbol{t}\right)\cdot\left(\Phi_1\left(\tfrac{(\boldsymbol{\Sigma}_s\boldsymbol{t})_2}{\sqrt{\sigma_{22}}}\right) - \Phi_2\left(\tfrac{(\boldsymbol{\Sigma}_s\boldsymbol{t})_2}{\sqrt{\sigma_{22}}}, -\tfrac{(\boldsymbol{\Sigma}_s\boldsymbol{t})_1}{\sqrt{\sigma_{11}}}; -\varrho_{12}^s\right)\right)$$

such that $\varrho_{12}^s = \sigma_{12}^s/\sqrt{\sigma_{11}\sigma_{22}}$. Hereinafter the symbol σ_{ij} (ϱ_{ij}) stands for the i,j-th element of matrix $\boldsymbol{\Sigma}_s$ ($\boldsymbol{\Lambda}_s\boldsymbol{R}\boldsymbol{\Lambda}_s$) defined in Remark 3 even though the index s is not mentioned in the notation. Next, recall the form of variables in this case

$$\boldsymbol{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}; \quad \boldsymbol{\Sigma}_s = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}; \quad \boldsymbol{\Sigma}_s\boldsymbol{t} = \begin{pmatrix} \sigma_{11}t_1 + \sigma_{12}t_2 \\ \sigma_{12}t_1 + \sigma_{22}t_2 \end{pmatrix}$$

and the quadratic term $\boldsymbol{t}^\top\boldsymbol{\Sigma}_s\boldsymbol{t} = \sigma_{11}t_1^2 + 2\sigma_{12}t_1t_2 + \sigma_{22}t_2^2$.

For the sake of clarity we denote

$$f(\boldsymbol{t}) = \exp\left(\tfrac{1}{2}\boldsymbol{t}^\top \boldsymbol{\Sigma}_s \boldsymbol{t}\right) \text{ and } g(\boldsymbol{t}) = \left(\Phi_1\left(\tfrac{(\boldsymbol{\Sigma}_s \boldsymbol{t})_2}{\sqrt{\sigma_{22}}}\right) - \Phi_2\left(\tfrac{(\boldsymbol{\Sigma}_s \boldsymbol{t})_2}{\sqrt{\sigma_{22}}}, -\tfrac{(\boldsymbol{\Sigma}_s \boldsymbol{t})_1}{\sqrt{\sigma_{11}}}; -\varrho_{12}\right)\right)$$

and omit indexing with s of f and g. Considering the properties of derivatives we have

$$\tfrac{\partial^2}{\partial t_1 \partial t_2} fg = \tfrac{\partial^2 f}{\partial t_1 \partial t_2}\, g + \tfrac{\partial f}{\partial t_1}\, \tfrac{\partial g}{\partial t_2} + \tfrac{\partial f}{\partial t_2}\, \tfrac{\partial g}{\partial t_1} + f\, \tfrac{\partial^2 g}{\partial t_1 \partial t_2}\,.$$

Thus, we compute

$$f(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = 1; \qquad \tfrac{\partial f}{\partial t_1}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = \tfrac{\partial f}{\partial t_2}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = 0; \qquad \tfrac{\partial^2}{\partial t_1 \partial t_2} f(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = \sigma_{12}\,.$$

$$\begin{aligned} g(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} &= \Phi_1(0) - \Phi_2(0,0,-\varrho_{12}) = (\text{Owen, 1980, (3.5)}) = 2T\left(0, \sqrt{\tfrac{1+\varrho_{12}}{1-\varrho_{12}}}\right) \\ &= (\text{Owen, 1980, (2.2)}) = \tfrac{1}{\pi} \arctan\left(\sqrt{\tfrac{1+\varrho_{12}}{1-\varrho_{12}}}\right). \end{aligned}$$

To do further calculations, we denote

$$h(\boldsymbol{t}) = \tfrac{(\boldsymbol{\Sigma}_s \boldsymbol{t})_2}{\sqrt{\sigma_{22}}}; \qquad k(\boldsymbol{t}) = -\tfrac{(\boldsymbol{\Sigma}_s \boldsymbol{t})_1}{\sqrt{\sigma_{11}}}\,.$$

and also compute following values of derivatives of h

$$h(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = 0; \quad \tfrac{\partial}{\partial t_1} h(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = \tfrac{\sigma_{12}}{\sqrt{\sigma_{22}}}; \quad \tfrac{\partial}{\partial t_2} h(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = \sqrt{\sigma_{22}}; \quad \tfrac{\partial^2}{\partial t_1 \partial t_2} h(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = 0$$

and same for k such that

$$k(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = 0; \quad \tfrac{\partial}{\partial t_1} k(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = -\sqrt{\sigma_{11}}; \quad \tfrac{\partial}{\partial t_2} k(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = -\tfrac{\sigma_{12}}{\sqrt{\sigma_{11}}}; \quad \tfrac{\partial^2}{\partial t_1 \partial t_2} h(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = 0\,.$$

We compute

$$\tfrac{\partial^2}{\partial t_1 \partial t_2} \Phi_1(h(\boldsymbol{t})) = \tfrac{\partial}{\partial t_2}\left(\varphi(h(\boldsymbol{t}))\, \tfrac{\partial h}{\partial t_1}\right) = \tfrac{\partial \varphi}{\partial h}\, \tfrac{\partial h}{\partial t_2}\, \tfrac{\partial h}{\partial t_1} + \varphi\, \tfrac{\partial^2 h}{\partial t_1 \partial t_2} \Rightarrow \tfrac{\partial^2}{\partial t_1 \partial t_2} \Phi_1(h(\boldsymbol{t}))\Big|_{\boldsymbol{t}=0} = 0$$

Last implication comes from formulas above and by realizing that $\frac{\partial \varphi}{\partial h}(0) = 0$.

We use chain rule such that

$$\begin{aligned} \tfrac{\partial^2}{\partial t_1 \partial t_2} \Phi_2 = {} & \tfrac{\partial^2 \Phi_2}{\partial h^2}\, \tfrac{\partial h}{\partial t_1}\, \tfrac{\partial h}{\partial t_2} + \tfrac{\partial^2 \Phi_2}{\partial h \partial k}\, \tfrac{\partial h}{\partial t_1}\, \tfrac{\partial k}{\partial t_2} + \tfrac{\partial \Phi_2}{\partial h}\, \tfrac{\partial^2 h}{\partial t_1 \partial t_2} \\ & + \tfrac{\partial^2 \Phi_2}{\partial h \partial k}\, \tfrac{\partial h}{\partial t_2}\, \tfrac{\partial k}{\partial t_1} + \tfrac{\partial^2 \Phi_2}{\partial k^2}\, \tfrac{\partial k}{\partial t_1}\, \tfrac{\partial k}{\partial t_2} + \tfrac{\partial \Phi_2}{\partial k}\, \tfrac{\partial^2 k}{\partial t_1 \partial t_2}\,. \end{aligned}$$

For the first partial derivatives of Φ we use following identities (Drezner and Wesolowsky, 1990, (4)):

$$\frac{\partial \Phi_2(h,k,-\varrho)}{\partial h} = \varphi(h)\,\Phi_1\big(\tfrac{k+h\varrho}{\sqrt{1-\varrho^2}}\big)\,, \tag{3}$$

$$\frac{\partial \Phi_2(h,k,-\varrho)}{\partial k} = \varphi(h)\,\Phi_1\big(\tfrac{h+k\varrho}{\sqrt{1-\varrho^2}}\big)\,. \tag{4}$$

This implies

$$\begin{aligned}
\frac{\partial^2 \Phi_2(h,k,-\varrho)}{\partial h^2} &= \frac{\partial\varphi}{\partial h}\varphi(h)\Phi_1\big(\tfrac{k+h\varrho}{\sqrt{1-\varrho^2}}\big) + \varphi(h)\varphi\big(\tfrac{k+h\varrho}{\sqrt{1-\varrho^2}}\big)\frac{\varrho}{\sqrt{1-\varrho^2}}\,,\\
\frac{\partial^2 \Phi_2(h,k,-\varrho)}{\partial h \partial k} &= \varphi(h)\varphi\big(\tfrac{k+h\varrho}{\sqrt{1-\varrho^2}}\big)\frac{1}{\sqrt{1-\varrho^2}}\,,\\
\frac{\partial^2 \Phi_2(h,k,-\varrho)}{\partial k^2} &= \frac{\partial\varphi}{\partial k}\varphi(k)\Phi_1\big(\tfrac{h+k\varrho}{\sqrt{1-\varrho^2}}\big) + \varphi(k)\varphi\big(\tfrac{h+k\varrho}{\sqrt{1-\varrho^2}}\big)\frac{\varrho}{\sqrt{1-\varrho^2}}\,.
\end{aligned}$$

Then we get

$$\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_2}\Phi_2\Big|_{\boldsymbol{t}=0} &= \frac{1}{2\pi}\frac{\varrho_{12}}{\sqrt{1-\varrho_{12}^2}}\,\sigma_{12} + \frac{1}{2\pi}\frac{1}{\sqrt{1-\varrho_{12}^2}}\big(-\frac{\sigma_{12}^2}{\sqrt{\sigma_{11}\sigma_{22}}}\big)\\
&\quad + \frac{1}{2\pi}\frac{1}{\sqrt{1-\varrho_{12}^2}}\big(-\sqrt{\sigma_{11}\sigma_{22}}\big) + \frac{1}{2\pi}\frac{\varrho_{12}}{\sqrt{1-\varrho_{12}^2}}\,\sigma_{12}\\
&= -\frac{1}{2\pi}\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\,.
\end{aligned}$$

We substitute $\varrho_{12} = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$ and get

$$\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_2}\Phi_2\Big|_{\boldsymbol{t}=0} &= \frac{1}{2\pi}\frac{1}{\sqrt{1-\varrho^2}}\big(\frac{\sigma_{12}^2}{\sqrt{\sigma_{11}\sigma_{22}}} - \frac{\sigma_{12}^2}{\sqrt{\sigma_{11}\sigma_{22}}} - \sqrt{\sigma_{11}\sigma_{22}} + \frac{\sigma_{12}^2}{\sqrt{\sigma_{11}\sigma_{22}}}\big) =\\
&= \frac{1}{2\pi}\frac{1}{\sqrt{1-\varrho^2}}\big(-\frac{\sigma_{11}\sigma_{22}-\sigma_{12}^2}{\sqrt{\sigma_{11}\sigma_{22}}}\big) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\varrho_{12}^2}}\big(-\frac{\sigma_{11}\sigma_{22}-\varrho_{12}^2\sigma_{11}\sigma_{22}}{\sqrt{\sigma_{11}\sigma_{22}}}\big) =\\
&= -\frac{1}{2\pi}\sqrt{\sigma_{11}\sigma_{22}}\sqrt{1-\varrho_{12}^2}\,.
\end{aligned}$$

And finally we have

$$\frac{\partial^2}{\partial t_1 \partial t_2} g(\boldsymbol{t})\big|_{\boldsymbol{t}=0} = \frac{\partial^2}{\partial t_1 \partial t_2}\Phi_1\Big|_{\boldsymbol{t}=0} - \frac{\partial^2}{\partial t_1 \partial t_2}\Phi_2\Big|_{\boldsymbol{t}=0} = \frac{1}{2\pi}\sqrt{\sigma_{11}\sigma_{22}}\sqrt{1-\varrho_{12}^2}$$

Thus considering properties of derivatives we get

$$\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_2}\,\mathrm{mgf}_{\boldsymbol{W}}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} &= \sum_{s\in\mathcal{S}(2)} \frac{\partial^2 f}{\partial t_1 \partial t_2}\,g + f\,\frac{\partial^2 g}{\partial t_1 \partial t_2}\\
&= \sum_{s\in\mathcal{S}(2)} \sigma_{12}\,\frac{1}{\pi}\arctan\big(\sqrt{\tfrac{1+\varrho_{12}}{1-\varrho_{12}}}\big) + \frac{1}{2\pi}\sqrt{\sigma_{11}\sigma_{22}}\sqrt{1-\varrho_{12}^2}
\end{aligned}$$

We realize that index s just change the sign of σ_{12} (or respectively ϱ_{12}), again considering $\varrho_{12} = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$ and we get

$$\frac{\partial^2}{\partial t_1 \partial t_2} \operatorname{mgf}_{\boldsymbol{W}}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = \frac{2\varrho_{12}\sqrt{\sigma_{11}\sigma_{22}}}{\pi}\Big(\arctan\big(\sqrt{\tfrac{1+\varrho_{12}}{1-\varrho_{12}}}\big) - \arctan\big(\sqrt{\tfrac{1-\varrho_{12}}{1+\varrho_{12}}}\big)\Big) + \frac{2\sqrt{\sigma_{11}\sigma_{22}}}{\pi}\sqrt{1-\varrho_{12}^2}\,.$$

We use formula for difference of arctan functions (Rektorys, 2000, Section 2.11, Proposition 2, Item 22) to get

$$\arctan\big(\sqrt{\tfrac{1+\varrho_{12}}{1-\varrho_{12}}}\big) - \arctan\big(\sqrt{\tfrac{1-\varrho_{12}}{1+\varrho_{12}}}\big) = \arctan\big(\tfrac{\varrho_{12}}{\sqrt{1-\varrho_{12}^2}}\big)$$

and finally we use the relation between arctan and arcsin functions (Rektorys, 2000, Section 2.11, Proposition 2, Item 12) to get the result such that

$$\frac{\partial^2}{\partial t_1 \partial t_2} \operatorname{mgf}_{\boldsymbol{W}}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} = \frac{2\sqrt{\sigma_{11}\sigma_{22}}}{\pi}\big(\varrho_{12}\arcsin\varrho_{12} + \sqrt{1-\varrho_{12}^2}\big)\,.$$

□

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