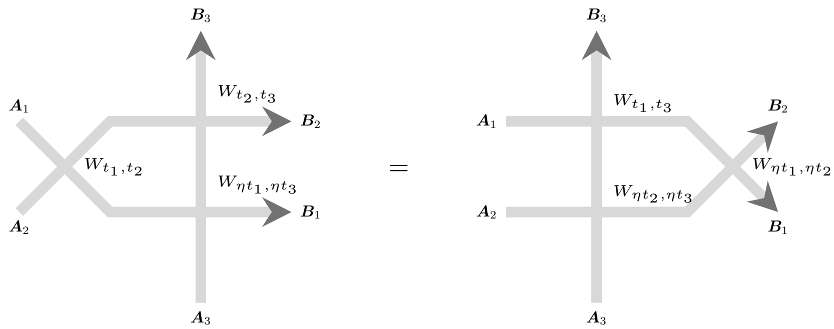
Appendix A. Appendix: deformed Yang-Baxter equations

The aim of this section is to provide proofs of (2.14) and (2.15). We start with the former.

Proposition A.1. *For any parameters t_1, t_2, t_3, η and any boundary conditions the following equality of rational functions holds*

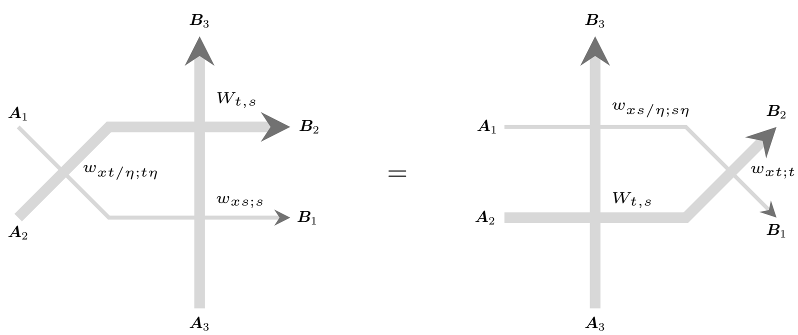


Proof. The identity follows at once after noticing that

$$W_{\eta t,\eta s}(\boldsymbol{A},\boldsymbol{B};\boldsymbol{C},\boldsymbol{D}) = \frac{(\eta^2 t^2;q)_{|\boldsymbol{D}|}}{(\eta^2 s^2;q)_{|\boldsymbol{A}|}} \frac{(s^2;q)_{|\boldsymbol{A}|}}{(t^2;q)_{|\boldsymbol{D}|}} W_{t,s}(\boldsymbol{A},\boldsymbol{B};\boldsymbol{C},\boldsymbol{D})$$

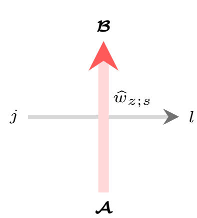
and applying the Yang-Baxter equation (2.12). □

Proposition A.2. *For any x, s, t, η and any boundary conditions the following equality of rational functions holds*



Proof. The proof is based on an analytic continuation of certain compositions appearing in the non-deformed version (2.13), followed by a non-integral shift.

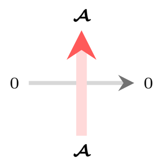
First we introduce a family of vertex weights denoted by

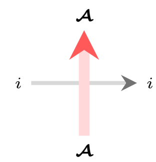


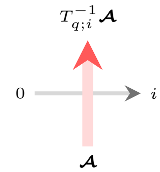
$$= \widehat{w}_{z;s}(\boldsymbol{\mathcal{A}}, j, \boldsymbol{\mathcal{B}}, l),$$

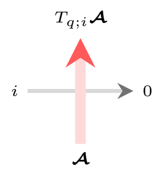
where $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_n) \in \mathbb{C}^n$ and $\boldsymbol{\mathcal{B}} = (\mathcal{B}_1, \dots, \mathcal{B}_n) \in \mathbb{C}^n$ denote *analytically continued* compositions. The explicit values of these weights are summarized in a table similar to (2.6):

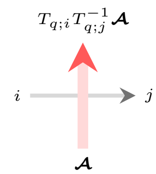
(A.1)

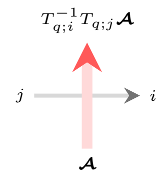
<table>
<tr>
<td>

$$\frac{1 - sz\mathcal{A}_{[1;n]}}{1 - sz}$$</td>
<td>

$$\frac{(s^2\mathcal{A}_i - sz)\mathcal{A}_{[i+1;n]}}{1 - sz}$$</td>
<td>

$$\frac{sz(\mathcal{A}_i - 1)\mathcal{A}_{[i+1;n]}}{1 - sz}$$</td>
</tr>
<tr>
<td>

$$\frac{1 - s^2\mathcal{A}_{[1;n]}}{1 - sz}$$</td>
<td>

$$\frac{sz(\mathcal{A}_j - 1)\mathcal{A}_{[j+1;n]}}{1 - sz}$$</td>
<td>

$$\frac{s^2(\mathcal{A}_i - 1)\mathcal{A}_{[i+1;n]}}{1 - sz}$$</td>
</tr>
</table>

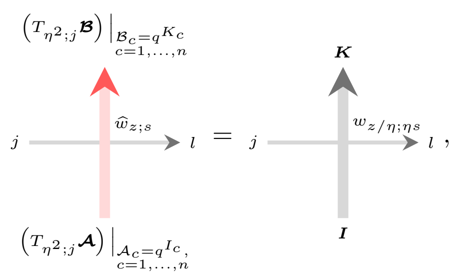
Here $1 \le i < j \le n$, for the analytically continued composition $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_n) \in \mathbb{C}^n$ we set $\mathcal{A}_{[a,b]} := \prod_{i=a}^{b} \mathcal{A}_i$ and $T_{q;i}$ (resp. $T_{q;i}^{-1}$) denotes the operator multiplying the ith component by q (resp. by q^{-1}). For the configurations not listed above we assume that the weight $\widehat{w}_{z;s}(\boldsymbol{\mathcal{A}}, j; \boldsymbol{\mathcal{B}}, l)$ vanishes, so that the conservation law holds:

$$T_{q;j}\boldsymbol{\mathcal{A}} = T_{q;l}\boldsymbol{\mathcal{B}},$$

where we define $T_{q;0}$ as the identity operator.

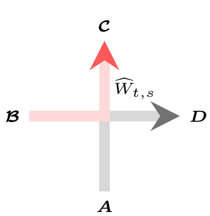
Comparing (2.6) and (A.1) we get the following relation for any $0 \le j, l \le n$ and compositions $\boldsymbol{I}, \boldsymbol{K}$:

(A.2)



where in the left hand-side we specialize $\boldsymbol{\mathcal{A}} = (q^{I_1}, \dots, q^{I_n}), \boldsymbol{\mathcal{B}} = (q^{K_1}, \dots, q^{K_n})$ and, as before, $T_{\eta^2;j}$ denotes the operator multiplying the jth position by η^2, acting by identity if $j = 0$. For $\eta = 1$ the relation (A.2) explains the interpretation of the weights $\widehat{w}_{z;s}$ as an analytic continuation of the weights $w_{z;s}$.

The next step is an analytic continuation of the weights $W_{t,s}(\boldsymbol{A}, \boldsymbol{B}; \boldsymbol{C}, \boldsymbol{D})$. Here we make use of the fact that these weights actually do not depend on $\boldsymbol{B}$ and $\boldsymbol{C}$ as long as the conservation law holds, so we define

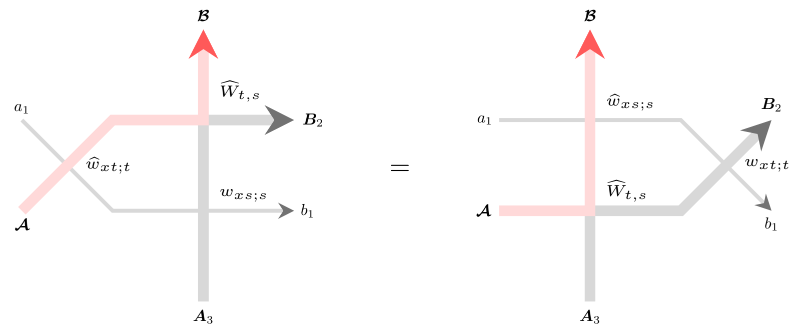


$$= \mathbb{1}_{q^{\boldsymbol{A}}\boldsymbol{\mathcal{B}} = q^{\boldsymbol{D}}\boldsymbol{\mathcal{C}}} \; (s^2/t^2)^{|\boldsymbol{D}|} \frac{(s^2/t^2; q)_{|\boldsymbol{A}| - |\boldsymbol{D}|}(t^2; q)_{|\boldsymbol{D}|}}{(s^2; q)_{|\boldsymbol{A}|}} q^{\sum_{i<j} D_i(A_j - D_j)} \prod_{i=1}^{n} \binom{A_i}{D_i}_q ,$$

where $\boldsymbol{\mathcal{B}} = (\mathcal{B}_1, \dots, \mathcal{B}_n)$ and $\boldsymbol{\mathcal{C}} = (\mathcal{C}_1, \dots, \mathcal{C}_n)$ are analytically continued compositions, and the conservation law $q^{\boldsymbol{A}}\boldsymbol{\mathcal{B}} = q^{\boldsymbol{D}}\boldsymbol{\mathcal{C}}$ reads as $q^{A_i}\mathcal{B}_i = q^{D_i}\mathcal{C}_i$ for every i.

Using the analytically continued weights $\widehat{w}_{z;s}$ and $\widehat{W}_{s,t}$, the Yang-Baxter equation (2.13) can be written as

(A.3)



where $\boldsymbol{\mathcal{A}} = q^{\boldsymbol{X}} := (q^{X_1}, \dots, q^{X_n})$ for sufficiently large $\boldsymbol{X} = (X_1, \dots, X_n) \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{\mathcal{B}}$ satisfies

$$T_{q;a_1} q^{\boldsymbol{A}_3} \boldsymbol{\mathcal{A}} = T_{q;b_1} q^{\boldsymbol{B}_2} \boldsymbol{\mathcal{B}}.$$

As before, the diagrams in (A.3) denote sums over all configurations not violating the conservation law for the weights $w, \widehat{w}$ and $\widehat{W}$. For fixed boundary conditions there are at most $n+1$ such configurations for each side of (A.3): each allowed configuration is uniquely determined by the label of the internal thin edge, which has $n+1$ possibilities. Since all weights depend rationally on $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$, both sides of (A.3) are rational functions in $\boldsymbol{\mathcal{A}}$ coinciding for $\boldsymbol{\mathcal{A}} = (q^{X_1}, \dots, q^{X_n})$, which implies that they are equal. So the analytically continued equation (A.3) holds for any $\boldsymbol{\mathcal{A}}$.

The proof is finished by applying $T_{\eta^2;a_1}$ to $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ and then setting back $\boldsymbol{\mathcal{A}} = q^{\boldsymbol{A}_2}$ and $\boldsymbol{\mathcal{B}} = q^{\boldsymbol{B}_3}$. By (A.2) the resulting equation coincides with the claim. □

Appendix B. Appendix: Computations for Theorem 8.4

Here we provide computational arguments for several statements used in the proof of Theorem 8.4. Recall that for fixed real parameters $\{\sigma_1, \dots, \sigma_k\}$, $\{\rho_1, \dots, \rho_k\}$, $\{\omega_1, \dots, \omega_k\}$, $\{\alpha_1, \dots, \alpha_k\}$, $\{\beta_1, \dots, \beta_k\}$, $\{\gamma_1, \dots, \gamma_k\}$, θ satisfying

$$\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i = \sum_{i=1}^k \gamma_i = 1, \qquad \alpha_i, \beta_i, \gamma_i \geq 0,$$

$$\omega_d < \rho_j < \sigma_i < \theta \qquad \text{for any } i, j, d,$$

we have defined

$$x = \sum_i \beta_i \Psi_1(\theta - \rho_i) - \sum_i \gamma_i \Psi_1(\theta - \omega_i), \qquad y = \sum_i \alpha_i \Psi_1(\theta - \sigma_i) - \sum_i \gamma_i \Psi_1(\theta - \omega_i),$$

$$I = x \sum_i \alpha_i \Psi(\theta - \sigma_i) - y \sum_i \beta_i \Psi(\theta - \rho_i) + (y - x) \sum_i \gamma_i \Psi(\theta - \omega_i),$$

$$h(z) = Iz - \sum_i x\alpha_i \ln\Gamma(z - \sigma_i) + \sum_i y\beta_i \ln\Gamma(z - \rho_i) - \sum_i (y - x)\gamma_i \ln\Gamma(z - \omega_i),$$

where

$$\Psi(z) = \frac{d}{dz} \ln\Gamma(z), \qquad \Psi_k(z) = \frac{d^{k+1}}{dz^{k+1}} \ln\Gamma(z).$$

For the computational purposes it is convenient to plug the expressions for x, y, I into $h(z)$ to obtain

$$
\begin{aligned}
h(z) = &-\left(\sum_i \beta_i \Psi_1(\theta-\rho_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i)\right) \sum_i \alpha_i \left(\ln\Gamma(z-\sigma_i) - z\Psi(\theta-\sigma_i)\right) \\
&+ \left(\sum_i \alpha_i \Psi_1(\theta-\sigma_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i)\right) \sum_i \beta_i \left(\ln\Gamma(z-\rho_i) - z\Psi(\theta-\rho_i)\right) \\
&- \left(\sum_i \alpha_i \Psi_1(\theta-\sigma_i) - \sum_i \beta_i \Psi_1(\theta-\rho_i)\right) \sum_i \gamma_i \left(\ln\Gamma(z-\omega_i) - z\Psi(\theta-\omega_i)\right).
\end{aligned}
$$

The last expression can be rewritten as a convex sum of simpler functions, corresponding to $k = 1$ case:

$$
h(z) = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{d=1}^{k} \alpha_i \beta_j \gamma_d h_{\sigma_i,\rho_j,\omega_d}(z), \tag{B.1}
$$

$$
\begin{aligned}
h_{\sigma,\rho,\omega}(z) = &-\left(\Psi_1(\theta-\rho) - \Psi_1(\theta-\omega)\right)\left(\ln\Gamma(z-\sigma) - z\Psi(\theta-\sigma)\right) \\
&+ \left(\Psi_1(\theta-\sigma) - \Psi_1(\theta-\omega)\right)\left(\ln\Gamma(z-\rho) - z\Psi(\theta-\rho)\right) \\
&- \left(\Psi_1(\theta-\sigma) - \Psi_1(\theta-\rho)\right)\left(\ln\Gamma(z-\omega) - z\Psi(\theta-\omega)\right).
\end{aligned}
$$

We will mostly work with the latter functions $h_{\sigma,\rho,\omega}(z)$, which for further convenience we rewrite as

$$
\begin{aligned}
h_{\sigma,\rho,\omega}(z) = &\left(\Psi_1(\theta-\rho) - \Psi_1(\theta-\omega)\right)\left(\ln\Gamma(z-\rho) - \ln\Gamma(z-\sigma) - z\Psi(\theta-\rho) + z\Psi(\theta-\sigma)\right) \\
&+ \left(\Psi_1(\theta-\sigma) - \Psi_1(\theta-\rho)\right)\left(\ln\Gamma(z-\rho) - \ln\Gamma(z-\omega) - z\Psi(\theta-\rho) + z\Psi(\theta-\omega)\right).
\end{aligned}
$$

In the following computations we frequently use the following series representations:

$$
\Psi(z) - \Psi(\theta) = \sum_{n\geq 0} -\frac{1}{n+z} + \frac{1}{n+\theta} = \sum_{n\geq 0} \frac{z-\theta}{(n+\theta)(n+z)}, \tag{B.2}
$$

$$
\Psi_k(z) = \sum_{n\geq 0} \frac{(-1)^{k+1}k!}{(n+z)^{k+1}}. \tag{B.3}
$$

In particular, the functions $(-1)^k\Psi_k(z)$ are increasing functions from $(0,\infty)$ to $(-\infty, 0)$, and

$$
\Psi_k(z+1) = \Psi_k(z) - \frac{(-1)^{k+1}k!}{z^{k+1}}. \tag{B.4}
$$

B.1. **Derivatives of $h(z)$.** This section is devoted to the signs of $h^{(d)}(\theta)$.

Lemma B.1. *For any $d \geq 3$ we have $(-1)^{d+1}h^{(d)}(\theta) > 0$.*

Proof. By (B.1) it is enough to prove that $(-1)^{d+1}h^{(d)}_{\sigma,\rho,\omega}(\theta) > 0$ for $\omega < \rho < \sigma$. By a slight abuse of notation let h denote $h_{\sigma,\rho,\omega}$ during the proof.

Our approach follows [BC15b, Lemma 5.3]. For $d \geq 3$ we have

$$
\begin{aligned}
h^{(d)}(\theta) = &\left(\Psi_1(\theta-\rho) - \Psi_1(\theta-\omega)\right)\left(\Psi_{d-1}(\theta-\rho) - \Psi_{d-1}(\theta-\sigma)\right) \\
&+ \left(\Psi_1(\theta-\sigma) - \Psi_1(\theta-\rho)\right)\left(\Psi_{d-1}(\theta-\rho) - \Psi_{d-1}(\theta-\omega)\right).
\end{aligned}
$$

Thus, we need to prove

$$
\frac{(-1)^{d-1}\Psi_{d-1}(\theta-\rho) - (-1)^{d-1}\Psi_{d-1}(\theta-\omega)}{\Psi_1(\theta-\rho) - \Psi_1(\theta-\omega)} > \frac{(-1)^{d-1}\Psi_{d-1}(\theta-\sigma) - (-1)^{d-1}\Psi_{d-1}(\theta-\rho)}{\Psi_1(\theta-\sigma) - \Psi_1(\theta-\rho)}.
$$

By the Cauchy's mean value theorem this inequality is equivalent to

$$
\frac{(-1)^{d-1}\Psi_d(\zeta)}{\Psi_2(\zeta)} > \frac{(-1)^{d-1}\Psi_d(\xi)}{\Psi_2(\xi)}
$$

for some $\xi \in (\theta-\sigma, \theta-\rho)$ and $\zeta \in (\theta-\rho, \theta-\omega)$. Since $\xi < \zeta$, it is enough to prove that the function $\frac{(-1)^{d-1}\Psi_d(z)}{\Psi_2(z)}$ is increasing on $\mathbb{R}_{>0}$. Taking derivative, it is enough to prove that

$$(-1)^{d-1}\left(\Psi_{d+1}(z)\Psi_2(z) - \Psi_d(z)\Psi_3(z)\right) > 0, \qquad z > 0.$$

Using series representation (B.3), the left hand side can be rewritten as

$$\begin{aligned}(-1)^{d-1}\left(\Psi_{d+1}(z)\Psi_2(z) - \Psi_d(z)\Psi_3(z)\right) &= \sum_{n,m\geq 0} \frac{2(d+1)!}{(n+z)^{d+2}(m+z)^3} - \frac{6\ d!}{(n+z)^{d+1}(m+z)^4} \\ &= 2d! \sum_{n,m\geq 0} \frac{(d+1)(m+z)^{d-1} - 3(n+z)(m+z)^{d-2}}{(n+z)^{d+2}(m+z)^{d+2}} \geq 6d! \sum_{n,m\geq 0} \frac{(m-n)(m+z)^{d-2}}{(n+z)^{d+2}(m+z)^{d+2}},\end{aligned}$$

where we have used $d \geq 3$ to establish inequality. The proof is finished by symmetrizing the last summation:

$$2\sum_{n,m\geq 0} \frac{(m-n)(m+z)^{d-2}}{(n+z)^{d+2}(m+z)^{d+2}} = \sum_{n,m\geq 0} \frac{(m-n)\left((m+z)^{d-2} - (n+z)^{d-2}\right)}{(n+z)^{d+2}(m+z)^{d+2}} > 0,$$

where for the last inequality we have used that the function z^{d-2} is increasing for $d \geq 3$. □

B.2. **Proof of Proposition 8.1.** It turns out that the two parts Proposition 8.1 are closely related. In the first part we want to prove that $x(\theta)/y(\theta)$ is increasing as a function of $\theta \in (\max_i \sigma_i, \infty)$, where

$$x = \sum_i \beta_i \Psi_1(\theta-\rho_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i), \qquad y = \sum_i \alpha_i \Psi_1(\theta-\sigma_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i).$$

Taking derivative, we need to prove that

$$x'(\theta)y(\theta) - x(\theta)y'(\theta) > 0.$$

After algebraic manipulations the left-hand side of the inequality above is equal to

$$-x(\theta)\sum_i \alpha_i \Psi_2(\theta-\sigma_i) + y(\theta)\sum_i \beta_i \Psi_2(\theta-\rho_i) - (y(\theta)-x(\theta))\sum_i \gamma_i \Psi_2(\theta-\omega_i),$$

which is exactly how we have defined $2c(\theta)^3$. So, the monotonicity from the first part boils down to showing that $c(\theta)^3 > 0$, which would immediately imply that $c(\theta)$ can be chosen to be positive. But from the expression above it is clear that $\frac{c(\theta)^3}{3} = h'''(\theta) > 0$, with the inequality following from Lemma B.1.

The limits of $x(\theta)/y(\theta)$ as $\theta \to \max_i \sigma_i, \infty$ are readily obtained from the definitions and the limit behavior of the trigamma function:

$$\lim_{z\to 0} \Psi_1(z) = \infty, \qquad \Psi_1(z) = \frac{1}{z} + \frac{1}{2z^2} + O(z^{-3}).$$

□

B.3. **Proof of Proposition 8.6.** Here we give a slightly modified version of the proof of the corresponding fact in [BC15b]. Recall that we want to show that $\mathrm{Re}[h(\theta+b\mathbf{i})]$ decreases for positive b and increases for negative b. Since $h(z)$ is a convex combination of $h_{\sigma,\rho,\omega}(z)$, it is enough to consider the latter functions. By an abuse of notation we again denote the function $h_{\sigma,\rho,\omega}(z)$ simply by $h(z)$. By symmetry, it is also enough to consider the case $\mathrm{Im}(z) > 0$, so we want to prove that

$$\mathrm{Re}[\mathbf{i}h'(\theta+b\mathbf{i})] < 0, \qquad b > 0.$$

We have

$$\begin{aligned}\mathrm{Re}[\mathbf{i}h'(\theta+b\mathbf{i})] = -\,\mathrm{Im}[&(\Psi_1(\theta-\rho)-\Psi_1(\theta-\omega))(\Psi(\theta+b\mathbf{i}-\rho)-\Psi(\theta+b\mathbf{i}-\sigma)) \\ &+ (\Psi_1(\theta-\sigma)-\Psi_1(\theta-\rho))(\Psi(\theta+b\mathbf{i}-\rho)-\Psi(\theta+b\mathbf{i}-\omega))].\end{aligned}$$

From (B.2) it follows that for any real $A_1, A_2 > 0$

$$\mathrm{Im}[\Psi(A_1+b\mathbf{i}) - \Psi(A_2+b\mathbf{i})] = \sum_{n\geq 0} \frac{b}{(n+A_1)^2+b^2} - \frac{b}{(n+A_2)^2+b^2}.$$

Thus, using that $b > 0$ and setting

$$\Phi(A) = \sum_{n\ge 0} \frac{1}{(n+A)^2+b^2},$$

the claim is reduced to showing

$$-(\Psi_1(\theta-\rho)-\Psi_1(\theta-\omega))(\Phi(\theta-\rho)-\Phi(\theta-\sigma)) - (\Psi_1(\theta-\sigma)-\Psi_1(\theta-\rho))(\Phi(\theta-\rho)-\Phi(\theta-\omega)) < 0.$$

This inequality can be rewritten in more convenient form

$$\frac{\Psi_1(\theta-\rho)-\Psi_1(\theta-\sigma)}{\Phi(\theta-\rho)-\Phi(\theta-\sigma)} > \frac{\Psi_1(\theta-\omega)-\Psi_1(\theta-\rho)}{\Phi(\theta-\omega)-\Phi(\theta-\rho)},$$

which by Cauchy's mean value theorem is equivalent to

$$\frac{\Psi_2(\xi)}{\Phi'(\xi)} > \frac{\Psi_2(\zeta)}{\Phi'(\zeta)}$$

for some $\xi \in (\theta-\sigma, \theta-\rho)$ and $\zeta \in (\theta-\rho, \theta-\omega)$. Since $0 < \xi < \zeta$, it is enough to show that the function $\frac{\Psi_2(z)}{\Phi'(z)}$ is decreasing for $z > 0$, and by taking derivative this is equivalent to

$$\Psi_2(z)\Phi''(z) - \Psi_3(z)\Phi'(z) > 0, \qquad z > 0.$$

Using the definition of $\Phi(x)$ and (B.3) we obtain

$$\begin{aligned}\Psi_2(z)\Phi''(z) - \Psi_3(z)\Phi'(z) &= \sum_{n,m\ge 0} -\frac{2}{(n+z)^3}\frac{6(z+m)^2-2b^2}{((m+z)^2+b^2)^3} + \frac{6}{(n+z)^4}\frac{2(m+z)}{((m+z)^2+b^2)^2} \\ &= 12\sum_{n,m\ge 0} \frac{(m+z)^3+b^2(m+z)-(n+z)(m+z)^2+\frac{b^2}{3}(n+z)}{(n+z)^4((m+z)^2+b^2)^3}.\end{aligned}$$

Clearly the terms in the numerator containing b^2 are positive, while for the remaining terms we have

$$\sum_{n,m\ge 0} \frac{(m+z)^3-(n+z)(m+z)^2}{(n+z)^4((m+z)^2+b^2)^3} = \frac{1}{2}\sum_{n,m\ge 0} \frac{(m-n)(m+z)^2}{(n+z)^4((m+z)^2+b^2)^3} + \frac{(n-m)(n+z)^2}{(m+z)^4((n+z)^2+b^2)^3},$$

where the equality follows by symmetrization. To finish the proof it is enough to show that each summand in the last expression is nonnegative for any $n, m, z \ge 0$, which is true:

$$\begin{aligned}&\frac{(m-n)(m+z)^2}{(n+z)^4((m+z)^2+b^2)^3} + \frac{(n-m)(n+z)^2}{(m+z)^4((n+z)^2+b^2)^3} \ge 0 \\ &\qquad\Leftrightarrow (m-n)\left[\left(\frac{(m+z)^2}{((m+z)^2+b^2)}\right)^3 - \left(\frac{(n+z)^2}{((n+z)^2+b^2)}\right)^3\right] \ge 0 \\ &\qquad\qquad\Leftrightarrow (m-n)\left[\frac{1}{\left(1+\frac{b^2}{(m+z)^2}\right)^3} - \frac{1}{\left(1+\frac{b^2}{(n+z)^2}\right)^3}\right] \ge 0.\end{aligned}$$

The last inequality holds since $\left(1+\frac{b^2}{(z+m)^2}\right)^{-3}$ is increasing in m. $\square$

B.4. **Proof of Proposition 8.7.** From now on we are working under Assumption 8.3, so $\theta \in \left(0, \frac{1}{2}\right)$ and the function $h(z)$ is a convex combination of the functions $h_{0,-1,\omega}(z)$ for $\omega < -1$. The latter functions are simpler, and for further convenience we provide an explicit expression for them:

$$h_{0,-1,\omega}(z) = (\Psi_1(\theta+1)-\Psi_1(\theta-\omega))\left(\ln(z)-\frac{z}{\theta}\right) - \frac{1}{\theta^2}\left(\ln\Gamma(z-\omega) - \ln\Gamma(z+1) - z\Psi(\theta-\omega) + z\Psi(\theta+1)\right).$$

Note that to get this expression we use (B.4) in the form $\Psi_1(\theta+1)-\Psi_1(\theta) = -\frac{1}{\theta^2}$.

Recall that we want to show that $\mathrm{Re}[h(\theta e^{\phi\mathbf{i}})]$ is increasing for $\phi > 0$ and decreasing for $\phi < 0$. Due to assumptions it is enough to consider the functions $h_{0,-1,\omega}(z)$, so by an abuse of notation we let h denote a function $h_{0,-1,\omega}(z)$ for the duration of the proof. We consider only the case $\phi > 0$, the other case follows by symmetry.

We need to prove that

$$\mathrm{Re}[\mathbf{i}\theta e^{\phi\mathbf{i}} h'(\theta e^{\phi\mathbf{i}})] > 0, \qquad \phi \in (0,\pi).$$

Computing the derivative and applying (B.2),(B.3) we obtain

$$\begin{aligned}
h'(z) &= (\Psi_1(\theta+1) - \Psi_1(\theta-\omega))\left(\frac{1}{z} - \frac{1}{\theta}\right) - \frac{1}{\theta^2}(\Psi(z-\omega) - \Psi(z+1) - \Psi(\theta-\omega) + \Psi(\theta+1)) \\
&= \sum_{n\geq 0}\left[\frac{\theta - z}{z\theta}\left(\frac{1}{(n+\theta+1)^2} - \frac{1}{(n+\theta-\omega)^2}\right) - \frac{1}{\theta^2}\left(\frac{z-\theta}{(n+\theta-\omega)(n+z-\omega)} - \frac{z-\theta}{(n+\theta+1)(n+z+1)}\right)\right] \\
&= \frac{(z-\theta)^2}{z\theta^2}\sum_{n\geq 0}\left[\frac{n+1}{(n+\theta+1)^2(n+z+1)} - \frac{n-\omega}{(n+\theta-\omega)^2(n+z-\omega)}\right].
\end{aligned}$$

Note that for any real τ

$$\mathrm{Re}\left[\mathbf{i}\theta e^{\phi\mathbf{i}}\frac{(e^{\phi\mathbf{i}}-1)^2}{\theta e^{\phi\mathbf{i}}}\frac{1}{\theta e^{\phi\mathbf{i}}+\tau}\right] = \left|1-e^{\phi\mathbf{i}}\right|^2\mathrm{Re}\left[-\mathbf{i}\frac{e^{\phi\mathbf{i}}}{\theta e^{\phi\mathbf{i}}+\tau}\right] = \left|1-e^{\phi\mathbf{i}}\right|^2\frac{\tau\sin\phi}{\tau^2+\theta^2+2\theta\tau\cos\phi},$$

so plugging $z = \theta e^{\phi\mathbf{i}}$ in the expression for $h'(z)$ and factoring out positive terms we need to prove that

$$\sum_{n\geq 0}\left[\frac{(n+1)^2}{((n+1)^2+\theta^2+2\theta(n+1)\cos\phi)(n+1+\theta)^2} - \frac{(n-\omega)^2}{((n-\omega)^2+\theta^2+2\theta(n-\omega)\cos\phi)(n-\omega+\theta)^2}\right] > 0.$$

It is enough to show that each summand above is positive, which can be rewritten as

$$\frac{X_1^2}{(X_1^2+\theta^2+2\theta X_1\cos\phi)(X_1+\theta)^2} > \frac{X_2^2}{(X_2^2+\theta^2+2\theta X_2\cos\phi)(X_2+\theta)^2}$$

for $X_1 = n+1$ and $X_2 = n-\omega$. Since $\omega < -1$ and both sides of the last inequality are clearly positive, we can equivalently show that

$$f(X) = (X^2+\theta^2+2\theta X\cos\phi)\left(1+\frac{\theta}{X}\right)^2$$

is increasing in $X \geq 1$ for any fixed ϕ. Taking derivative and using $\theta \in \left(0,\frac{1}{2}\right)$ we get

$$\begin{aligned}
f'(X) &= 2(X+\theta\cos\phi)\left(1+\frac{\theta}{X}\right)^2 - \frac{2\theta}{X^2}(X^2+\theta^2+2\theta X\cos\phi)\left(1+\frac{\theta}{X}\right) \\
&= 2\left(1-\frac{\theta^2}{X^2}\right)\left(X+\theta(1+\cos\phi)+\frac{\theta^2}{X}\right) > 0.
\end{aligned}$$

□

B.5. **Proof of Proposition 8.8.** For convenience, we repeat the statement:

Proposition B.2. *Suppose that Assumption 8.3 is satisfied. Let* $\psi(\varepsilon) = \arg(v_\varepsilon - \theta)$*, where* v_ε *is the intersection point of* $\mathcal{C}_\theta$ *with the circle of radius* $\varepsilon > 0$ *around* θ *satisfying* $\mathrm{Im}[v_\varepsilon] > 0$*. Then, for sufficiently small fixed* $\varepsilon > 0$ *there exists a depending on* ε *constant* $b > 0$ *such that*

$$\mathrm{Re}\left[h(\theta+\varepsilon e^{\pm\phi\mathbf{i}}) - h(\theta)\right] > b, \qquad \phi \in [\psi(\varepsilon), 2\pi/3].$$

Proof. Recall that $\theta \in \left(0,\frac{1}{2}\right)$. Due to the symmetry, it is enough to consider $\mathrm{Re}[h(\theta+\varepsilon e^{\phi\mathbf{i}}) - h(\theta)]$. Since h is a convex sum of finitely many $h_{0,-1,\omega}$, it is enough to consider $h(z) \equiv h_{0,-1,\omega}(z)$.

We consider only $\varepsilon < \theta/2$. By Taylor expansion there exists a constant $\Xi > 0$ such that

$$\left|h(\theta+\varepsilon e^{\phi\mathbf{i}}) - h(\theta) - \frac{h'''(\theta)}{6}\varepsilon^3 e^{3\phi\mathbf{i}} - \frac{h^{(4)}(\theta)}{24}\varepsilon^4 e^{4\phi\mathbf{i}}\right| < \Xi\varepsilon^5$$

Taking real part and using trigonometric equalities $\cos(3\phi) = \sin(3(\phi-\pi/2))$ and $\cos(4\phi) = \cos(4(\phi-\pi/2))$, we get

$$\mathrm{Re}\left[h(\theta+\varepsilon e^{\phi\mathbf{i}}) - h(\theta)\right] \geq \frac{h'''(\theta)}{6}\varepsilon^3\sin(3(\phi-\pi/2)) + \frac{h^{(4)}(\theta)}{24}\varepsilon^4\cos(4(\phi-\pi/2)) - \Xi\varepsilon^5$$

Since $h'''(\theta) > 0$ by Lemma B.1, for any $\phi \in [5\pi/8, 2\pi/3]$ we clearly have

$$\operatorname{Re}\left[h(\theta + \varepsilon e^{\phi\mathbf{i}}) - h(\theta)\right] \geq \frac{h'''(\theta)}{6}\varepsilon^3 \sin(3\pi/8) - \Lambda_1\varepsilon^4 > \Lambda_2\varepsilon^3$$

for some constants $\Lambda_1, \Lambda_2 > 0$ and sufficiently small ε. So it is enough to consider the claim for $\phi \in [\psi(\varepsilon), 5\pi/8]$. Moreover, note that for $\phi \in [\pi/2, 5\pi/8]$ and fixed ε the function

$$\frac{h'''(\theta)}{6}\varepsilon^3 \sin(3(\phi - \pi/2)) + \frac{h^{(4)}(\theta)}{24}\varepsilon^4 \cos(4(\phi - \pi/2)) - \Xi\varepsilon^5 \tag{B.5}$$

is increasing as a function of ϕ, where we have again used Lemma B.1 to get $h^{(4)}(\theta) < 0$. Hence the claim reduces to showing that the function above is positive at $\phi = \psi(\varepsilon)$.

Recall that $\psi(\varepsilon)$ was defined using v_ε, which can be explicitly described:

$$v_\varepsilon = \frac{2\theta^2 - \varepsilon^2}{2\theta} + \varepsilon\sqrt{1 - \frac{\varepsilon^2}{4\theta^2}}\mathbf{i}.$$

On the other hand, $v_\varepsilon = \theta + \varepsilon e^{\psi(\varepsilon)\mathbf{i}}$, so we obtain

$$\frac{v_\varepsilon - \theta}{\varepsilon} = -\frac{\varepsilon}{2\theta} + \sqrt{1 - \frac{\varepsilon^2}{4\theta^2}}\mathbf{i} = \cos(\psi(\varepsilon)) + \sin(\psi(\varepsilon))\mathbf{i}.$$

This implies that $\psi(\varepsilon) = \frac{\pi}{2} + \frac{\varepsilon}{2\theta} + O(\varepsilon^2)$ as $\varepsilon \to 0$. Plugging it into the studied function (B.5) we get

$$\frac{h'''(\theta)}{6}\varepsilon^3 \sin(3(\psi(\varepsilon) - \pi/2)) + \frac{h^{(4)}(\theta)}{24}\varepsilon^4 \cos(4(\psi(\varepsilon) - \pi/2)) - \Xi\varepsilon^5 > \varepsilon^4\left(\frac{h'''(\theta)}{4\theta} + \frac{h^{(4)}(\theta)}{24}\right) - \Lambda_3\varepsilon^5$$

for some constant $\Lambda_3 > 0$. So the claim follows as long as

$$\frac{h'''(\theta)}{\theta} + \frac{h^{(4)}(\theta)}{6} > 0.$$

We prove the last inequality by direct computation. Recall that

$$h(z) = (\Psi_1(\theta+1) - \Psi_1(\theta-\omega))\left(\ln(z) - \frac{z}{\theta}\right) - \frac{1}{\theta^2}(\ln\Gamma(z-\omega) - \ln\Gamma(z+1) - z\Psi(\theta-\omega) + z\Psi(\theta+1)),$$

where $\omega < -1$. The derivatives are given by

$$h'''(\theta) = \frac{2}{\theta^3}(\Psi_1(\theta+1) - \Psi_1(\theta-\omega)) - \frac{1}{\theta^2}(\Psi_2(\theta-\omega) - \Psi_2(\theta+1)),$$
$$h^{(4)}(\theta) = -\frac{6}{\theta^4}(\Psi_1(\theta+1) - \Psi_1(\theta-\omega)) - \frac{1}{\theta^2}(\Psi_3(\theta-\omega) - \Psi_3(\theta+1)).$$

Rearranging terms we obtain

$$\frac{h'''(\theta)}{\theta} + \frac{h^{(4)}(\theta)}{6} = f(-\omega) - f(1)$$

where

$$f(\tau) = -\frac{1}{\theta^4}\Psi_1(\theta+\tau) - \frac{1}{\theta^3}\Psi_2(\theta+\tau) - \frac{1}{6\theta^2}\Psi_3(\theta+\tau).$$

So, it is enough to prove that $f(\tau)$ is increasing in $\tau \geq 1$. Using series representations (B.3) write

$$f(\tau) = \frac{1}{\theta^4}\sum_{n\geq 0} -\frac{1}{(n+\theta+\tau)^2} + \frac{2\theta}{(n+\theta+\tau)^3} - \frac{\theta^2}{(n+\theta+\tau)^4} = \sum_{n\geq 0}\frac{-(n+\tau)^2}{\theta^4(n+\theta+\tau)^4}$$

The claim follows since $-\frac{X^2}{(X+\theta)^4}$ is increasing in $X \geq 1$:

$$\left(-\frac{X^2}{(X+\theta)^4}\right)' = \frac{-2X(X+\theta) + 4X^2}{(X+\theta)^5} = \frac{2X(X-\theta)}{(X+\theta)^5} > 0.$$

$\square$