Appendices of "Orlicz risks for assessing stochastic streamflow environments: a static optimization approach"

Appendix A. Proofs of Propositions

Proof of Proposition 1

We firstly consider the case (i). The admissible set $\bar{\mathcal{D}}$ is convex. By the assumption, for each $k>0$, we have

$$\int_0^{+\infty}\exp_q\left(\frac{1}{w}\Phi\left(\frac{f(x)}{k}\right)\right)g(x)\mathrm{d}x=\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{1}{kx}\right)^p\right)g_\pi(x)\mathrm{d}x\in(0,+\infty). \tag{A1}$$

Indeed, we have the order estimate $\exp_q\left(\frac{1}{w}\left(\frac{1}{kx}\right)^p\right)=O\left(x^{-\frac{p}{1-q}}\right)$ for $x>0$ small and hence $g_\pi(x)\exp_q\left(\frac{1}{w}\left(\frac{1}{kx}\right)^p\right)=O\left(x^{\alpha_\pi-\frac{p}{1-q}-1}\right)$ for $x>0$ small. Further, $\exp_q\left(\frac{1}{w}\left(\frac{1}{kx}\right)^p\right)=O(1)$ (i.e., bounded) for $x>0$ large. These findings combined with the assumption $p\in(1,\alpha_\pi)$, and $q\in\left(0,1-\frac{p}{\alpha_\pi}\right)$ imply that the integrand of (A1) is integrable with respect to g_π over $x>0$. The Radon–Nikodym derivative $\phi=\bar{\phi}_k$ is therefore well-defined and admissible: i.e., $\bar{\phi}_k\in\bar{\mathcal{D}}$ for all $k>0$. Then, we can follow Theorem 3.1 of Ma and Tian (2021) and its proof, and conclude that the maximizer of **Problem U1** is given by $\phi=\bar{\phi}_k$. After an elementary calculation, we obtain

$$G(k)=\int_0^{+\infty}\left(\frac{1}{kx}\right)^p\left\{\bar{\phi}_k(x)\right\}^q g_\pi(x)\mathrm{d}x-wD_q\left(\bar{\phi}_k g_\pi|g_\pi\right)=w\ln_q\left(\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{1}{kx}\right)^p\right)g_\pi(x)\mathrm{d}x\right). \tag{A2}$$

Now, we show that there exists a unique solution $k=\bar{k}>0$ to the nonlinear equation (21). We follow Proof of Lemma 5 of Bellini et al. (2018). A difference between their and our problems is that the former deals with bounded random variables while ours does not. We overcome this difficulty by invoking an explicit expression (A2). We set $\hat{G}:[0,+\infty)\to\mathbb{R}$ as

$$\hat{G}(k)=\sup_{\phi\in\bar{\mathcal{D}}}\left\{\int_0^{+\infty}\Phi(kf(x))\{\phi(x)\}^q g(x)\mathrm{d}x-wD_q(\phi g|g)\right\}=w\ln_q\left(\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x\right). \tag{A3}$$

Then, we have

$$
\begin{aligned}
\hat{G}(0) &= \sup_{\phi\in\bar{\mathcal{D}}}\left\{\int_0^{+\infty}\left(\frac{0}{x}\right)^p \{\phi(x)\}^q g_\pi(x)\mathrm{d}x - wD_q\left(\phi g_\pi \middle| g_\pi\right)\right\} \\
&= \sup_{\phi\in\bar{\mathcal{D}}}\left\{-wD_q\left(\phi g_\pi \middle| g_\pi\right)\right\} \\
&= -w\inf_{\phi\in\bar{\mathcal{D}}}\left\{D_q\left(\phi g_\pi \middle| g_\pi\right)\right\} \\
&= 0
\end{aligned}, \tag{A4}
$$

where the last equality is achieved by $\phi(x)=1$ ($x>0$) that is admissible. In addition, by the last line of (A3), we have

$$
\hat{G}(k) = w\ln_q\left(\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x\right) < +\infty \quad \text{for each} \quad k \geq 0 \tag{A5}
$$

due to

$$
\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x < +\infty \quad \text{for each} \quad k \geq 0, \tag{A6}
$$

showing that $\hat{G}$ is bounded at each $k \geq 0$, and it is continuous and strictly increasing for $k>0$. Indeed, for $k>0$ we have (we use the notation $\exp_q^q(\cdot)=\{\exp_q(\cdot)\}^q$)

$$
\begin{aligned}
&\int_0^{+\infty}\left|\frac{\partial}{\partial k}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)\right|g_\pi(x)\mathrm{d}x \\
&= \int_0^{+\infty}\frac{\partial}{\partial k}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x \\
&= \int_0^{+\infty}\frac{1}{w}\left(\frac{1}{x}\right)^p pk^{p-1}\exp_q^q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x \\
&= pk^{p-1}\frac{1}{w}\int_0^{+\infty}\left(\frac{1}{x}\right)^p \exp_q^q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x \\
&\leq pk^{p-1}\frac{1}{w}\int_0^{+\infty}\left(\frac{1}{x}\right)^p C_1\left(1+\left(\frac{1}{w}\right)^{\frac{q}{1-q}}\left(\frac{k}{x}\right)^{\frac{qp}{1-q}}\right)g_\pi(x)\mathrm{d}x \\
&= pk^{p-1}\frac{1}{w}C_1\int_0^{+\infty}\left(\left(\frac{1}{x}\right)^p+\left(\frac{1}{w}\right)^{\frac{q}{1-q}}k^{\frac{qp}{1-q}}\left(\frac{1}{x}\right)^{\frac{p}{1-q}}\right)g_\pi(x)\mathrm{d}x \\
&= pk^{p-1}\frac{1}{w}C_1\left\{\left(\frac{1}{\beta_\pi}\right)^p\frac{\Gamma(\alpha_\pi-p)}{\Gamma(\alpha_\pi)}+\left(\frac{1}{w}\right)^{\frac{q}{1-q}}k^{\frac{qp}{1-q}}\left(\frac{1}{\beta_\pi}\right)^{\frac{p}{1-q}}\frac{\Gamma\left(\alpha_\pi-\dfrac{p}{1-q}\right)}{\Gamma(\alpha_\pi)}\right\} \\
&< +\infty
\end{aligned} \tag{A7}
$$

due to $q\in\left(0,1-\dfrac{p}{\alpha_\pi}\right)$, namely $\alpha_\pi > \dfrac{p}{1-q}$, with a sufficiently large constant $C_1>0$ depending only on $q\in(0,1)$ such that (See, **Lemma A.1**)

$$\exp_q^q(x)=(1+(1-q)x)^{\frac{q}{1-q}}\le C_1\left(1+x^{\frac{q}{1-q}}\right),\quad x\ge 0. \tag{A8}$$

We further have, by the classical Lebesgue convergence theorem (Proposition 6 of Zorich (2016)), that

$$\begin{aligned}\frac{\partial}{\partial k}\hat{J}(k)&=w\frac{\partial}{\partial k}\ln_q\left(\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x\right)\\&=\frac{\partial}{\partial k}\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x\times\left(\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x\right)^{-\alpha}\\&=\int_0^{+\infty}\frac{\partial}{\partial k}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x\times\left(\int_0^{+\infty}\exp_q\left(\frac{1}{w}\left(\frac{k}{x}\right)^p\right)g_\pi(x)\mathrm{d}x\right)^{-\alpha}\\&\in(0,+\infty)\end{aligned}. \tag{A9}$$

Moreover, the classical monotone convergence theorem suggests $\hat{G}(+\infty)=+\infty$ due to (A5) and

$$\begin{aligned}\lim_{k\to+\infty}\hat{G}(k)&=\lim_{k\to+\infty}\sup_{\phi\in\mathcal{D}}\left\{\int_0^{+\infty}\Phi(kf(x))\{\phi(x)\}^q g(x)\mathrm{d}x-wD_q(\phi g|g)\right\}\\&\ge\lim_{k\to+\infty}\int_0^{+\infty}\left(\frac{k}{x}\right)^p g_\pi(x)\mathrm{d}x\\&\ge\int_0^{+\infty}\left(\frac{1}{x}\right)^p g_\pi(x)\mathrm{d}x\times\lim_{k\to+\infty}k^p\\&=+\infty\end{aligned}. \tag{A10}$$

Therefore, the equation $\hat{G}(k)=1$ admits a unique positive solution. By the definition of $\hat{G}$, **Problem U2** admits a unique positive solution $k=\bar{k}>0$. By turning to the representation formula (A2), it is clear that a minimizer of (20) for $k=\bar{k}$ is $\phi=\bar{\phi}_{\bar{k}}$.

The case (ii) is easier to handle because the singularity issue for x small does not arise for all $m\in\mathbb{N}$. Then, the existence as well as the unique solvability of **Problem U1** follows based on the argument of Theorem 3.1 of Ma and Tian (2021), and the unique existence of the solution to **Problem U2** follows again based on Proof of Lemma 5 of Bellini et al. (2018).

□

Lemma A.1

It follows that for $q\in(0,1)$ there exists a constant $C_1>0$ such that

$$\exp_q^q(x)\le C_1\left(1+x^{\frac{q}{1-q}}\right),\quad x\ge 0. \tag{A11}$$

Proof of Lamma A

Due to

$$(1+(1-q)x)^{\frac{q}{1-q}}\le(1+x)^{\frac{q}{1-q}},\quad x\ge 0, \tag{A12}$$

it suffices to show that there exists a constant $C_1 > 0$ such that

$$C_1 \geq g(x) := \frac{(1+x)^{\frac{q}{1-q}}}{1+x^{\frac{q}{1-q}}}, \quad x \geq 0. \tag{A13}$$

We have

$$\begin{aligned} \frac{\mathrm{d}g(x)}{\mathrm{d}x} &= \frac{\frac{q}{1-q}(1+x)^{\frac{q}{1-q}-1}\left(1+x^{\frac{q}{1-q}}\right)-(1+x)^{\frac{q}{1-q}}\frac{q}{1-q}x^{\frac{q}{1-q}-1}}{\left(1+x^{\frac{q}{1-q}}\right)^2} \\ &= \frac{q}{1-q}\frac{(1+x)^{\frac{q}{1-q}-1}}{\left(1+x^{\frac{q}{1-q}}\right)^2}\left(1-x^{\frac{2q-1}{1-q}}\right) \end{aligned}, \quad x \geq 0. \tag{A14}$$

The function g satisfies $g(0) = \lim_{x\to+\infty} g(x) = 1$ and is globally maximized (resp., globally minimized) at $x = 1$ if $q \in (1/2, 1)$ (resp., if $q \in (0, 1/2)$). We have $g(1) = 2^{\frac{2q-1}{1-q}}$. It is therefore sufficient to set $C_1 = \max\{1, 2^{\frac{2q-1}{1-q}}\}$.

□

Proof of Proposition 2

We only show the proof for the case (i) as the case (ii) can be handled in essentially the same way. The admissible set $\mathcal{D}$ is clearly convex. By the assumption of case (i), for each $k > 0$, we have

$$\int_0^{+\infty} \exp_q\left(-\frac{1}{w}\Phi^{(-1)}\left(\frac{f(x)}{k}\right)\right) g(x)\mathrm{d}x = \int_0^{+\infty} \exp_q\left(-\frac{1}{w}\left(\frac{1}{kx}\right)^{\frac{1}{p}}\right) g_\pi(x)\mathrm{d}x \in (0,1). \tag{A15}$$

This holds true because of assuming $q \geq 1$ and $\exp_q\left(-\frac{1}{w}\left(\frac{1}{kx}\right)^{\frac{1}{p}}\right)$ is uniformly bounded due to

$$0 \leq \exp_q\left(-\frac{1}{w}\left(\frac{1}{kx}\right)^{\frac{1}{p}}\right) = \left(1+(q-1)\frac{1}{w}\left(\frac{1}{kx}\right)^{\frac{1}{p}}\right)^{-\frac{1}{q-1}} \leq 1. \tag{A16}$$

Then, we can follow Theorem 3.1 of Ma and Tian (2021) and its proof to obtain that the minimizer of **Problem D1** is given by $\phi = \underline{\phi}_k$ that is admissible. Consequently, we have

$$
\begin{aligned}
J(k) &= \int_0^{+\infty} \left(\frac{1}{kx}\right)^{\frac{1}{p}} \left\{\underline{\phi}_k(x)\right\}^q g_\pi(x)\mathrm{d}x + wD_q\left(\underline{\phi}_k g_\pi \middle| g_\pi\right) \\
&= -w\ln_q\left(\int_0^{+\infty} \exp_q\left(-\frac{1}{w}\left(\frac{1}{kx}\right)^{\frac{1}{p}}\right) g_\pi(x)\mathrm{d}x\right)
\end{aligned}
. \tag{A17}
$$

Now, we show that there exists a unique solution $k = \underline{k} > 0$ to the nonlinear equation (28). We again follow Proof of Lemma 5 of Bellini et al. (2018). We set $\hat{J}:[0,+\infty)\to\mathbb{R}$ as

$$
\begin{aligned}
\hat{J}(k) &= \inf_{\phi\in\mathcal{D}}\left\{\int_0^{+\infty} \Phi^{(-1)}(kf(x))\{\phi(x)\}^q g(x)\mathrm{d}x + wD_q(\phi g|g)\right\} \\
&= -w\ln_q\left(\int_0^{+\infty} \exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right) g_\pi(x)\mathrm{d}x\right)
\end{aligned}
. \tag{A18}
$$

We have

$$
\hat{J}(0) = \inf_{\phi\in\mathcal{D}}\left\{\int_0^{+\infty}\left(\frac{0}{x}\right)^{\frac{1}{p}}\{\phi(x)\}^q g_\pi(x)\mathrm{d}x + wD_q(\phi g_\pi|g_\pi)\right\} = \inf_{\phi\in\mathcal{D}}\left\{wD_q(\phi g_\pi|g_\pi)\right\} = 0, \tag{A19}
$$

where the last equality is achieved by $\phi(x)=1$ ($x>0$) that is admissible. In addition, by the last line of (A18), we have

$$
\hat{J}(k) < +\infty \tag{A20}
$$

due to

$$
\int_0^{+\infty} \exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right) g_\pi(x)\mathrm{d}x \in (0,1) \quad \text{for each} \quad k \geq 0, \tag{A21}
$$

showing that $\hat{J}$ is bounded at each $k \geq 0$, and it is continuous and strictly increasing for $k > 0$. Indeed, we have

$$
\begin{aligned}
\int_0^{+\infty}\left|\frac{\partial}{\partial k}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)\right| g_\pi(x)\mathrm{d}x &= \int_0^{+\infty}\left|\frac{1}{w}\left(\frac{1}{x}\right)^{\frac{1}{p}}\frac{1}{p}k^{\frac{1}{p}-1}\exp_q^q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)\right| g_\pi(x)\mathrm{d}x \\
&= \frac{1}{p}k^{\frac{1}{p}-1}\frac{1}{w}\int_0^{+\infty}\left(\frac{1}{x}\right)^{\frac{1}{p}}\exp_q^q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right) g_\pi(x)\mathrm{d}x \\
&\leq \frac{1}{p}k^{\frac{1}{p}-1}\frac{1}{w}\int_0^{+\infty}\left(\frac{1}{x}\right)^{\frac{1}{p}} g_\pi(x)\mathrm{d}x \\
&= \frac{1}{p}k^{\frac{1}{p}-1}\frac{1}{w}\left(\frac{1}{\beta_\pi}\right)^{\frac{1}{p}}\frac{\Gamma\left(\alpha_\pi-\frac{1}{p}\right)}{\Gamma(\alpha_\pi)} \\
&< +\infty
\end{aligned}
, \quad k > 0. \tag{A22}
$$

Hence, by the classical Lebesgue convergence theorem (Proposition 6 in Chapter 17.2.3 of Zorich (2016)) we can exchange the order of the integration and partial differentiation. Thereby, for $k > 0$ we have

$$
\begin{aligned}
\frac{\partial}{\partial k}\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x &= \int_0^{+\infty}\frac{\partial}{\partial k}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x \\
&= -\frac{1}{p}k^{\frac{1}{p}-1}\frac{1}{w}\int_0^{+\infty}\left(\frac{1}{x}\right)^{\frac{1}{p}}\exp_q^q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x
\end{aligned}
\tag{A23}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial k}\hat{J}(k) &= -w\frac{\partial}{\partial k}\ln_q\left(\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x\right) \\
&= -\frac{\partial}{\partial k}\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x\times\left(\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x\right)^{-\alpha} \\
&= \frac{1}{p}k^{\frac{1}{p}-1}\frac{1}{w}\int_0^{+\infty}\left(\frac{1}{x}\right)^{\frac{1}{p}}\exp_q^q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x \\
&\quad\times\left(\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x\right)^{-\alpha} \\
&\in(0,+\infty)
\end{aligned}
\quad . \tag{A24}
$$

Moreover, the classical monotone convergence theorem suggests $\hat{J}(+\infty)=+\infty$ by (A20) and

$$
\begin{aligned}
\lim_{k\to+\infty}\hat{J}(k) &= -\lim_{k\to+\infty}\frac{1}{1-q}\left(\left(\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x\right)^{1-q}-1\right) \\
&= \frac{1}{q-1}\lim_{k\to+\infty}\left(\left(\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x\right)^{-(q-1)}-1\right) \\
&= +\infty
\end{aligned}
\tag{A25}
$$

due to

$$
\lim_{k\to+\infty}\int_0^{+\infty}\exp_q\left(-\frac{1}{w}\left(\frac{k}{x}\right)^{\frac{1}{p}}\right)g_\pi(x)\mathrm{d}x=0\,. \tag{A26}
$$

Therefore, the equation $\hat{J}(k)=1$ admits a unique positive solution. By the definition of $\hat{J}$, **Problem D2** admits a unique positive solution $k=\underline{k}>0$. By turning to the representation formula (A17), it is clear that a minimizer of (27) for $k=\underline{k}$ is $\phi=\underline{\phi}_k$.

□

Proof of Proposition 3

The proof of the case (a) is due to the following equivalent representation of $\bar{O}_{g,\Phi,f}$ for given $w>0$:

$$\bar{O}_{g,\Phi,f} = \sup_{\phi\in\bar{\mathcal{D}}}\left(\frac{\int_0^{+\infty}(f(x))^p\{\phi(x)\}^q g(x)\mathrm{d}x}{1+wD_q(\phi g|g)}\right)^{\frac{1}{p}}, \tag{A27}$$

which follows from a static version of Example 9 in Bellini et al. (2021). By the representation (A27), it follows that

$$\bar{O}_{g,\Phi,f}\Big|_{w=w_2} \le \bar{O}_{g,\Phi,f}\Big|_{w=w_1} \quad \text{for } 0<w_1\le w_2 . \tag{A28}$$

Further, by (A27), for any $w>0$ we have

$$\bar{O}_{g,\Phi,f} \ge \left(\frac{\int_0^{+\infty}(f(x))^p g(x)\mathrm{d}x}{1+wD_q(g|g)}\right)^{\frac{1}{p}} = \left(\int_0^{+\infty}(f(x))^p g(x)\mathrm{d}x\right)^{\frac{1}{p}} \ge \int_0^{+\infty} f(x)g(x)\mathrm{d}x, \tag{A29}$$

where we used the Classical Jensen's inequality. Combining (A28) and (A29) proves the statement of the proposition for the case (a).

The proof for the case (b) is similar to that presented above with a slight modification. We have the following equivalent representation of $\underline{O}_{g,\Phi,f}$ for given $w>0$; we must ensure $1-wD_q(\phi g|g)>0$

$$\underline{O}_{g,\Phi,f} = \inf_{\phi\in\underline{\mathcal{D}}}\left(\frac{\int_0^{+\infty}(f(x))^{\frac{1}{p}}\{\phi(x)\}^q g(x)\mathrm{d}x}{1-wD_q(\phi g|g)}\right)^p . \tag{A30}$$

Here, we restrict ourselves to ϕ such that $1-wD_q(\phi g|g)>0$, which is possible because of

$$\underline{O}_{g,\Phi,f} \le \left(\frac{\int_0^{+\infty}(f(x))^{\frac{1}{p}} g(x)\mathrm{d}x}{1-wD_q(g|g)}\right)^p = \left(\int_0^{+\infty}(f(x))^{\frac{1}{p}} g(x)\mathrm{d}x\right)^p \le \int_0^{+\infty} f(x)g(x)\mathrm{d}x . \tag{A31}$$

Then, by the representation formula (A30), it follows that

$$\underline{O}_{g,\Phi,f}\Big|_{w=w_1} \le \underline{O}_{g,\Phi,f}\Big|_{w=w_2} \quad \text{for } 0<w_1\le w_2 . \tag{A32}$$

where we again used the classical Jensen's inequality. Combining (A31) and (A32) proves the statement of the proposition for the case (b).

□

Proof of Proposition 4

We firstly prove the case (a). By the assumption and the classical Hölder inequality, for any $\phi\in\bar{\mathcal{D}}$, we have

$$\left(\frac{\int_0^{+\infty}(f(x))^{p_1}\{\phi(x)\}^q g(x)\mathrm{d}x}{M}\right)^{\frac{1}{p_1}} \le \left(\frac{\int_0^{+\infty}(f(x))^{p_2}\{\phi(x)\}^q g(x)\mathrm{d}x}{M}\right)^{\frac{1}{p_2}} \tag{A33}$$

or equivalently

$$\left(\int_0^{+\infty}\left(f(x)\right)^{p_1}\{\phi(x)\}^q g(x)\mathrm{d}x\right)^{\frac{1}{p_1}} \leq \left(\int_0^{+\infty}\left(f(x)\right)^{p_2}\{\phi(x)\}^q g(x)\mathrm{d}x\right)^{\frac{1}{p_2}} M^{\frac{1}{p_1}-\frac{1}{p_2}} \tag{A34}$$

with

$$M := \int_0^{+\infty}\{\phi(x)\}^q g(x)\mathrm{d}x\,. \tag{A35}$$

Again, by the Hölder inequality considering $q\in(0,1)$, we have

$$M^{\frac{1}{q}} = \left(\int_0^{+\infty}\{\phi(x)\}^q g(x)\mathrm{d}x\right)^{\frac{1}{q}} \leq \int_0^{+\infty}\phi(x)g(x)\mathrm{d}x = 1\,. \tag{A36}$$

Hence, we obtain $0 < M \leq 1$. Thereby, considering $\frac{1}{p_1}-\frac{1}{p_2}\geq 0$ due to $p_2 \geq p_1$, we have

$$\frac{\left(\int_0^{+\infty}\left(f(x)\right)^{p_1}\{\phi(x)\}^q g(x)\mathrm{d}x\right)^{\frac{1}{p_1}}}{\left(1+wD_q\left(\phi g|g\right)\right)^{\frac{1}{p_1}}} \leq \frac{\left(\int_0^{+\infty}\left(f(x)\right)^{p_2}\{\phi(x)\}^q g(x)\mathrm{d}x\right)^{\frac{1}{p_2}}}{\left(1+wD_q\left(\phi g|g\right)\right)^{\frac{1}{p_1}}} \tag{A37}$$

and further

$$\begin{aligned}\left(\frac{\int_0^{+\infty}\left(f(x)\right)^{p_1}\{\phi(x)\}^q g(x)\mathrm{d}x}{1+wD_q\left(\phi g|g\right)}\right)^{\frac{1}{p_1}} &\leq \left(\frac{\int_0^{+\infty}\left(f(x)\right)^{p_2}\{\phi(x)\}^q g(x)\mathrm{d}x}{1+wD_q\left(\phi g|g\right)}\right)^{\frac{1}{p_2}} \frac{1}{\left(1+wD_q\left(\phi g|g\right)\right)^{\frac{1}{p_1}-\frac{1}{p_2}}} \\ &\leq \left(\frac{\int_0^{+\infty}\left(f(x)\right)^{p_2}\{\phi(x)\}^q g(x)\mathrm{d}x}{1+wD_q\left(\phi g|g\right)}\right)^{\frac{1}{p_2}}\end{aligned} \tag{A38}$$

due to $\left(1+wD_q\left(g|g\right)\right)^{\frac{1}{p_1}-\frac{1}{p_2}} \geq 1$. We then obtain

$$\left(\frac{\int_0^{+\infty}\left(f(x)\right)^{p_1}\{\phi(x)\}^q g(x)\mathrm{d}x}{1+wD_q\left(\phi g|g\right)}\right)^{\frac{1}{p_1}} \leq \bar{O}_{g,\Phi,f}\Big|_{p=p_2} \tag{A39}$$

and thus

$$\bar{O}_{g,\Phi,f}\Big|_{p=p_1} \leq \bar{O}_{g,\Phi,f}\Big|_{p=p_2}\,, \tag{A40}$$

which is the statement of the case (a).

We prove the statement of the case (b). In the rest of this proof, we consider only $\phi\in\mathcal{D}$ such that $1-wD_q\left(\phi g|g\right)>0$. Assuming this additional condition is harmless as our focus is only on the optimized ϕ that clearly satisfies it. Indeed, if $1-wD_q\left(\phi g|g\right)\leq 0$ at $\phi=\underline{\phi}$, then we must have

$$\begin{aligned}1 = J\left(\bar{k}\right) &= \inf_{\phi\in\mathcal{D}}\left\{\int_0^{+\infty}\Phi^{(-1)}\left(\frac{f(x)}{k}\right)\{\phi(x)\}^q g(x)\mathrm{d}x + wD_q\left(\phi g|g\right)\right\} \\ &\geq \int_0^{+\infty}\Phi^{(-1)}\left(\frac{f(x)}{\underline{k}}\right)\{\underline{\phi}(x)\}^q g(x)\mathrm{d}x + 1\end{aligned}\,, \tag{A41}$$

leading to the contradiction $\underline{\phi} \equiv 0$ or $\underline{k} = +\infty$.

Fix some $\phi \in \underline{\mathcal{D}}$ such that $1 - wD_q(\phi g|g) > 0$. By the assumption and the Jensen inequality, by $p_2 \geq p_1 \geq 1$ and thus $\frac{1}{p_2} \leq \frac{1}{p_1}$, we have

$$\left(\frac{\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_2}} \{\phi(x)\}^q g(x) \mathrm{d}x}{M} \right)^{p_2} \leq \left(\frac{\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_1}} \{\phi(x)\}^q g(x) \mathrm{d}x}{M} \right)^{p_1} \tag{A42}$$

with M given by (A35), or equivalently

$$\left(\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_2}} \{\phi(x)\}^q g(x) \mathrm{d}x \right)^{p_2} \leq \left(\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_1}} \{\phi(x)\}^q g(x) \mathrm{d}x \right)^{p_1} M^{p_2 - p_1} \tag{A43}$$

and hence

$$\left(\frac{\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_2}} \{\phi(x)\}^q g(x) \mathrm{d}x}{1 - wD_q(\phi g|g)} \right)^{p_2} \leq \left(\frac{\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_1}} \{\phi(x)\}^q g(x) \mathrm{d}x}{1 - wD_q(\phi g|g)} \right)^{p_1} \frac{M^{p_2 - p_1}}{\left(1 - wD_q(\phi g|g)\right)^{p_2 - p_1}}. \tag{A44}$$

By $q \geq 1$, we have

$$M^{\frac{1}{q}} = \left(\int_0^{+\infty} \{\phi(x)\}^q g(x) \mathrm{d}x \right)^{\frac{1}{q}} \geq \int_0^{+\infty} \phi(x) g(x) \mathrm{d}x = 1. \tag{A45}$$

Hence, we obtain $M \geq 1$. Then, we have

$$\begin{aligned} \underline{O}_{g,\Phi,f}\Big|_{p=p_2} &= \inf_{\phi \in \underline{\mathcal{D}}} \left(\frac{\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_2}} \{\phi(x)\}^q g(x) \mathrm{d}x}{1 - wD_q(\phi g|g)} \right)^{p_2} \\ &\leq \left(\frac{\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_1}} \{\phi(x)\}^q g(x) \mathrm{d}x}{1 - wD_q(\phi g|g)} \right)^{p_1} \frac{M^{p_2 - p_1}}{\left(1 - wD_q(\phi g|g)\right)^{p_2 - p_1}} \end{aligned} \tag{A46}$$

and hence

$$\frac{\left(1 - wD_q(\phi g|g)\right)^{p_2 - p_1}}{M^{p_2 - p_1}} \times \underline{O}_{g,\Phi,f}\Big|_{p=p_2} \leq \left(\frac{\int_0^{+\infty} \left(f(x) \right)^{\frac{1}{p_1}} \{\phi(x)\}^q g(x) \mathrm{d}x}{1 - wD_q(\phi g|g)} \right)^{p_1}, \tag{A47}$$

leading to

$$\inf_{\phi \in \underline{\mathcal{D}}} \left\{ \frac{\left(1 - wD_q(\phi g|g)\right)^{p_2 - p_1}}{M^{p_2 - p_1}} \right\} \times \underline{O}_{g,\Phi,f}\Big|_{p=p_2} \leq \underline{O}_{g,\Phi,f}\Big|_{p=p_1}. \tag{A48}$$

Note that we have, by $p_2 \geq p_1$,

$$\begin{aligned}
\inf_{\phi\in\mathcal{D}}\left\{\frac{\left(1-wD_q\left(\phi g|g\right)\right)^{p_2-p_1}}{M^{p_2-p_1}}\right\} &= \inf_{\phi\in\mathcal{D}}\left(\frac{1-wD_q\left(\phi g|g\right)}{M}\right)^{p_2-p_1} \\
&= \left(\inf_{\phi\in\mathcal{D}}\left(\frac{1-wD_q\left(\phi g|g\right)}{M}\right)\right)^{p_2-p_1} \\
&= \left(\inf_{\phi\in\mathcal{D}}\left(\frac{1-w\frac{1}{1-q}\left(1-M\right)}{M}\right)\right)^{p_2-p_1} \\
&= \left(\inf_{\phi\in\mathcal{D}}\left(\frac{1-w\frac{1}{q-1}+w\frac{1}{q-1}M}{M}\right)\right)^{p_2-p_1} \\
&= \left(\inf_{\phi\in\mathcal{D}}\left(\left(1-w\frac{1}{q-1}\right)\frac{1}{M}\right)+w\frac{1}{q-1}\right)^{p_2-p_1}
\end{aligned} \quad . \tag{A49}$$

By $M\geq 1$, we have $0<M^{-1}\leq 1$. Hence, if $1-w\dfrac{1}{q-1}\geq 0$, then we obtain the bound

$$\inf_{\phi\in\mathcal{D}}\left(\left(1-w\frac{1}{q-1}\right)\frac{1}{M}\right)=\left(1-w\frac{1}{q-1}\right)\inf_{\phi\in\mathcal{D}}\frac{1}{M}\geq 0\ . \tag{A50}$$

Substituting (A50) into (A49) yields

$$\inf_{\phi\in\mathcal{D}}\left\{\frac{\left(1-wD_q\left(\phi g|g\right)\right)^{p_2-p_1}}{M^{p_2-p_1}}\right\}\geq\left(w\frac{1}{q-1}\right)^{p_2-p_1} . \tag{A51}$$

If $1-w\dfrac{1}{q-1}<0$, then by $M\geq 1$ we obtain the bound

$$\begin{aligned}
\inf_{\phi\in\mathcal{D}}\left(\left(1-w\frac{1}{q-1}\right)\frac{1}{M}\right) &= \inf_{\phi\in\mathcal{D}}\left(-\left(w\frac{1}{q-1}-1\right)\frac{1}{M}\right) \\
&= \left(w\frac{1}{q-1}-1\right)\inf_{\phi\in\mathcal{D}}\left(-\frac{1}{M}\right) \\
&= \left(w\frac{1}{q-1}-1\right)\times\left(-1\right) \\
&= 1-w\frac{1}{q-1}
\end{aligned} \tag{A52}$$

and hence

$$\inf_{\phi\in\mathcal{D}}\left\{\frac{\left(1-wD_q\left(\phi g|g\right)\right)^{p_2-p_1}}{M^{p_2-p_1}}\right\}=1. \tag{A53}$$

Combining the cases $1-w\dfrac{1}{q-1}\geq 0$ and $1-w\dfrac{1}{q-1}<0$ yields

$$\inf_{\phi \in \mathcal{D}} \left\{ \frac{\left(1 - wD_q\left(\phi g \middle| g\right)\right)^{p_2 - p_1}}{M^{p_2 - p_1}} \right\} \geq \min\left\{1, \left(w\frac{1}{q-1}\right)^{p_2 - p_1}\right\}. \quad \text{(A54)}$$

Substituting (A51) into (A48) yields

$$\min\left\{1, \left(w\frac{1}{q-1}\right)^{p_2 - p_1}\right\} \times \underline{O}_{g,\Phi,f}\Big|_{p=p_2} \leq \underline{O}_{g,\Phi,f}\Big|_{p=p_1}. \quad \text{(A55)}$$

The positivity of $\underline{O}_{g,\Phi,f}\Big|_{p=p_2}$ is trivial. Hence, the statement of the case (b) was proven.

□

Appendix B. The components S **and** X

The components S and X are visually presented in this appendix for the self-contentedness of the paper.

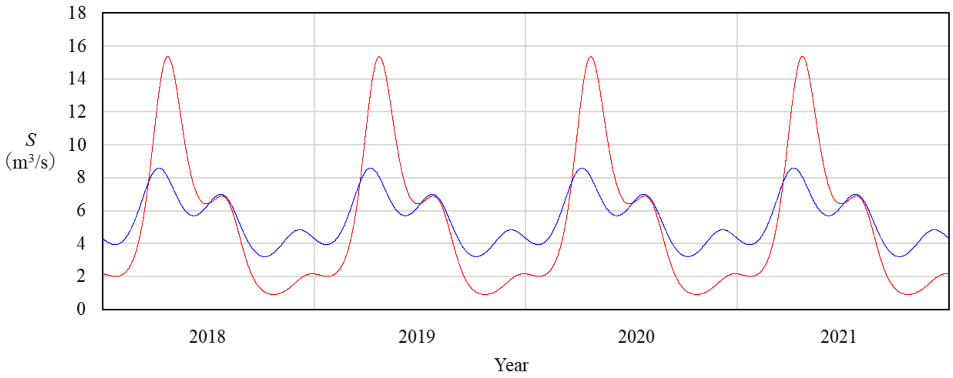

Figure B1. The component S at Kazarashi (red) and Shimono (blue).

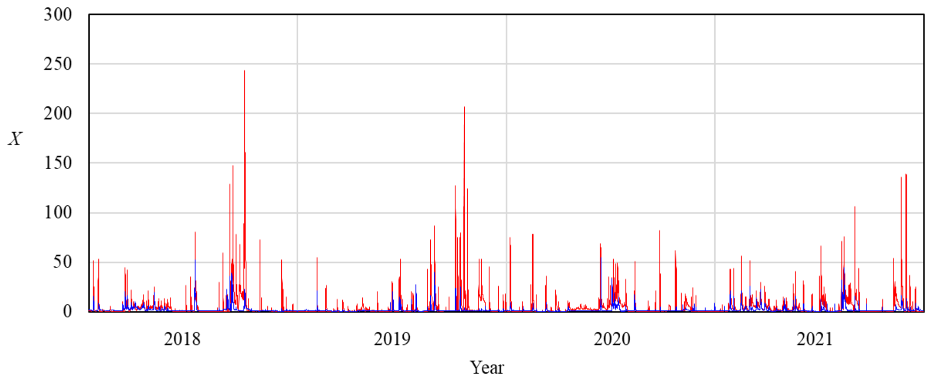

Figure B2. The component X at Kazarashi (red) and Shimono (blue).

Appendix C. Numerical algorithm for the lower Orlicz risk

Algorithm 2 presented below aims at computing the lower-Orlicz risk $\underline{O}_{g,\Phi,f}$ and the associated worstcase Radon–Nikodym derivative $\underline{\phi}$ for the specific choice $\Phi(x)=x^p$ ($p \geq 1$). As in **Algorithm 1**, we can use the following algorithm based on the representation formula (e.g., Example 9 in Bellini et al. (2021))

$$\underline{O}_{g,\Phi,f} = \inf_{\phi \in \underline{\mathcal{D}}} \left(\frac{\int_0^{+\infty} \left(f(x)\right)^{\frac{1}{p}} \left\{\phi(x)\right\}^q g(x)\mathrm{d}x}{1-wD_q\left(\phi g | g\right)} \right)^p . \tag{B1}$$

Algorithm 2

1. Set g, Φ, f, the admissible set $\underline{\mathcal{D}}$, and the weighting factor w.
2. Set the initial guess $\underline{\phi}^{(0)}(x)=1$ ($x>0$) and $\underline{k}^{(0)}>0$.
3. Set the iteration count $m=0$.
4. Compute $\underline{\phi}^{(m+1)}(x)=\left\{\int_0^{+\infty} \exp_q\left(-\frac{1}{w}\left(\frac{f(x)}{\underline{k}^{(m)}}\right)^{\frac{1}{p}}\right) g(x)\mathrm{d}x\right\}^{-1} \exp_q\left(-\frac{1}{w}\left(\frac{f(x)}{\underline{k}^{(m)}}\right)^{\frac{1}{p}}\right)$.
5. Compute $\underline{k}^{(m+1)} = \left\{ \frac{\int_0^{+\infty} \left(f(x)\right)^{\frac{1}{p}} \left\{\underline{\phi}^{(m+1)}(x)\right\}^q g(x)\mathrm{d}x}{1-wD_q\left(\underline{\phi}^{(m+1)} g | g\right)} \right\}^p$.
6. If $\left|\underline{\phi}^{(m+1)}-\underline{\phi}^{(m)}\right| \leq \mathrm{Err}$, then output $\underline{k}^{(m+1)}$ as the approximation of $\underline{O}_{g,\Phi,f}$ and $\underline{\phi}^{(m+1)}$ as the approximation of $\underline{\phi}$, and go to 7. If it is not, set $m \to m+1$ and go to 4.
7. Terminate the algorithm.

We also present **Algorithms 3** and **4** to compute the upper and lower Orlicz risks with generic Φ. In these algorithms, we use $\delta=1$ and $\mathrm{Err}=10^{-10}$. The initial guesses are $\overline{k}^{(0)}=20$ and $\underline{k}^{(0)}=1$.

Algorithm 3

1. Set g, Φ, f and the weighting factor w.
2. Set the increment $\delta>0$ and the error tolerance $\mathrm{Err}>0$.
3. Set the initial guess $\overline{\phi}^{(0)}(x)=1$ ($x>0$) and $\overline{k}^{(0)}>0$.
4. Set the iteration count $m=0$.
5. Compute $\overline{\phi}^{(m+1)}(x)=\left\{\int_0^{+\infty} \exp_q\left(\frac{1}{w}\Phi\left(\frac{f(x)}{\overline{k}^{(m)}}\right)\right) g(x)\mathrm{d}x\right\}^{-1} \exp_q\left(\frac{1}{w}\Phi\left(\frac{f(x)}{\overline{k}^{(m)}}\right)\right)$.
6. Compute $\overline{k}^{(m+1)}=\overline{k}^{(m)}-\delta\left\{\int_0^{+\infty}\Phi\left(\frac{f(x)}{\overline{k}^{(m)}}\right)\left\{\overline{\phi}^{(m)}(x)\right\}^q g(x)\mathrm{d}x - wD_q\left(\overline{\phi}^{(m)} g | g\right)-1\right\}$.
7. If $\left|\overline{k}^{(m+1)}-\overline{k}^{(m)}\right| \leq \mathrm{Err}$, then output $\overline{k}^{(m+1)}$ as the approximation of $\overline{O}_{g,\Phi,f}$ and $\overline{\phi}^{(m+1)}$ as the

approximation of $\bar{\phi}$, and go to 8. If it is not, set $m \to m+1$ and go to 5.

8. Terminate the algorithm.

Algorithm 4

1. Set g, Φ, f and the weighting factor w.
2. Set the increment $\delta > 0$ and the error tolerance $\mathrm{Err} > 0$.
3. Set the initial guess $\underline{\phi}^{(0)}(x) = 1$ ($x > 0$) and $\underline{k}^{(0)} > 0$.
4. Set the iteration count $m = 0$.
5. Compute $\underline{\phi}^{(m+1)}(x) = \left\{ \int_0^{+\infty} \exp_q \left(-\frac{1}{w} \Phi^{(-1)} \left(\frac{f(x)}{\underline{k}^{(m)}} \right) \right) g(x) \mathrm{d}x \right\}^{-1} \exp_q \left(-\frac{1}{w} \Phi^{(-1)} \left(\frac{f(x)}{\underline{k}^{(m)}} \right) \right)$.
6. Compute $\underline{k}^{(m+1)} = \underline{k}^{(m)} - \delta \left\{ \int_0^{+\infty} \Phi^{(-1)} \left(\frac{f(x)}{\underline{k}^{(m)}} \right) \left\{ \underline{\phi}^{(m)}(x) \right\}^q g(x) \mathrm{d}x + w D_q \left(\underline{\phi}^{(m)} g \middle| g \right) - 1 \right\}$.
7. If $\left| \underline{k}^{(m+1)} - \underline{k}^{(m)} \right| \le \mathrm{Err}$, then output $\underline{k}^{(m+1)}$ and $\underline{\phi}^{(m+1)}$ as the approximation of $\underline{O}_{g,\Phi,f}$ and $\underline{\phi}^{(m+1)}$ as the approximation of $\underline{\phi}$, and go to 8. If it is not, set $m \to m+1$ and go to 5.
8. Terminate the algorithm.

References

Ma H, Tian D (2021) Generalized entropic risk measures and related BSDEs. Stat Probab Lett 174: 109110. https://doi.org/10.1016/j.spl.2021.109110

Bellini F, Laeven RJ, Rosazza Gianin E (2018) Robust return risk measures. Math Financ Econ 12: 5-32. https://doi.org/10.1007/s11579-017-0188-x

Bellini F, Laeven RJ, Gianin ER (2021) Dynamic robust Orlicz premia and Haezendonck–Goovaerts risk measures. European J Oper Res 291(2): 438-446. https://doi.org/10.1016/j.ejor.2019.08.049

Zorich VA (2016) Mathematical Analysis II. Springer, Berlin.