

New large Condorcet domains on 10 and 11 alternatives, their TRS and Properties

1 Introduction to the properties of Condorcet domains

The distance between two permutations is the minimal number of neighbor inversions needed to convert one permutation to another. A *median graph* is a bidirectional graph formed by all the permutations in a Condorcet domain (CD) as vertices in which two permutations with minimal distance are connected by an edge (i.e. an edge connects two permutations that can be made identical by swapping two neighbor alternatives in one of them)

The width of the CD is the longest possible distance between two permutations in the median graph. A Condorcet domain has *maximal width* if it contains a pair of completely reversed linear orders ($[1, 2, 3, \dots, n]$ and $[n, n-1, n-2, \dots, 1]$). A Condorcet domain is *connected* if every two orders from the domain can be obtained from each other by a sequence of transpositions of neighboring alternatives, such that all orders from the connecting path belong to the domain.

Fig. 1 shows the median graph for a Condorcet domain on four alternatives. Each connected permutation only differs at two neighbor positions, and the connection remains unchanged if the alternatives at these positions are exchanged. For instance, permutations 1243 and 1234 will be equivalent if alternatives 3 and 4 are swapped in one of them. The width of this graph is, by definition, six as there are six edges between permutation 1234 and 4321.

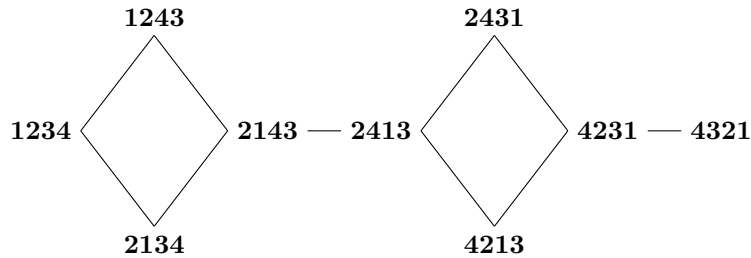


Fig. 1 The median graph for a Condorcet domain on 4 alternatives. The distance between permutations 1234 and 4321 is the longest among all pairs of permutations. The shortest path between them has a length of 6 so the width of this graph is 6.

The definition of the width, radius and centre point of a Condorcet domain is the same as those of their median graph. The width (also called diameter) of a Condorcet domain is the longest possible distance between any two permutations in it. The radius of a Condorcet domain is the minimum distance between the identity permutation and any other permutation within the Condorcet domain. A centre point is a permutation whose distance to the identity permutation is equal to the radius.

In Table 1, we display the sizes, widths, radius, the number of centre points and the number of their isomorphic Condorcet domains for the Fishburn domain on 4 to 13 alternatives. While the median graphs have been extensively discussed in the literature, to the best of our knowledge, the data for the radius and the number of centre points have not been published. The width and radius grow monotonically as the number of alternatives in the graph increases. The number of centre points generally follows that trend, but there are 93 centre points in the 11 alternative median graph, which is less than the 10 alternative ones. Not all Fishburn domains shown in this table have the same number

of isomorphic Condorcet domains as their sizes. The number of isomorphic Condorcet domains for Fishburn domains on the odd number of alternatives is only half of their size.

Table 1 The size, width, radius, the number of centre points, and the number of isomorphic Condorcet domain of the Fishburn domains on 4 to 13 alternatives.

Num of alternatives (n)	4	5	6	7	8	9	10	11	12	13
Size	9	20	45	100	222	488	1069	2324	5034	10840
Width	6	10	15	21	28	36	45	55	66	78
Radius	3	5	8	11	14	18	23	28	33	39
Num of centre points	1	2	6	10	9	16	57	102	93	172
Num of isomorphic CDs	9	10	45	50	222	244	1069	1162	5034	5420

2 The properties of the n=10 CD of size 1082

We found 242 CDs on 10 alternatives larger than the Fishburn domain and by definition, all the median graphs built upon these CDs are connected. So far, the CDs on 10 alternatives with the highest width and radius are created by the alternating scheme, which has a width of 45 and a radius of 23. Although the sizes of the newly found large CDs are larger than those of the Fishburn domain, their widths and radius are smaller. The number of centre points in the graph for the Fishburn domain is 57, the same as the domains of size 1070 and 1080, the smallest and the largest CDs that broke the record found by our search algorithm.

Table 2 The size and the number of equivalent CDs of the 10 alternative large CDs we found, and the width of, the radius of, and the number of centre points in their median graph.

New CD size	1070	1071	1072	1074	1078	1079	1082
Width	41	43	41	42	42	43	41
Radius	21	22	21	21	21	22	21
Num of centre points	57	56	38	24	28	56	57
Num of isomorphic CDs	1070	1071	1072	1074	1078	1079	1082

3 The properties of the n=11 CD of size 2349

We discovered 543 CDs in 5 different sizes larger than the Fishburn domain on 11 alternatives. Their properties are presented in Table 3. The largest CD among them is of size 2349, which contains 25 more permutations than the Fishburn domain, which is of size 2324. The width and radius of all the new CDs are less than those of the Fishburn domain. However, the number of all the equivalent CDs of the newly discovered CDs is the same as their size, differing from the Fishburn domain, where the number of isomorphic CDs is exactly half its size.

Table 3 The size and the number of isomorphic CDs of the record-breaking CDs for 11 alternatives and the width, the radius, and the number of centre points in their median graph.

New CD size	2328	2329	2334	2337	2349
Width	53	51	51	52	52
Radius	27	26	26	26	26
Num of centre points	102	102	102	51	51
Num of isomorphic CDs	2328	2329	2334	2337	2349

4 The triples and their never condition (NC) for the n=10 CD of size 1082

Table 4 The TRS that led to the CD of size 1082 on 10 alternatives. The ordering of the triples displayed is another static ordering that differs from the ordering used in the search algorithm and is defined such that (x_1, y_1, z_1) is before (x_2, y_2, z_2) if $x_1 \leq x_2$ and $y_1 \leq y_2$ and $z_1 \leq z_2$ out of consideration for the ease of looking up the table manually. There are 1082 isomorphic versions of this CD. The 10 alternative CD built by these never conditions is presented in Sect. 6.

Triples	NC	Triples	NC	Triples	NC	Triples	NC
(1, 2, 3)	2N3	(1, 7, 8)	1N3	(2, 7, 10)	1N3	(4, 6, 7)	3N1
(1, 2, 4)	2N3	(1, 7, 9)	1N3	(2, 8, 9)	1N3	(4, 6, 8)	3N1
(1, 2, 5)	1N3	(1, 7, 10)	1N3	(2, 8, 10)	1N3	(4, 6, 9)	1N3
(1, 2, 6)	1N3	(1, 8, 9)	2N1	(2, 9, 10)	3N1	(4, 6, 10)	3N1
(1, 2, 7)	1N3	(1, 8, 10)	2N1	(3, 4, 5)	1N3	(4, 7, 8)	3N1
(1, 2, 8)	3N1	(1, 9, 10)	3N1	(3, 4, 6)	1N3	(4, 7, 9)	1N3
(1, 2, 9)	2N3	(2, 3, 4)	2N3	(3, 4, 7)	1N3	(4, 7, 10)	3N1
(1, 2, 10)	2N3	(2, 3, 5)	1N3	(3, 4, 8)	1N3	(4, 8, 9)	1N3
(1, 3, 4)	2N3	(2, 3, 6)	1N3	(3, 4, 9)	1N3	(4, 8, 10)	3N1
(1, 3, 5)	1N3	(2, 3, 7)	1N3	(3, 4, 10)	1N3	(4, 9, 10)	1N3
(1, 3, 6)	1N3	(2, 3, 8)	1N3	(3, 5, 6)	3N1	(5, 6, 7)	3N1
(1, 3, 7)	1N3	(2, 3, 9)	3N1	(3, 5, 7)	3N1	(5, 6, 8)	1N3
(1, 3, 8)	3N1	(2, 3, 10)	3N1	(3, 5, 8)	3N1	(5, 6, 9)	2N1
(1, 3, 9)	3N1	(2, 4, 5)	1N3	(3, 5, 9)	1N3	(5, 6, 10)	1N3
(1, 3, 10)	3N1	(2, 4, 6)	1N3	(3, 5, 10)	3N1	(5, 7, 8)	1N3
(1, 4, 5)	1N3	(2, 4, 7)	1N3	(3, 6, 7)	3N1	(5, 7, 9)	2N1
(1, 4, 6)	1N3	(2, 4, 8)	1N3	(3, 6, 8)	3N1	(5, 7, 10)	1N3
(1, 4, 7)	1N3	(2, 4, 9)	3N1	(3, 6, 9)	1N3	(5, 8, 9)	2N1
(1, 4, 8)	3N1	(2, 4, 10)	3N1	(3, 6, 10)	3N1	(5, 8, 10)	2N1
(1, 4, 9)	3N1	(2, 5, 6)	3N1	(3, 7, 8)	3N1	(5, 9, 10)	3N1
(1, 4, 10)	3N1	(2, 5, 7)	3N1	(3, 7, 9)	1N3	(6, 7, 8)	2N1
(1, 5, 6)	3N1	(2, 5, 8)	3N1	(3, 7, 10)	3N1	(6, 7, 9)	2N1
(1, 5, 7)	3N1	(2, 5, 9)	1N3	(3, 8, 9)	1N3	(6, 7, 10)	2N1
(1, 5, 8)	1N3	(2, 5, 10)	1N3	(3, 8, 10)	3N1	(6, 8, 9)	2N1
(1, 5, 9)	1N3	(2, 6, 7)	3N1	(3, 9, 10)	1N3	(6, 8, 10)	2N1
(1, 5, 10)	1N3	(2, 6, 8)	3N1	(4, 5, 6)	3N1	(6, 9, 10)	3N1
(1, 6, 7)	3N1	(2, 6, 9)	1N3	(4, 5, 7)	3N1	(7, 8, 9)	2N1
(1, 6, 8)	1N3	(2, 6, 10)	1N3	(4, 5, 8)	3N1	(7, 8, 10)	2N1
(1, 6, 9)	1N3	(2, 7, 8)	3N1	(4, 5, 9)	1N3	(7, 9, 10)	3N1
(1, 6, 10)	1N3	(2, 7, 9)	1N3	(4, 5, 10)	3N1	(8, 9, 10)	3N1

5 The triples and their never condition for the n=11 CD of size 2349

Table 5 The TRS that constructed CD of size 2349 on 11 alternatives. The 11 alternative CD built by these never conditions is presented in Sect. 6.

Triples	NC	Triples	NC	Triples	NC	Triples	NC
(1, 2, 3)	1N3	(1, 9, 10)	3N1	(3, 4, 8)	3N1	(4, 8, 11)	2N1
(1, 2, 4)	2N3	(1, 9, 11)	3N1	(3, 4, 9)	1N3	(4, 9, 10)	3N1
(1, 2, 5)	2N3	(1, 10, 11)	2N1	(3, 4, 10)	3N1	(4, 9, 11)	2N3
(1, 2, 6)	2N3	(2, 3, 4)	1N3	(3, 4, 11)	1N3	(4, 10, 11)	2N1
(1, 2, 7)	3N1	(2, 3, 5)	1N3	(3, 5, 6)	3N1	(5, 6, 7)	2N1
(1, 2, 8)	1N3	(2, 3, 6)	1N3	(3, 5, 7)	3N1	(5, 6, 8)	2N1
(1, 2, 9)	1N3	(2, 3, 7)	3N1	(3, 5, 8)	3N1	(5, 6, 9)	2N1
(1, 2, 10)	1N3	(2, 3, 8)	3N1	(3, 5, 9)	1N3	(5, 6, 10)	2N1
(1, 2, 11)	1N3	(2, 3, 9)	2N3	(3, 5, 10)	3N1	(5, 6, 11)	2N1
(1, 3, 4)	1N3	(2, 3, 10)	3N1	(3, 5, 11)	1N3	(5, 7, 8)	2N1
(1, 3, 5)	1N3	(2, 3, 11)	2N3	(3, 6, 7)	3N1	(5, 7, 9)	2N1
(1, 3, 6)	1N3	(2, 4, 5)	3N1	(3, 6, 8)	3N1	(5, 7, 10)	2N1
(1, 3, 7)	1N3	(2, 4, 6)	3N1	(3, 6, 9)	1N3	(5, 7, 11)	2N1
(1, 3, 8)	3N1	(2, 4, 7)	1N3	(3, 6, 10)	3N1	(5, 8, 9)	2N1
(1, 3, 9)	2N3	(2, 4, 8)	1N3	(3, 6, 11)	1N3	(5, 8, 10)	2N1
(1, 3, 10)	3N1	(2, 4, 9)	1N3	(3, 7, 8)	3N1	(5, 8, 11)	2N1
(1, 3, 11)	2N3	(2, 4, 10)	1N3	(3, 7, 9)	1N3	(5, 9, 10)	3N1
(1, 4, 5)	3N1	(2, 4, 11)	1N3	(3, 7, 10)	3N1	(5, 9, 11)	2N3
(1, 4, 6)	3N1	(2, 5, 6)	3N1	(3, 7, 11)	1N3	(5, 10, 11)	2N1
(1, 4, 7)	3N1	(2, 5, 7)	1N3	(3, 8, 9)	1N3	(6, 7, 8)	2N1
(1, 4, 8)	1N3	(2, 5, 8)	1N3	(3, 8, 10)	3N1	(6, 7, 9)	2N1
(1, 4, 9)	1N3	(2, 5, 9)	1N3	(3, 8, 11)	1N3	(6, 7, 10)	2N1
(1, 4, 10)	1N3	(2, 5, 10)	1N3	(3, 9, 10)	1N3	(6, 7, 11)	2N1
(1, 4, 11)	1N3	(2, 5, 11)	1N3	(3, 9, 11)	3N1	(6, 8, 9)	2N1
(1, 5, 6)	3N1	(2, 6, 7)	1N3	(3, 10, 11)	1N3	(6, 8, 10)	2N1
(1, 5, 7)	3N1	(2, 6, 8)	1N3	(4, 5, 6)	3N1	(6, 8, 11)	2N1
(1, 5, 8)	1N3	(2, 6, 9)	1N3	(4, 5, 7)	1N3	(6, 9, 10)	3N1
(1, 5, 9)	1N3	(2, 6, 10)	1N3	(4, 5, 8)	1N3	(6, 9, 11)	2N3
(1, 5, 10)	1N3	(2, 6, 11)	1N3	(4, 5, 9)	2N1	(6, 10, 11)	2N1
(1, 5, 11)	1N3	(2, 7, 8)	2N1	(4, 5, 10)	1N3	(7, 8, 9)	2N1
(1, 6, 7)	3N1	(2, 7, 9)	2N1	(4, 5, 11)	2N1	(7, 8, 10)	2N1
(1, 6, 8)	1N3	(2, 7, 10)	2N1	(4, 6, 7)	1N3	(7, 8, 11)	2N1
(1, 6, 9)	1N3	(2, 7, 11)	2N1	(4, 6, 8)	1N3	(7, 9, 10)	3N1
(1, 6, 10)	1N3	(2, 8, 9)	2N1	(4, 6, 9)	2N1	(7, 9, 11)	2N3
(1, 6, 11)	1N3	(2, 8, 10)	2N1	(4, 6, 10)	1N3	(7, 10, 11)	2N1
(1, 7, 8)	1N3	(2, 8, 11)	2N1	(4, 6, 11)	2N1	(8, 9, 10)	3N1
(1, 7, 9)	1N3	(2, 9, 10)	3N1	(4, 7, 8)	2N1	(8, 9, 11)	2N3
(1, 7, 10)	1N3	(2, 9, 11)	3N1	(4, 7, 9)	2N1	(8, 10, 11)	2N1
(1, 7, 11)	1N3	(2, 10, 11)	2N1	(4, 7, 10)	2N1	(9, 10, 11)	1N3
(1, 8, 9)	2N1	(3, 4, 5)	3N1	(4, 7, 11)	2N1		
(1, 8, 10)	2N1	(3, 4, 6)	3N1	(4, 8, 9)	2N1		
(1, 8, 11)	2N1	(3, 4, 7)	3N1	(4, 8, 10)	2N1		

6 The Condorcet domains

The large Condorcet domains we discovered for 10 and 11 alternatives are demonstrated here. We use hexadecimal numbers A and B to represent the domain's decimal numbers 10 and 11.

The Condorcet domain of size 1082 on 10 alternatives

123456789A 12345678A9 123456798A 12345679A8 123456978A 12345697A8 1234569A78 1234569A87 123459678A 12345967A8
1234596A78 1234596A87 123459A678 123459A687 123459A876 123459A867 123495678A 12349567A8 1234956A78 1234956A87
123495A678 123495A687 123495A867 123495A876 123496578A 12349657A8 1234965A78 1234965A87 123496758A 12349675A8
12349A5678 12349A5687 12349A5867 12349A5876 12349A8567 12349A8576 123546789A 12354678A9 123546798A 12354679A8
123546978A 12354697A8 1235469A78 1235469A87 123549678A 12354967A8 1235496A78 1235496A87 123549A678 123549A687
123549A867 123549A876 123564789A 12356478A9 123564798A 12356479A8 123564978A 12356497A8 1235649A78 1235649A87
123567489A 12356748A9 123567498A 12356749A8 123567849A 12356784A9 1235678A49 123945678A 12394567A8 1239456A78

5

The Condorcet domain of size 2349 on 11 alternatives

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8

93BA1248765 93BA1284567 93BA1284576 93BA1284756 93BA1284765 93BA1287456 93BA1287465 93BA1824567 93BA1824576
93BA1824756 93BA1824765 93BA1827456 93BA1827465 93BA1872456 93BA1872465 93BA8124567 93BA8124576 93BA8124756
93BA8124765 93BA8127456 93BA8127465 93BA8172456 93BA8172465 9B31245678A 9B3124567A8 9B312456A78 9B312456A87
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9B312A45876 9B312A48567 9B312A48576 9B312A48756 9B312A48765 9B312A84567 9B312A84576 9B312A84756 9B312A84765
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9B31A248765 9B31A284567 9B31A284576 9B31A284756 9B31A284765 9B31A287456 9B31A287465 9B31A824567 9B31A824576
9B31A824756 9B31A824765 9B31A827456 9B31A827465 9B31A872456 9B31A872465 9B3A1245678 9B3A1245687 9B3A1245867
9B3A1245876 9B3A1248567 9B3A1248576 9B3A1248756 9B3A1248765 9B3A1284567 9B3A1284576 9B3A1284756 9B3A1284765
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9B3A1872465 9B3A8124567 9B3A8124576 9B3A8124756 9B3A8124765 9B3A8127456 9B3A8127465 9B3A8172456 9B3A8172465