Extended Washburn equation with slip boundary condition

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From the Navier-Stokes momentum conservation equation, considering unidirectional Hele-Shaw flow between parallel plates at steady state is given by,

$$\frac{dp}{dx} = \mu \frac{\partial^2 U}{\partial y^2} \tag{1}$$

Integrating the above equation twice to obtain velocity gives,

$$\frac{dU}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1 \tag{2}$$

$$U = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2. \tag{3}$$

Boundary conditions are $U = 0$ at $y = 0$ and at $y = h + 2\lambda$ [where the slip lengths end] gives $C_2 = 0$ and $C_1 = -\frac{1}{2\mu}\frac{dp}{dx}(h + 2\lambda)$.

Substituting the constants (C_1, C_2) for U gives,

$$U = \frac{1}{2\mu}\frac{dp}{dx}y[y - (h + 2\lambda)]. \tag{4}$$

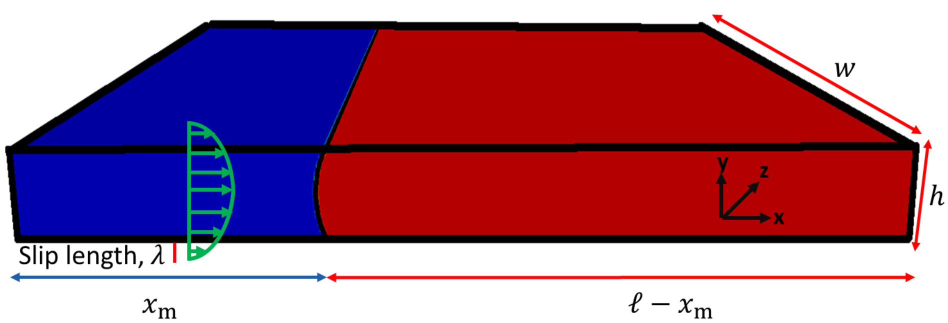

Figure 1: Illustration of the problem set-up.

The average velocity within the channel is given by

$$U_{avg} = \frac{Q}{wh} = \frac{1}{wh}\int_{\lambda}^{h+\lambda} \frac{1}{2\mu}\frac{dp}{dx} y[y-(h+2\lambda)]dy \int_0^w dz \tag{5}$$

Solving the above equation gives

$$U_{avg} = -\frac{1}{12\mu}\frac{dp}{dx}[h^2 + 6h\lambda + 6\lambda^2] \tag{6}$$

and the flow rate is

$$Q = \frac{wh}{12\mu}\frac{dp}{dx}[h^2 + 6h\lambda + 6\lambda^2]. \tag{7}$$

The maximum velocity occurs at the centre of the channel at a height of $h/2 + \lambda$. Substituting $h/2 + \lambda$ in Eq. 4 gives

$$U_{max} = -\frac{1}{8\mu}\frac{dp}{dx}(2\lambda + h)^2. \tag{8}$$

The ratio of U_{max} to U_{avg} is obtained by dividing Eq. 8 with Eq.6 giving

$$U_{max} = 1.5U_{avg}\frac{(2\lambda + h)^2}{h^2 + 6h\lambda + 6\lambda^2}. \tag{9}$$

Active forces are the capillary and viscous.
Capillary force is given by

$$\mathbf{F}_{\text{cap}} = 2w\sigma\cos(\theta). \tag{10}$$

The viscous force is given by

$$\mathbf{F}_{\text{vis}} = \int_0^A \mu\nabla U dA = 2w\int_0^l \mu\nabla U dl. \tag{11}$$

From Hele-Shaw equation,

$$\nabla U = \frac{h}{2\mu}\nabla p. \tag{12}$$

Substituting ∇p from Eq. 6 gives,

$$\nabla U = \frac{6Uh}{h^2 + 6h\lambda + 6\lambda^2}. \tag{13}$$

Substituting ∇U in Eq.11 gives

$$\mathbf{F}_{\text{vis}} = \frac{12Uwh}{h^2 + 6h\lambda + 6\lambda^2}(\mu_w x_m + \mu_n(\ell - x_m)). \tag{14}$$

Balancing the active forces (Eq, 10 and Eq. 14) results in

$$\frac{h^2 + 6h\lambda + 6\lambda^2}{6h}\sigma\cos(\theta)dt = ((\mu_w - \mu_n)x_m + \mu_n\ell)dx_m \tag{15}$$

Integrating on both sides results in requiring to solve for the positive root of the following quadratic equation to find the position of the meniscus,

$$\frac{\mu_w - \mu_n}{2} x_m^2 + \mu_n \ell x_m - \frac{h^2 + 6h\lambda + 6\lambda^2}{6h} \sigma \cos\theta t = 0. \tag{16}$$

The position of the meniscus is given by

$$x_m = \frac{-\mu_n \ell + \sqrt{(\mu_n \ell)^2 + \frac{\mu_w - \mu_n}{3} \frac{h^2 + 6h\lambda + 6\lambda^2}{h} \sigma \cos\theta t}}{\mu_w - \mu_n} \tag{17}$$

To determine the velocity of the meniscus, $U = dx_m/dt$. Solving the above differential equation after some algebraic transformation gives,

$$U = \frac{h^2 + 6h\lambda + 6\lambda^2}{6h\mu_n \ell \sqrt{1 + 2(\frac{1}{M} - 1) \frac{h^2 + 6h\lambda + 6\lambda^2}{6h\mu_n \ell^2} \sigma \cos\theta t}} \sigma \cos\theta, \tag{18}$$

where M is the viscosity ratio defined as μ_n/μ_w.

The reference velocity is,

$$U_{ref} = \frac{h^2 + 6h\lambda + 6\lambda^2}{6h\mu_n \ell} \sigma \cos\theta, \tag{19}$$

and the reference time is,

$$t_{ref} = \frac{6h\mu_n \ell^2}{(h^2 + 6h\lambda + 6\lambda^2)\sigma \cos\theta}. \tag{20}$$

Example:

For a slip length $\lambda = 0.1\mu m$, with channel height $h = 10\mu m$, channel length $\ell = 400\mu m$ with the surface tension $\sigma = 0.01 kg/s^2$ and contact angle $\theta = 45$, the reference velocity is $U_{ref} = 0.0312 m/s$.