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Supplementary Materials to: Galerkin–Chebyshev approximation of Gaussian random fields on compact Riemannian manifolds

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SM1 Series bounds

Lemma SM1.1 *Let $m \in \mathbb{R}$, $m \neq -1$ and $n \in \mathbb{N}$. Then,*

$$\frac{1}{m+1}\left(1-\frac{1}{n^{m+1}}\right)+\frac{1}{n^{\max\{1,m+1\}}} \leq \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^m \leq \frac{1}{m+1}\left(1-\frac{1}{n^{m+1}}\right)+\frac{1}{n^{\min\{1,m+1\}}}$$

Proof Let $n \geq 1$ and let S_n denote the sum $S_n = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^m$.
First, assume that $m \leq 0$. Then, for any $k \in [\![1, n-1]\!]$ and any $t \in [k/n, (k+1)/n]$,

$$\left(\frac{k+1}{n}\right)^m \leq t^m \leq \left(\frac{k}{n}\right)^m.$$

Integrating both inequalities over $[k/n, (k+1)/n]$ and summing them for $k \in [\![1, n-1]\!]$ gives:

$$\frac{1}{n}\left(S_n - \frac{1}{n^m}\right) \leq I_n \leq \frac{1}{n}(S_n - 1),$$

where $I_n = \int_{1/n}^{1} t^m \, \mathrm{d}t = \left(1-n^{-(m+1)}\right)/(m+1)$. Hence, we have

$$I_n + \frac{1}{n} \leq \frac{1}{n}S_n \leq I_n + \frac{1}{n^{m+1}}.$$

Similarly, if $m \geq 0$ we get

$$I_n + \frac{1}{n^{m+1}} \leq \frac{1}{n}S_n \leq I_n + \frac{1}{n}.$$

So, for any $m \neq -1$, we have

$$I_n + \frac{1}{n^{\max\{1,m+1\}}} \leq \frac{1}{n}S_n \leq I_n + \frac{1}{n^{\min\{1,m+1\}}}.$$

Lemma SM1.2 *Let $m \in \mathbb{R}, m > 1$ and let $J \in \mathbb{N}$, $J \geq 1$. Then,*

$$\frac{1}{(m-1)(J+1)^{m-1}} \leq \sum_{j=J+1}^{\infty} j^{-m} \leq \frac{1}{(m-1)J^{m-1}}$$

Proof This result is obtained straightforwardly by upper-bounding and lower-bounding the integrals $\int_1^J t^{-m}\mathrm{d}t$, $\int_1^{J+1} t^{-m}\mathrm{d}t$ and $\int_{J+1}^{+\infty} t^{-m}\mathrm{d}t$.

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SM2 Additional properties of the Galerkin discretization

Lemma SM2.1 *Let $\boldsymbol{C}$ and $\boldsymbol{R}$ be the mass and stiffness matrices defined in* (3.2)*. Then, $\boldsymbol{C}$ is a symmetric positive definite matrix and $\boldsymbol{R}$ is a symmetric positive semi-definite matrix.*

Proof On one hand, note that $\boldsymbol{C}$ is symmetric since the functions $\{\psi_k\}_{1\le k\le n}$ are real-valued. Also, for any $\boldsymbol{x}\in\mathbb{R}^n$,

$$\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}=\sum_{k=1}^n\sum_{l=1}^n x_k x_l(\psi_k,\psi_l)_0=\left\|\sum_{k=1}^n x_k\psi_k\right\|_0^2\ge 0.$$

Given that the functions $\{\psi_k\}_{1\le k\le n}$ are linearly independent, this quantity is zero if and only if $\boldsymbol{x}=\boldsymbol{0}$. Hence, $\boldsymbol{C}$ is positive definite.
On the other hand, $\boldsymbol{R}$ is by definition symmetric. And, for any $\boldsymbol{x}\in\mathbb{R}^n$,

$$\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}=\left(\sum_{k=1}^n x_k\nabla_{\mathscr{M}}\psi_k,\sum_{l=1}^n x_l\nabla_{\mathscr{M}}\psi_l\right)_0=\left\|\sum_{k=1}^n x_k\nabla_{\mathscr{M}}\psi_k\right\|_0^2\ge 0.$$

Hence $\boldsymbol{R}$ is positive semi-definite.

Proposition SM2.1. *The definition of $\gamma(-\Delta_n)$ in* (3.3) *does not depend on the choice of orthonormal basis $\{e_k^{(n)}\}_{1\le k\le n}$ of eigenfunctions of $-\Delta_n$ satisfying for any $k\in[\![1,n]\!]$, $-\Delta_n e_k^{(n)}=\lambda_k^{(n)}e_k^{(n)}$.*

Proof Let $\{e_k^{(n)}\}_{1\le k\le n}$ and $\{\tilde e_k^{(n)}\}_{1\le k\le n}$ denote two orthonormal bases of V_n such that for any $k\in[\![1,n]\!]$, $-\Delta_n e_k^{(n)}=\lambda_k^{(n)}e_k^{(n)}$ and $-\Delta_n\tilde e_k^{(n)}=\lambda_k^{(n)}\tilde e_k^{(n)}$. Assume that $\gamma(-\Delta_n)$ is defined by (3.3).

Let $\boldsymbol{A}$ be the change-of-basis matrix between $\{e_k^{(n)}\}_{1\le k\le n}$ and $\{\tilde e_k^{(n)}\}_{1\le k\le n}$, i.e., for any $k\in[\![1,n]\!]$, $e_k^{(n)}=\sum_{l=1}^n A_{kl}\tilde e_l^{(n)}$.

Note that, since $\{\tilde e_k^{(n)}\}_{1\le k\le n}$ is orthonormal, we have for any $k,k'\in[\![1,n]\!]$,

$$(e_k^{(n)},e_{k'}^{(n)})_0=\sum_{l,l'=1}^n A_{kl}A_{k'l'}(\tilde e_l^{(n)},\tilde e_{l'}^{(n)})_0=\sum_{l=1}^n A_{kl}A_{k'l}=[\boldsymbol{A}\boldsymbol{A}^T]_{kk'}.$$

Therefore, since $\{e_k^{(n)}\}_{1\le k\le n}$ is also orthonormal, we have $\boldsymbol{A}\boldsymbol{A}^T=\boldsymbol{I}_n=\boldsymbol{A}^T\boldsymbol{A}$.

Then, recall that $\{e_k^{(n)}\}_{1\le k\le n}$ and $\{\tilde e_k^{(n)}\}_{1\le k\le n}$ are eigenfunctions of $-\Delta_n$. Hence, for any $k\in[\![1,n]\!]$,

$$-\Delta_n e_k^{(n)}=\lambda_k^{(n)}e_k^{(n)}=\sum_{l=1}^n\lambda_k^{(n)}A_{kl}\tilde e_l^{(n)},$$

and, by linearity of $-\Delta_n$,

$$-\Delta_n e_k^{(n)}=\sum_{l=1}^n A_{kl}(-\Delta_n\tilde e_l^{(n)})=\sum_{l=1}^n\lambda_l^{(n)}A_{kl}\tilde e_l^{(n)}.$$

Consequently, by identification of both formulas, for any $k,l\in[\![1,n]\!]$,

$$\lambda_k^{(n)}A_{kl}=\lambda_l^{(n)}A_{kl}.$$

A proof by contradiction then gives that for any $k,l\in[\![1,n]\!]$, the following also holds:

$$\gamma(\lambda_k^{(n)})A_{kl}=\gamma(\lambda_l^{(n)})A_{kl},$$

and therefore,

$$\gamma(\boldsymbol{\Lambda})\boldsymbol{A}=\boldsymbol{A}\gamma(\boldsymbol{\Lambda}),\quad\text{where}\quad\gamma(\boldsymbol{\Lambda}):=\mathrm{Diag}\left(\gamma(\lambda_1^{(n)}),\dots,\gamma(\lambda_n^{(n)})\right).$$

Finally, note that, by definition of $\gamma(-\Delta_n)$, we have for every $\varphi\in V_n$,

$$\begin{aligned}\gamma(-\Delta_n)\varphi&=\sum_{k=1}^n\gamma(\lambda_k^{(n)})\big(\varphi,e_k^{(n)}\big)_0\,e_k^{(n)}=\sum_{k,l,l'=1}^n\gamma(\lambda_k^{(n)})A_{kl}A_{kl'}\big(\varphi,\tilde e_l^{(n)}\big)_0\tilde e_{l'}^{(n)}\\&=\sum_{l,l'=1}^n[\boldsymbol{A}^T\gamma(\boldsymbol{\Lambda})\boldsymbol{A}]_{ll'}\big(\varphi,\tilde e_l^{(n)}\big)_0\tilde e_{l'}^{(n)},\end{aligned}$$

and since we proved that $\gamma(\boldsymbol{\Lambda})\boldsymbol{A} = \boldsymbol{A}\gamma(\boldsymbol{\Lambda})$,

$$\begin{aligned}\gamma(-\Delta_n)\varphi &= \sum_{l,l'=1}^{n} [\boldsymbol{A}^T\boldsymbol{A}\gamma(\boldsymbol{\Lambda})]_{ll'}\left(\varphi,\tilde{e}_l^{(n)}\right)_0\tilde{e}_{l'}^{(n)} = \sum_{l,l'=1}^{n} [\boldsymbol{I}_n\gamma(\boldsymbol{\Lambda})]_{ll'}\left(\varphi,\tilde{e}_l^{(n)}\right)_0\tilde{e}_{l'}^{(n)} \\ &= \sum_{l=1}^{n}\gamma(\lambda_l^{(n)})\left(\varphi,\tilde{e}_l^{(n)}\right)_0\tilde{e}_l^{(n)},\end{aligned}$$

which proves the result.

SM3 Proof of Theorem 5.7 of the main article

We know provide a proof of Theorem 5.7 of the main article, which we first recall.

Theorem. Let Assumptions 2.4 and 5.1 be satisfied. Then, there exists some $N_2 \in \mathbb{N}$ such that for any $n > N_2$, the covariance error between the random field $\mathscr{Z}$ and its discretization $\mathscr{Z}_n$ satisfies, for any $\theta, \varphi \in H$,

$$\begin{aligned}&\left|\mathrm{Cov}\left((\mathscr{Z},\theta)_0,(\mathscr{Z},\varphi)_0\right) - \mathrm{Cov}\left((\mathscr{Z}_n,\theta)_0,(\mathscr{Z}_n,\varphi)_0\right)\right| \\ &\qquad\lesssim \begin{cases} n^{-\min\{s;\,r;\,(2\alpha\beta-1)\}}\log n & \text{if } (2\alpha\beta-1)=s, \\ n^{-\min\{s;\,r;\,(2\alpha\beta-1)\}}\log n & \text{if } (2\alpha\beta-1)=r \text{ and } q>1, \\ n^{-\min\{s;\,r;\,(2\alpha\beta-1)\}} & \text{else.}\end{cases}\end{aligned}$$

Proof Let $n > \max\{M_0; N_0\}$, $\theta, \varphi \in H$ and let $R(\theta,\varphi)$ be defined by

$$\begin{aligned}R(\theta,\varphi) &= \left|\mathrm{Cov}\left((\mathscr{Z},\theta)_0,(\mathscr{Z},\varphi)_0\right) - \mathrm{Cov}\left((\mathscr{Z}_n,\theta)_0,(\mathscr{Z}_n,\varphi)_0\right)\right| \\ &= \left|\mathbb{E}\left[(\mathscr{Z},\theta)_0(\mathscr{Z},\varphi)_0\right] - \mathbb{E}\left[(\mathscr{Z}_n,\theta)_0(\mathscr{Z}_n,\varphi)_0\right]\right|,\end{aligned}$$

where the last equality follows from the fact that $\mathscr{Z}$ and $\mathscr{Z}_n$ are centered.

To prove the error estimate of this theorem, we proceed in the same way as in Theorem 5.6 by splitting

$$\begin{aligned}R(\theta,\varphi) &\le \left|\mathbb{E}\left[(\mathscr{Z},\theta)_0(\mathscr{Z},\varphi)_0\right] - \mathbb{E}[(\mathscr{Z}^{(n)},\theta)_0(\mathscr{Z}^{(n)},\varphi)_0]\right| \\ &\quad + \left|\mathbb{E}[(\mathscr{Z}^{(n)},\theta)_0(\mathscr{Z}^{(n)},\varphi)_0] - \mathbb{E}\left[(\mathscr{Z}_n,\theta)_0(\mathscr{Z}_n,\varphi)_0\right]\right| \\ &= R_T(\theta,\varphi) + R_D(\theta,\varphi),\end{aligned}$$

where $\mathscr{Z}^{(n)}$ denotes the truncation of $\mathscr{Z}$ after n terms.

Truncation error term $R_T(\theta,\varphi)$: Note that

$$\mathbb{E}\left[(\mathscr{Z},\theta)_0(\mathscr{Z},\varphi)_0\right] = \mathbb{E}\left[\sum_{k,l\in\mathbb{N}}\gamma(\lambda_k)\gamma(\lambda_l)W_kW_l(e_k,\theta)_0(e_l,\varphi)_0\right] = \sum_{k\in\mathbb{N}}\gamma(\lambda_k)^2(e_k,\theta)_0(e_k,\varphi)_0$$

which gives

$$R_T(\theta,\varphi) \le \sum_{k>n}\gamma(\lambda_k)^2|(e_k,\theta)_0(e_k,\varphi)_0|.$$

Using the Cauchy–Schwartz inequality on the terms $(e_k,\theta)_0$ and $(e_k,\varphi)_0$, the orthonormality of $\{e_k\}_{k\in\mathbb{N}}$, and Definition 2.3, we obtain

$$R_T(\theta,\varphi) \le \|\theta\|_0\|\varphi\|_0\sum_{k>n}\gamma(\lambda_k)^2 \lesssim \|\theta\|_0\|\varphi\|_0\sum_{k>n}\lambda_k^{-2\beta}.$$

Finally, Proposition 2.1 yields

$$R_T(\theta,\varphi) \lesssim \|\theta\|_0\|\varphi\|_0\sum_{k>n}k^{-2\alpha\beta} \lesssim \|\theta\|_0\|\varphi\|_0\, n^{-(2\alpha\beta-1)}.$$

Discretization error $R_D(\theta,\varphi)$: From the triangle inequality,

$$\begin{aligned}R_D(\theta,\varphi) &\le \left|\mathbb{E}[(\mathscr{Z}^{(n)},\theta)_0(\mathscr{Z}^{(n)},\varphi)_0] - \mathbb{E}[(\widetilde{\mathscr{Z}}^{(n)},\theta)_0(\widetilde{\mathscr{Z}}^{(n)},\varphi)_0]\right| \\ &\quad + \left|\mathbb{E}[(\widetilde{\mathscr{Z}}^{(n)},\theta)_0(\widetilde{\mathscr{Z}}^{(n)},\varphi)_0] - \mathbb{E}\left[(\mathscr{Z}_n,\theta)_0(\mathscr{Z}_n,\varphi)_0\right]\right| \\ &= R_D^{(1)}(\theta,\varphi) + R_D^{(2)}(\theta,\varphi),\end{aligned}$$

where $\widetilde{\mathscr{Z}}^{(n)}$ is defined as

$$\widetilde{\mathscr{Z}}^{(n)} = \sum_{k=1}^{n} \gamma(\lambda_k) W_k e_k^{(n)}.$$

The first term can be bounded by

$$\begin{aligned} &R_D^{(1)}(\theta,\varphi) \\ &\leq \sum_{k=1}^{n} \gamma(\lambda_k)^2 \big|(e_k,\theta)_0(e_k,\varphi)_0 - (e_k^{(n)},\theta)_0(e_k^{(n)},\varphi)_0\big| \\ &= \sum_{k=1}^{n} \gamma(\lambda_k)^2 \big|(e_k - e_k^{(n)},\theta)_0(e_k,\varphi)_0 + (e_k,\theta)_0(e_k - e_k^{(n)},\varphi)_0 - (e_k - e_k^{(n)},\theta)_0(e_k - e_k^{(n)},\varphi)_0\big| \end{aligned}$$

and satisfies further by the triangle and the Cauchy–Schwartz inequality:

$$R_D^{(1)}(\theta,\varphi) \lesssim \|\theta\|_0\|\varphi\|_0 \left(\sum_{k=1}^{n} \gamma(\lambda_k)^2 \|e_k - e_k^{(n)}\|_0 + \sum_{k=1}^{n} \gamma(\lambda_k)^2 \|e_k - e_k^{(n)}\|_0^2\right).$$

Using Proposition 2.1, Definition 2.3 Assumption 5.1, and the fact that $\alpha q \leq 2s$, the first sum can be bounded by

$$\begin{aligned} \sum_{k=1}^{n} \gamma(\lambda_k)^2 \|e_k - e_k^{(n)}\|_0 &\lesssim n^{-s}\left(\sum_{k=1}^{n} k^{-2\alpha\beta+\alpha q/2}\right) \lesssim n^{-s}\left(\sum_{k=1}^{n} k^{s-2\alpha\beta}\right) \\ &\lesssim \begin{cases} n^{-s}\log n & \text{if } 2\alpha\beta - 1 = s, \\ n^{-\min\{s,2\alpha\beta-1\}} & \text{else.} \end{cases} \end{aligned}$$

Similarly, we prove that

$$\sum_{k=1}^{n} \gamma(\lambda_k)^2 \|e_k - e_k^{(n)}\|_0^2 \lesssim \begin{cases} n^{-2s}\log n & \text{if } 2\alpha\beta - 1 = 2s, \\ n^{-\min\{2s,2\alpha\beta-1\}} & \text{else.} \end{cases}$$

We conclude then by considering the term with the slowest convergence that

$$R_D^{(1)}(\theta,\varphi) \lesssim \|\theta\|_0\|\varphi\|_0 \begin{cases} n^{-s}\log n & \text{if } 2\alpha\beta - 1 = s, \\ n^{-\min\{s,2\alpha\beta-1\}} & \text{else.} \end{cases}$$

For $R_D^{(2)}(\theta,\varphi)$ we observe that

$$\begin{aligned} R_D^{(2)}(\theta,\varphi) &\leq \sum_{k=1}^{n} \big|\gamma(\lambda_k)^2 - \gamma(\lambda_k^{(n)})^2\big| \big|(e_k^{(n)},\theta)_0(e_k^{(n)},\varphi)_0\big| \\ &\leq \|\theta\|_0\|\varphi\|_0 \sum_{k=1}^{n} \big|\gamma(\lambda_k)^2 - \gamma(\lambda_k^{(n)})^2\big|, \end{aligned}$$

where we used that $\{e_k^{(n)}\}_{1\leq k\leq n}$ is orthonormal. Applying the mean value theorem we get, for any $k \in [\![1,n]\!]$,

$$\begin{aligned} |\gamma(\lambda_k)^2 - \gamma(\lambda_k^{(n)})^2| &= |\gamma(\lambda_k) + \gamma(\lambda_k^{(n)})||\gamma(\lambda_k) - \gamma(\lambda_k^{(n)})| \\ &\leq |\gamma(\lambda_k) + \gamma(\lambda_k^{(n)})||\lambda_k^{(n)} - \lambda_k| \sup_{\theta\in(0,1)} |\gamma'(\theta\lambda_k^{(n)} + (1-\theta)\lambda_k)|. \end{aligned}$$

Using the same arguments as the ones used in the proof for Theorem 5.6, we can find some $N_1 \in \mathbb{N}$ such that for any $n > N_1$, and any $k \in [\![N_1,n]\!]$, $\min\{\lambda_k;\, \lambda_k^{(n)}\} \geq l_\lambda c_\lambda k^\alpha \geq L_\gamma$, where L_γ is defined in Definition 2.3. For any such k, we then have, still according to Definition 2.3, $|\gamma(\lambda_k) + \gamma(\lambda_k^{(n)})| \lesssim (\min\{\lambda_k;\, \lambda_k^{(n)}\})^{-\beta} \lesssim k^{-\alpha\beta}$ and for any $\theta \in (0,1)$,

$$|\gamma'(\theta\lambda_k^{(n)} + (1-\theta)\lambda_k)| \lesssim \left(\min\{\lambda_k;\, \lambda_k^{(n)}\}\right)^{-(1+\beta)} \lesssim k^{-\alpha(1+\beta)},$$

which in turn gives

$$|\gamma(\lambda_k)^2 - \gamma(\lambda_k^{(n)})^2| \lesssim |\lambda_k^{(n)} - \lambda_k| k^{-(1+2\beta)}.$$

And if $k \in [\![M_0+1, N_1-1]\!]$, we can simply take $|\gamma(\lambda_k)^2 - \gamma(\lambda_k^{(n)})^2| \lesssim |\lambda_k^{(n)} - \lambda_k|$ as the other terms can be bounded by constants independent of n. In conclusion, using Assumption 5.1 and Proposition 2.1, we get

$$R_D^{(2)}(\theta,\varphi) \lesssim \|\theta\|_0\|\varphi\|_0 \left(\sum_{k=M_0+1}^{N_1-1} |\lambda_k^{(n)} - \lambda_k| + \sum_{k=N_1}^{n} |\lambda_k^{(n)} - \lambda_k| k^{-(1+2\beta)}\right),$$
$$\lesssim \|\theta\|_0\|\varphi\|_0 n^{-r}\left(1 + \sum_{k=N_1}^{n} k^{\alpha(q-(1+2\beta))}\right),$$

If $q=1$, since $2\alpha\beta > 1$, we get $R_D^{(2)}(\theta,\varphi) \lesssim \|\theta\|_0\|\varphi\|_0 n^{-r}$. If now $q>1$, since $\alpha(q-1) \le r$,

$$R_D^{(2)}(\theta,\varphi) \lesssim \|\theta\|_0\|\varphi\|_0 n^{-r}\left(1 + \sum_{k=N_1}^{n} k^{r-2\alpha\beta}\right) \lesssim \|\theta\|_0\|\varphi\|_0 \begin{cases} n^{-r}\log n & \text{if } 2\alpha\beta - 1 = r, \\ n^{-\min\{r;\, 2\alpha\beta-1\}} & \text{else.} \end{cases}$$

Total error : Combining the three error terms $R_T(\theta,\varphi)$, $R_D^{(1)}(\theta,\varphi)$, and $R_D^{(2)}(\theta,\varphi)$, and keeping the terms with the slowest convergence then gives the claim for the total error.

SM4 Sampling a Galerkin–Chebyshev approximation of a random field

In Section 4 of the main article, we present an approach to generate samples of the Galerkin–Chebyshev approximation of a random field defined on a Riemannian manifold. We provide here additional implementation details and pseudo-code for this approach.

SM4.1 An upper-bound for the eigenvalues of the stiffness matrix

In order to define the polynomial $P_{\gamma,K}$ used to approximate the power spectral density γ defining the random field, one needs to provide an upper-bound $\lambda_{\max}$ of the largest eigenvalue of the stiffness matrix $\boldsymbol{S}$. Let us denote by $\lambda_n^{(n)}$ this maximal eigenvalue. Recall from Theorem 3.1 and Corollary 3.2 of the main article that the eigenvalues of $\boldsymbol{S}$ are exactly those of the stencil $(\boldsymbol{R},\boldsymbol{C})$. As such, they can be upper-bounded by the maximum of the associated Rayleigh quotient, thus giving

$$\lambda_n^{(n)} \le \max_{\boldsymbol{x}\in\mathbb{R}^n, \|\boldsymbol{x}\|_2=1} \frac{\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}} \le \frac{\max_{\boldsymbol{x}\in\mathbb{R}^n, \|\boldsymbol{x}\|_2=1} \boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}}{\min_{\boldsymbol{x}\in\mathbb{R}^n, \|\boldsymbol{x}\|_2=1} \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}}.$$

We recognize on the right-hand side of the last inequality the ratio between two Rayleigh quotients. Hence, we can conclude that an upper-bound $\lambda_{\max}$ of the eigenvalues of $\boldsymbol{S}$ is obtained by taking the ratio

$$\lambda_{\max} = \frac{\lambda_{\max}(\boldsymbol{R})}{\lambda_{\min}(\boldsymbol{C})},$$

where $\lambda_{\max}(\boldsymbol{R})$ (resp. $\lambda_{\min}(\boldsymbol{C})$) is an upper-bound (resp. lower-bound) of the eigenvalues of the stiffness matrix $\boldsymbol{R}$ (resp. mass matrix $\boldsymbol{C}$). On the one hand, $\lambda_{\max}(\boldsymbol{R})$ can be obtained using the Gershgorin circle theorem, thus requiring only to sum the entries of $\boldsymbol{R}$ row-wise (or column-wise) to get the bound. On the other hand, $\lambda_{\min}(\boldsymbol{C})$ can be taken to be the inverse of an upper-bound $\lambda_{\max}(\boldsymbol{C}^{-1})$ of the eigenvalues of the inverse of $\boldsymbol{C}$. This upper-bound can in turn be obtained using a power iteration scheme which would require to solve linear systems defined by $\boldsymbol{C}$.

SM4.2 Workflow and pseudo-code

We now present the workflow used to generate samples of the Galerkin–Chebyshev approximation and some pseudo-code associated with the different steps of this workflow.

The overall workflow is presented in Workflow 1. The weights of the Galerkin–Chebyshev approximation can be sampled using Algorithm 1. This algorithm relies on the following two sub-algorithms:

- an algorithm $\boldsymbol{\Pi_S}$ taking as input a vector $\boldsymbol{x}$ and returning the product $\boldsymbol{\Pi_S}(\boldsymbol{x}) = \boldsymbol{S}\boldsymbol{x}$;
- an algorithm $\boldsymbol{\Pi}_{(\sqrt{\boldsymbol{C}})^{-T}}$ taking as input a vector $\boldsymbol{x}$ and returning the solution $\boldsymbol{y} = \boldsymbol{\Pi}_{(\sqrt{\boldsymbol{C}})^{-T}}(\boldsymbol{x})$ to the linear system $\left(\sqrt{\boldsymbol{C}}\right)^T\boldsymbol{y} = \boldsymbol{x}$.

In the most general case, we proposed implementations for these two algorithms that are recalled in Algorithms 2 and 3.

Workflow 1: Generate a sample of the discretized field $\widehat{\mathscr{Z}_{n,K}}$ in (6.1)

1. Compute the Cholesky factorization $\boldsymbol{C} = \boldsymbol{L}\boldsymbol{L}^T$ of the mass matrix $\boldsymbol{C}$;
2. Compute
   - an upper-bound $\lambda_{\max}(\boldsymbol{R})$ of the eigenvalues of $\boldsymbol{R}$ (using Gershgorin circle theorem),
   - an upper-bound $\lambda_{\max}(\boldsymbol{C}^{-1})$ of the eigenvalues of the inverse of $\boldsymbol{C}$ (using the Cholesky factors of $\boldsymbol{C}$ to solve the linear systems in a power iteration scheme);
3. Run Algorithm 1 using the implementations of $\boldsymbol{\Pi_S}$, and $\boldsymbol{\Pi}_{(\sqrt{\boldsymbol{C}})^{-T}}$ given in Algorithms 2 and 3, and taking $\lambda_{\max} = \lambda_{\max}(\boldsymbol{R})\lambda_{\max}(\boldsymbol{C}^{-1})$.

Note that in the two particular cases presented in Section 6.2 of the main article, the first step of Workflow 1 is no longer required, and the second step can be performed without requiring a power iteration scheme. Besides, the implementations of $\boldsymbol{\Pi_S}$ and $\boldsymbol{\Pi}_{(\sqrt{\boldsymbol{C}})^{-T}}$ can be replaced by single products with sparse or diagonal matrices. This speeds up greatly the time needed to generate samples. To illustrate this, we gave in Figure 7.2 of the main article the computational time (and corresponding orders of polynomial approximation) needed to generate the samples used in the numerical experiment presented in Section 7.1.2 of the main article.

Algorithm 1 Compute a sample of the discretized random field $\widehat{\mathscr{Z}_{n,K}}$ in (6.1)

Input:
Algorithms $\boldsymbol{\Pi_S}$ and $\boldsymbol{\Pi}_{(\sqrt{\boldsymbol{C}})^{-T}}$ to compute products by $\boldsymbol{S}$ and $\left(\sqrt{\boldsymbol{C}}\right)^{-T}$,
Estimate $\lambda_{\max}$ of the largest eigenvalue of $\boldsymbol{S}$,
Order of polynomial approximation K.
Output: Gaussian random weights $\widehat{\boldsymbol{Z}}$ of the discretized random field (6.1).

1: Set $\widehat{\boldsymbol{X}} \leftarrow \boldsymbol{0}$;
2: Sample $\boldsymbol{W} \sim \mathscr{N}(\boldsymbol{0}, \boldsymbol{I})$;
3: Compute the first $(K+1)$ coefficients $c_0, \dots, c_K$ of the Chebyshev series (4.5) of the function $t \in [-1,1] \mapsto \gamma\left(\frac{\lambda_{\max}}{2}(1+t)\right)$ using the FFT algorithm;
4: Set $\boldsymbol{y}^{(-2)} \leftarrow \boldsymbol{W}$;
5: Update $\widehat{\boldsymbol{X}} \leftarrow \widehat{\boldsymbol{X}} + c_0\boldsymbol{y}^{(-2)}$;
6: **if** $(K=0)$ **then**
7: Jump to Line 20;
8: **end if**
9: Set $\boldsymbol{y}^{(-1)} \leftarrow (2/\lambda_{\max})\boldsymbol{\Pi_S}(\boldsymbol{W}) - \boldsymbol{W}$;
10: Update $\widehat{\boldsymbol{X}} \leftarrow \widehat{\boldsymbol{X}} + c_1\boldsymbol{y}^{(-1)}$;
11: **if** $(K=1)$ **then**
12: Jump to Line 20;
13: **end if**
14: **for** $k=2$ **to** K **do**
15: Compute $\boldsymbol{y} = (4/\lambda_{\max})\boldsymbol{\Pi_S}(\boldsymbol{y}^{(-1)}) - 2\boldsymbol{y}^{(-1)} - \boldsymbol{y}^{(-2)}$;
16: Update $\widehat{\boldsymbol{X}} \leftarrow \widehat{\boldsymbol{X}} + c_k\boldsymbol{y}$;
17: Set $\boldsymbol{y}^{(-2)} \leftarrow \boldsymbol{y}^{(-1)}$;
18: Set $\boldsymbol{y}^{(-1)} \leftarrow \boldsymbol{y}$;
19: **end for**
20: Compute $\widehat{\boldsymbol{Z}} = \boldsymbol{\Pi}_{(\sqrt{\boldsymbol{C}})^{-T}}\left(\widehat{\boldsymbol{X}}\right)$;
21: **return** $\widehat{\boldsymbol{Z}}$.

SM4.3 Numerical experiment: Polynomial approximation error

The polynomial approximation error refers to the error due to the fact that the power spectral density γ defining the random field is in practice approximated by a polynomial. This polynomial is defined as a truncated Chebyshev

Algorithm 2 Implementation of $\boldsymbol{\Pi}_{(\sqrt{\boldsymbol{C}})^{-T}}$

Depends on: Cholesky factor $\boldsymbol{L}^T$ of $\boldsymbol{C}$.
Input: Vector $\boldsymbol{x}$.
Output: Vector $\boldsymbol{y}$ solution of $\left(\sqrt{\boldsymbol{C}}\right)^T \boldsymbol{y} = \boldsymbol{x}$.

Solve $\boldsymbol{L}^T \boldsymbol{y} = \boldsymbol{x}$ by backward substitution;
return $\boldsymbol{y}$.

Algorithm 3 Implementation of $\boldsymbol{\Pi}_{\boldsymbol{S}}$

Depends on:
Cholesky factor $\boldsymbol{L}$ of $\boldsymbol{C}$,
Stiffness matrix $\boldsymbol{R}$.
Input: Vector $\boldsymbol{x}$.
Output: Vector $\boldsymbol{y} = \boldsymbol{S}\boldsymbol{x}$.

Solve $\boldsymbol{L}^T \boldsymbol{u} = \boldsymbol{x}$ by backward substitution;
Update $\boldsymbol{u} \leftarrow \boldsymbol{R}\boldsymbol{u}$;
Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{u}$ by forward substitution;
return $\boldsymbol{y}$.

series of order K chosen by the practitioner. Theorem 5.8 of the main article ensures that this error converges to 0 as $K \to \infty$.

To get a feeling of how fast this convergence can be, we consider the same setting as the one described for the truncation error study (cf. Section 7.1.1 of the main article). Let n be a fixed truncation order. We compute for various choices of K, the approximation error

$$\|\mathscr{Z}_n - \widehat{\mathscr{Z}_{n,K}}\|^2_{L^2(\Omega;H)}$$

between the truncated expansion $\mathscr{Z}_n = \mathscr{Z}^{(n)}$ and its approximation $\widehat{\mathscr{Z}_{n,K}}$ obtained by replacing γ by a Chebyshev series of order K. Following the proof of Theorem 5.8, this error can in particular be computed without requiring any simulations since it has a closed form given by

$$\|\mathscr{Z}_n - \widehat{\mathscr{Z}_{n,K}}\|^2_{L^2(\Omega;H)} = \mathbb{E}\left[\|\mathscr{Z}_n - \widehat{\mathscr{Z}_{n,K}}\|_0^2\right] = \sum_{k=1}^{n} (\gamma(\lambda_k) - P_{\gamma,K}(\lambda_k))^2.$$

These approximation errors are computed for four scenarios corresponding to truncation orders $n \in \{1024, 10^4, 100489, 10^6\}$ and for the power spectral density given by the parameters $\nu = 1$ and $a = \pi/6$. The results are presented in Figure SM4.1 and show that even for large truncation orders, a Chebyshev series of order of around 1000 is enough to reach very small errors. Moreover, considering very large orders of approximation results in the error stagnating at the machine precision level.

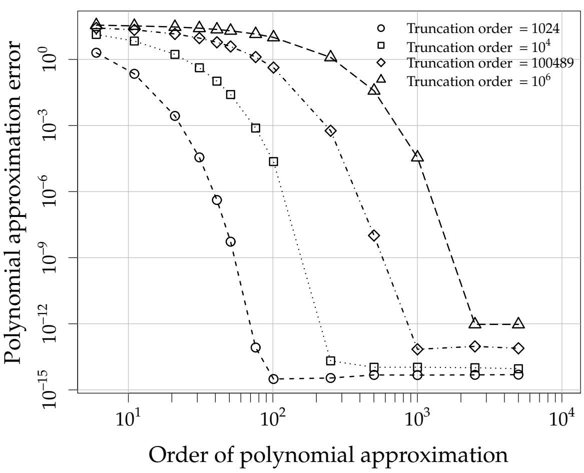

Fig. SM4.1: Polynomial approximation error $\|\mathscr{Z}_n - \widehat{\mathscr{Z}_{n,K}}\|^2_{L^2(\Omega;H)}$ on the sphere for the Matérn power spectral density with $\nu = 1$ and $a = \pi/6$.