Supplement 2: Statistical Transformation Method of Computing Expected Costs and Adaptation for use in Black-Scholes

Here we illustrate our method of computing expected cost functions and how we transform variance in order to use the Black-Scholes equation with non log-normal distributions. We demonstrate these techniques using the Merced River cumulative discharge time series as an example. We make the following assumptions:

1. the river system is undammed,
2. the time series has no trend,
3. the annual water that farmers need to withdraw from the river is 650 million cubic meters, and
4. any shortfall in this amount is made up with ground water purchased from neighboring districts at $0.73 per cubic meter.

Because the discharge series has actually been increasing slightly since 1917, we de-trended the series to simplify the analysis. We did this by fitting a line to the series, computing the residuals, and adding to each residual 600 million cubic meters, the value of the fitted line for the year 2014. The resultant de-trended series is shown in Figure 6 in the main paper.

Using this de-trended series, we canvassed the results to see in which years there was a shortfall of water (< 650 million cubic meters/year), how much the shortfall would cost to make it right (at $\$0.73/m^3$ of water), and the average cost for all 98 years of the de-trended record, given the shortfall for each year; if there was no shortfall, the cost was zero. The average was $94,852,747, which becomes the expected value for a *supply-to-need ratio* of 600 million/650 million, or 0.92. This becomes one point on an expected cost curve (Fig. S2-1).

This same point on the curve can be mimicked by the value of a put option (*P*) with a strike price (*K*) of 650 million cubic meters/year valued using the distribution of the de-trended series. To see this, consider the general option valuation approach for the expected value of a put at maturity as given by:

$$P_T = \int_0^\infty max\,[0,(K-S)]p(S)ds = \int_0^K (K-S)p(S)ds \qquad \text{[Equation S2-1]}$$

We then applied Equation S2-1 to the detrended Merced River socio-climate time series. The value *p(S)* is the probability of the realization of *S* in the series, i.e., the probability of a given *S*. Note that this method does not assume any particular form of the distribution (such as uniform, binomial, normal, log-normal, exponential, etc.) but instead allows the socio-climate time series to define the distribution.

For example, when considering the de-trended series, each observation (*S*) in the de-trended series provides information regarding the value at maturity. For realizations of *S* less than *K*, *K-S* > 0 and is equivalent to a shortfall in water. The probability of this shortfall is mimicked by sampling (with replacement) from the de-trended series. By taking each observation, evaluating it at *K-S* for values below *K*, and zero otherwise, and weighting each of these values equally, we can estimate the integral in Equation S2-1. (The equal-weighting of these observations captures the probability at each *S*; if there are multiple observations at a particular *S*, that point would get more weight, and observations close to each other in value create a higher density around those values of *S*.)

It is important to note that at this point we do not have an option *per se*, even though the equation above could be, and is used, in general option valuation approaches. As described below, for options, the *max* [*0*, *(K-S)*] term arises due to the *choice* of whether or not to exercise the option, while in this case, the *max* operator arises depending on whether or not nature provides sufficient bounty in the form of run-off so that there is no need to supplement water through purchase. Therefore, while the payoff function of the water supply problem mimics a payoff function of an option, in fact there is no strategic choice. The system requires a constant supply of 650 million cubic meters, and by definition there is no choice of whether or not to supply the water, and hence no option. Instead, there is an asymmetric cost structure of supplying water, which is deterministic once the state of the climate is realized. As is noted in more detail below, while this problem looks like it might be a real option problem, real options always contain a strategic choice while in this problem there is no choice at all.

To compute expected costs (mimicking put values) at other *supply-to-need ratios*, we performed a linear transformation of the original de-trended distribution (X) to create a new distribution (X'):

$$X' = a + bX \qquad \text{[Equation S2-2]}$$

This transformation is similar to the conversion of a temperature series from Celsius (C) to Fahrenheit (F) ($F = 32 + 9/5 \cdot C$). The process spreads out the data and shifts its mean, but the relative structure of the data remains the same, with all ordering and relative magnitude of spacing in the distribution preserved.

We started by adding (or subtracting) a fixed amount (a) to each value of the series, while holding $b = 1$. It was necessary to truncate to zero any values that were negative due to the subtraction (since there cannot be negative discharge). While the variance (or standard deviation) of the initial time series is not absolutely preserved during this procedure, the alteration tends to be small, and the procedure is accurate in that it removes the possibility of negative flows. For example, when we increased each annual discharge value by 50 million cubic meters (a=50,000,000; b=1), it reduced the number of years in which there is a shortfall in supply (and therefore an incurred water cost) by 5 years. By adding 50 million cubic meters to each observation in the de-trended series, we effectively created a new distribution, with a new mean (that is 50 million cubic meters higher) but the same variance. Even so, the probability for this new distribution at any given value S was now different from the previous one, and so the values and probabilities at any given S in Equation S2-1 have now changed and so Equation S2-1 needs to be re-run with the new data. Functionally, this means a new average of the value K-S for values below K, and zero otherwise. The expected cost due to this increase in discharge dropped to $74,440,921, and the *supply-to-need ratio* increased to 1.0 ({600 + 50 million}/650 million=1.0). By continuing to increment (or decrement) the initial de-trended time series (+100 million, +150 million, -50 million, -100 million, etc.) we were able create a complete new expected cost curve. We did this for *supply-to-need ratio* ratios ranging from 0.5 to 1.5.

In order to compute expected cost curves for higher or lower values of variance we then used the residuals we computed when de-trending the time series, multiplying each of these values by a factor, 0.5 and 1.5 in our case, to decrease or increase the variance. We then added back to each residual a fixed value of 600 million (the average of 98 time series de-trended values) to produce a series that has the original mean but a higher or lower variance.

Mathematically the new distribution (Y) was computed from the original:

$$Y = X - mean \qquad \text{[Equation S2-3]}$$

where the *mean* was from the original de-trended time series, and b=1. We then altered the standard deviation of this new distribution (Y), which has mean of zero, to produce a second distribution (Z):

$$Z = c \bullet Y \qquad \text{[Equation S2-4]}$$

where c = 0.5 or c = 1.5 for purposes of this paper.

Finally, we added back the mean, to create a new distribution with the same mean as the original but with a different standard deviation:

$$X' = mean + Z \qquad \text{[Equation S2-5]}$$

The new distribution, *Z,* retains the structure of the original distribution, but with an increased or decreased variance. Again we truncated at zero any negative values. Using the new, transformed time series, we then repeated the procedure described in the preceding paragraphs, producing (in our case) two additional expected cost functions, one for a standard deviation that is half the original, and another that is 1.5 times the original (see Figure 6, main text for these curves at difference values of variance). Each of these cost functions is effectively an implementation of Equation S2-1 (the general option valuation approach) for each of these newly created distributions.

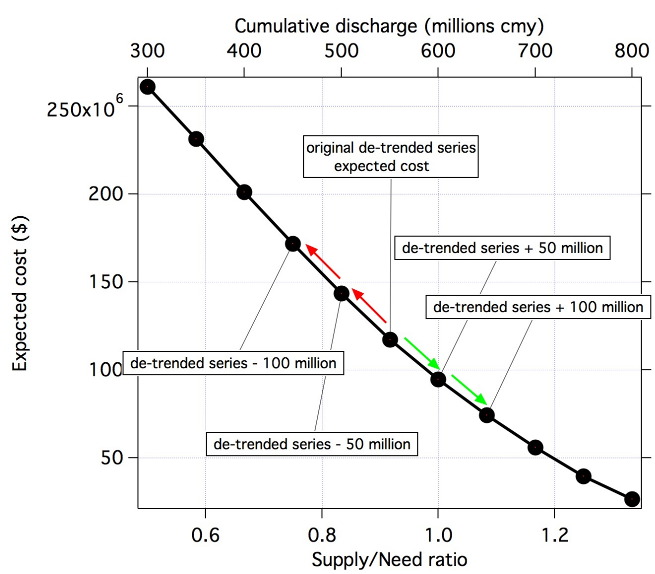

Figure S2-1: Computing expected costs by transforming the initial distribution incrementally. In order to produce cost curves for alternate values of variance, the same procedure shown here is repeated after applying Equations S2-3 through S2-5.

The previous paragraphs described estimating the costs (which mimic put values) using what is effectively a general option valuation approach. In a seminal paper, Black and Scholes (1973) created a closed form solution for valuing options for log-normal distributions. Since their results provide a closed form solution that is computationally fast and easy to use, we next determined for these socio-climatic distributions the variance values that we would need to map back into the Black-Scholes (1973) option pricing formula (called hereafter the B-S formula) in order to use it to compute expected costs. Again, while these systems are not an option *per se*, the realized cost functions mimic the asymmetric payoff functions of a put or call option. In the same way as before, we averaged all possible *ex post* values to get a single number, and the B-S formula ultimately collapses the asymmetric distribution into a single value as well – the value of the option. By appropriate parameterization and replacement, we can use the B-S formula to summarize the payoffs (or expected costs) of these asymmetric socio-climatic systems as well.

If the underlying distribution of S is log-normal, using the B-S formula will give a solution exactly equivalent to the more general option valuation approach, but since it is unlikely that socio-climate distributions are log-normal, below we demonstrate a heuristic we created that works well even if the underlying distribution is not log-normal.

The "option" at the heart of stock option is represented mathematically in the payoff function by $max[0, S\text{-}K]$ where S is the current stock price and K is the strike price. While there is not choice in these socio-climatic systems, a similar $max[0,S\text{-}K]$ arises, allowing us to use option valuation approaches such as the B-S formula (appropriately parameterized) to mimic the socio-climatic systems mathematically and therefore provide analogous results. Translated into a socio-climatic system, K becomes the threshold beyond which greater costs are incurred while utilizing the same free ecosystem service (or alternatively, beyond which the costs drop). S becomes the value of the climate metric that governs the way the system functions, and can either be the current value, or some prescribed value for which we need to know the expected costs. With S and K replaced in the B-S formula, it becomes a tool for rapidly summarizing the expected values of the asymmetric payoff function, i.e., we are using the formula by analogy to estimate expected costs.

More formally, the standard B-S formula prices the right (but not the obligation) to call a stock away from someone else at a specified time in the future at a specified price. It assumes returns on a stock are normally distributed with a mean expected return μ and a standard deviation σ. It describes the end result of the movement of stock prices over time and is parsimonious in that it only needs five inputs (S, K, $T\text{-}t$, r, and σ).

For a call option (C) the B-S formula is:

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$$
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$
$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]$$
$$= d_1 - \sigma\sqrt{T-t}$$
[Equations S2-6]

where *S, K*, σ and μ are defined as above, *T-t* is the time remaining to exercise the option, *r* is the instantaneous risk-free-rate, and *N( )* is the cumulative distribution function operator of the standard normal distribution *N*(0,1).

The price of a corresponding put option (*P*) is based on the put-call parity (Stoll, 1969):

$$P(S,t) = Ke^{-r(T-t)} - S + C(S,t)$$
$$= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S$$
[Equation S2-7]

To use the B-S formula, we need to map the parameters of our socio-climatic system onto the B-S formula parameters. As indicated above, the strike price, *K*, becomes the operational tipping point (OPT), at a supply-to-need-ratio of 1.0. This parameter reflects the business or monetary cost side of the system. Past the tipping point, costs are incurred at some rate set by commodity, transportation, or other prices (these must be specified by the user). The climate side of the system mimics the stock price and its volatility. For the input for *S*, we use the mean of the time series itself. $\sigma$is just the standard deviation of the climate time series when the series is lognormal. If it is not, then we must find the standard deviation of an equivalent log-normal distribution with a similar mean value (*S*) such that it preserves the sum of the products of the probability of the outcome times the expected losses (*S-K*) of the socio-climate distribution:

$$\int_0^K (K-S) * lnN(\mu,\sigma) * dS = \int_0^K (K-S) * SocioClimateProbability(S) * dS$$ [Eqn. S2-8]

where μ and σ are the mean and standard deviation of the underlying normal distribution such that [Eqns. S2-9]:

$$\mu = \ln\left(\frac{S}{\sqrt{1+\frac{v}{S^2}}}\right)$$

$$\sigma = \sqrt{ln\left(1+\frac{v}{S^2}\right)}$$

and

$$v = (f * standard\ deviation\ of\ socioclimate\ distribution)^2.$$

Note that the right hand side of Equations S2-1 and S2-8 are the same, i.e., both are the integral from zero to *K* of *K-S* times the probability of *S* in the socio-climate series.

The key is finding f, a factor used to adjust the standard deviation estimated from underlying socio-climate distribution so as to find a log-normal distribution where the integral between 0 and K gives the same value. If the socio-climate distribution is log-normal, then f equals 1. If not, finding the correct f can be done via a Monte Carlo simulation, by drawing 10,000 log-normally distributed random numbers using the formulas above and varying f until the average of the 10,000 random numbers and the average of the de-trended record are equal.

An alternate, faster heuristic method for finding the appropriate value of f is as follows: using the formula for μ, σ, and v above, find f such that the Black-Scholes Option Pricing formula gives the same answer for the value of a put option as the average value of the shortfall estimated from the socio-climate series (which was $94,852,747 in the case of the Merced River above). This factor can then be used to find the value of σ to be used as an input into Black-Scholes for other values of S, for example, to derive the Black-Scholes curves quickly for different possible climate means.

With this one-to-one parameter mapping we can then feed our values into any standard B-S formula model (widely available on spreadsheets and on-line) to compute the put price, or in our system, a cost function. This can be done without any reference to the transformation method described above. While theoretically the B-S formula requires a log-normal distribution, our tests (Fig. S2-2), which were applied to non-log normal distributions using the technique above, resulted in close agreements between the statistically transformed results and the B-S formula computations, though we have not tested this agreement in an exhaustive fashion.

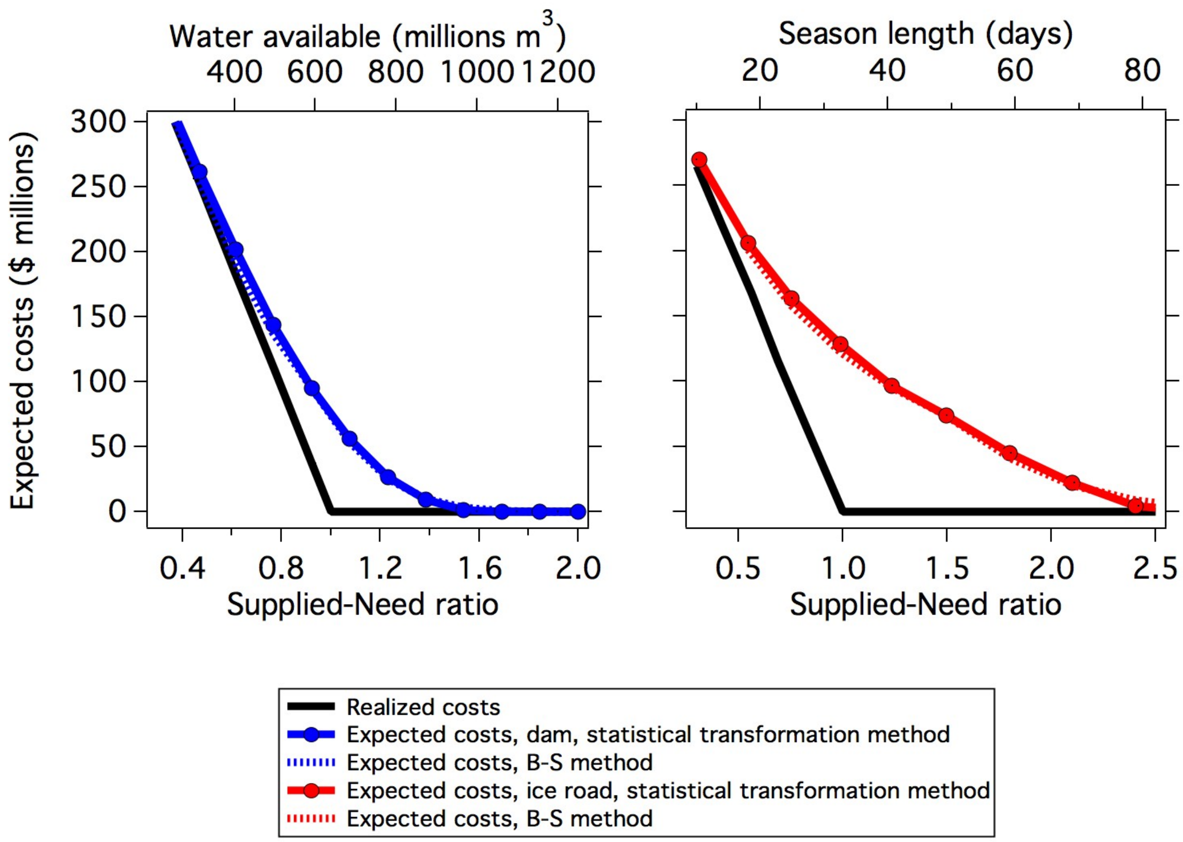

Figure S2-2: Comparison between expected costs computed using statistical transformation methods and the B-S formula (dashed lines). Results for the Merced River are in blue; for the ice road in red.

Again, while we used the B-S formula with certain modified parameters to get these values, the underlying socio-climate problem did not have any underlying choice, and was therefore not a true option. While the B-S formula was applied to a "real" as opposed to "financial" problem, this was not a strategic choice issue. The application of this technique to socio-climate problems is therefore notably different from standard real option problems, such as the choice to delay drilling in an oil field (see Chapter 22.11 in Ross et al. (2008), or the choice of whether or not to abandon a mine as in Chapter 23.4 of the same text). Without choice, these are not "options" *per se*, although the payoff structures, and therefore the mathematical valuation techniques, are similar.

The decision whether or not to keep a mine open or whether or not to build a dam is strategic, and therefore a real option. Once that choice is made, however, the cost of supplying the mine or the water to the ultimate user must be undertaken, yet has an asymmetric payoff that mimics an option without actually being an option. The functions are mathematically indistinguishable, and so option pricing methods can be used to value these payoffs.[1]

[1] In fact, in order to estimate the value of the real options where there is strategic choice, it is necessary to know the value of the transportation services or the value of the water downstream to estimate the values if the real option is undertaken or not. So to estimate the real options, the value of these asymmetric payoff functions must first be estimated.

Ultimately, it is the combination climate variability and a specific human need that induces the asymmetry in an expected cost function to mimic an option, and since it mimics an option payoff, it turns out it can be valued as if it were an option, even though it is not an option since there is no 'choice'. While Black-Scholes is only one such option valuation technique and requires certain assumptions, we show that we can relax those assumptions and by appropriate adjustments mimic reasonably closely the estimated value of the option using other methods. By showing that Black-Scholes will give a similar answer to more direct but more complex estimation methods in a system that is non-linear but without choice, we demonstrate that the system that has option-like payoffs does in fact act like different systems that are true options. It also potentially opens a vast area of financial research for use in climate studies.

References:

Ross, Stephen A., Randolph W. Westerfield, Jeffrey Jaffe, 2008, Corporate Finance (Eighth edition), McGraw-Hill/Irwin, New York, NY.