

Supplementary Material 3:

Analytical Challenges in Deriving Asymptotic Distributions for Nonlinear Unit Root Tests and the TFAF Solution: An example of ESTAR type Unit Root test

1 Introduction

Nonlinear unit root tests, particularly those based on the Exponential Smooth Transition Autoregressive (ESTAR) model, have become increasingly prevalent in econometric literature due to their ability to capture complex dynamics in economic time series (Kapetanios et al. (2003)). However, a significant challenge in these models is the analytical intractability of the stochastic integrals required to derive their asymptotic distributions. This supplementary material provides a detailed examination of this problem, focusing on the ESTAR model, and elucidates how the Triple Fractional Arctan-augmented Fourier (TFAF) method overcomes these challenges. We emphasize the necessity of the TFAF approach by contrasting it with existing methods, such as those proposed by Kapetanios et al. (2003) (KSS) and Kılıç (2011), and demonstrate its practical and theoretical superiority.

The document is structured as follows: Section 2 describes the ESTAR model and the analytical challenges in deriving its asymptotic distribution. Section 3 provides a detailed mathematical analysis of why the required integrals are intractable. Section 4 discusses the limitations of existing approaches like KSS and Kılıç's inf-t test. Section 5 introduces the TFAF method and explains how it resolves the intractability issue. Section 6 presents empirical validation of the TFAF-based unrestricted ESTAR test, and Section 7 summarizes why the TFAF method is essential. Finally, Section 8 concludes with an important note.

2 The ESTAR Model and Asymptotic Distribution Problem

The ESTAR model is widely used to capture nonlinear dynamics in economic time series, such as real exchange rates, by allowing smooth transitions between regimes. It is defined as:

$$\Delta y_t = \theta y_{t-1} \cdot (1 - \exp \{-\gamma(y_{t-1} - c)^2\}) + \varepsilon_t, \quad (1)$$

where $\varepsilon_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2)$, $\gamma > 0$ is the transition speed parameter, and $c \in \mathbb{R}$ is the threshold parameter Kapetanios et al. (2003). Under the null hypothesis ($H_0 : \theta = 0$), the process reduces to a random walk:

$$\Delta y_t = \varepsilon_t \quad \Rightarrow \quad y_t = y_{t-1} + \varepsilon_t. \quad (2)$$

The transition function is:

$$G(y_{t-1}; \gamma, c) = 1 - \exp \{ -\gamma(y_{t-1} - c)^2 \}, \quad (3)$$

which governs the nonlinear adjustment of the process, transitioning smoothly between regimes based on the deviation of y_{t-1} from the threshold c .

The primary objective in nonlinear unit root testing is to derive the asymptotic distribution of the test statistic $t_\theta^{\text{ESTAR}} = \frac{\hat{\theta}}{\text{se}(\hat{\theta})}$. Under H_0 , the general asymptotic distribution takes the form:

$$t_\theta^{\text{ESTAR}} \xrightarrow{d} \frac{\int_0^1 W(r) \cdot G(W(r); \gamma, c) dW(r)}{\left(\int_0^1 [W(r) \cdot G(W(r); \gamma, c)]^2 dr \right)^{1/2}}, \quad (4)$$

where $W(r)$ is a standard Brownian motion on $[0, 1]$, and $G(W(r); \gamma, c) = 1 - \exp \{ -\gamma(W(r) - c)^2 \}$. This distribution is non-standard because $G(\cdot)$ is a nonlinear function, and γ and c are not identifiable under the null hypothesis, leading to the so-called Davies problem Davies (1987).

To compute the asymptotic distribution, several key integrals must be evaluated, as follows:

$$g_0 = \int_0^1 \mathbb{E} \left(1 - \exp \{ -\gamma(W(r) - c)^2 \} \right) dr, \quad (5)$$

$$g_1 = \int_0^1 r \mathbb{E} \left(1 - \exp \{ -\gamma(W(r) - c)^2 \} \right) dr, \quad (6)$$

$$g_2 = \int_0^1 \mathbb{E} \left(1 - \exp \{ -\gamma(W(r) - c)^2 \} \right)^2 dr. \quad (7)$$

Remark 3: Since $W(r)$ is a stochastic process, the integrals g_0, g_1, g_2 are random variables. In asymptotic distribution theory, we need the expected values of these random variables to compute the moments of the test statistic. By Fubini's theorem, $\mathbb{E} \left[\int f(r, W(r)) dr \right] = \int \mathbb{E}[f(r, W(r))] dr$ holds under suitable conditions, allowing us to interchange the expectation and integral operators. Thus, we first compute $\mathbb{E}[\cdot]$ for each r , then take the deterministic integral over r to obtain g_0, g_1, g_2 as deterministic.

These integrals are critical for constructing the asymptotic distribution but pose significant analytical challenges due to the nonlinear nature of $G(\cdot)$, the stochastic properties of $W(r)$, and the parameter dependence on γ and c .

3 Why the Integrals Are Analytically Intractable

To approximately address the mean and variance of the stochastic integral, we define g_0, g_1 , and g_2 , though the ultimate goal is to demonstrate that a closed-form solution for

the stochastic integral does not exist. The integrals in Equations (5), (6), and (7) are analytically intractable due to the combination of stochastic processes and nonlinear functions. Let us examine the g_0 integral in detail to illustrate the fundamental mathematical barriers.

Consider the integral:

$$g_0 = \int_0^1 \mathbb{E} [1 - \exp \{-\gamma(W(r) - c)^2\}] dr \quad (8)$$

Since $W(r) \sim N(0, r)$ is a standard Brownian motion, we have $W(r) - c \sim N(-c, r)$. The probability density function of $W(r) - c$ at a given r is:

$$f_{W(r)-c}(x) = \frac{1}{\sqrt{2\pi r}} \exp \left\{ -\frac{(x+c)^2}{2r} \right\} \quad (9)$$

Expressing g_0 as an expectation over the distribution of $W(r)$:

$$g_0 = \int_0^1 \int_{-\infty}^{\infty} [1 - \exp \{-\gamma x^2\}] \frac{1}{\sqrt{2\pi r}} \exp \left\{ -\frac{(x+c)^2}{2r} \right\} dx dr \quad (10)$$

The inner integral becomes:

$$\int_{-\infty}^{\infty} [1 - \exp \{-\gamma x^2\}] \frac{1}{\sqrt{2\pi r}} \exp \left\{ -\frac{(x+c)^2}{2r} \right\} dx \quad (11)$$

3.1 Inner Integral Analysis

While the inner integral involving the product of exponential terms initially appears tractable, we can apply the standard technique of completing the square. The product of exponential terms is:

$$\exp \{-\gamma x^2\} \exp \left\{ -\frac{(x+c)^2}{2r} \right\} = \exp \left\{ -\gamma x^2 - \frac{(x+c)^2}{2r} \right\} \quad (12)$$

Expanding the exponent:

$$-\gamma x^2 - \frac{(x+c)^2}{2r} = -x^2 \left(\gamma + \frac{1}{2r} \right) - \frac{cx}{r} - \frac{c^2}{2r} \quad (13)$$

Let $\alpha = \gamma + \frac{1}{2r}$ and complete the square:

$$-\alpha x^2 - \frac{cx}{r} = -\alpha \left(x^2 + \frac{cx}{r\alpha} \right) \quad (14)$$

$$= -\alpha \left[\left(x + \frac{c}{2r\alpha} \right)^2 - \frac{c^2}{4r^2\alpha^2} \right] \quad (15)$$

$$= -\alpha \left(x + \frac{c}{2r\alpha} \right)^2 + \frac{c^2}{4r^2\alpha} \quad (16)$$

The complete exponent becomes:

$$-\alpha \left(x + \frac{c}{2r\alpha} \right)^2 + \frac{c^2}{4r^2\alpha} - \frac{c^2}{2r} \quad (17)$$

Using the standard Gaussian integral $\int_{-\infty}^{\infty} \exp\{-\alpha(x + \beta)^2\} dx = \sqrt{\pi/\alpha}$, the inner integral evaluates to:

$$\int_{-\infty}^{\infty} \exp\left\{-\gamma x^2 - \frac{(x + c)^2}{2r}\right\} dx = \sqrt{\frac{\pi}{\alpha}} \exp\left\{\frac{c^2}{4r^2\alpha} - \frac{c^2}{2r}\right\} \quad (18)$$

where $\alpha = \gamma + \frac{1}{2r}$.

Therefore, the function $h(r; \gamma, c)$ representing the inner integral becomes:

$$h(r; \gamma, c) = \frac{1}{\sqrt{2\pi r}} \left[\sqrt{2\pi r} - \sqrt{\frac{\pi}{\gamma + \frac{1}{2r}}} \exp\left\{\frac{c^2}{4r^2(\gamma + \frac{1}{2r})} - \frac{c^2}{2r}\right\} \right] \quad (19)$$

Simplifying:

$$h(r; \gamma, c) = 1 - \frac{1}{\sqrt{2r}} \sqrt{\frac{\pi}{\gamma + \frac{1}{2r}}} \exp\left\{\frac{c^2}{4r^2(\gamma + \frac{1}{2r})} - \frac{c^2}{2r}\right\} \quad (20)$$

3.2 The Fundamental Intractability: The Outer Integral

While the inner integral can be solved analytically using completing the square, the outer integral presents insurmountable analytical challenges:

$$g_0 = \int_0^1 h(r; \gamma, c) dr \quad (21)$$

The function $h(r; \gamma, c)$ exhibits multiple mathematical complexities that prevent analytical integration:

Singular Behavior: The term $r^{-1/2}$ creates singular behavior as $r \rightarrow 0^+$, requiring careful handling of the integration bounds.

Mixed Rational-Algebraic Terms: The expression $\sqrt{\gamma + \frac{1}{2r}}$ combines the parameter γ with r^{-1} , creating a rational-algebraic function that lacks elementary antiderivatives.

Double Exponential Structure: The exponential term contains both r^{-2} and r^{-1} dependencies through:

$$\exp\left\{\frac{c^2}{4r^2(\gamma + \frac{1}{2r})} - \frac{c^2}{2r}\right\} \quad (22)$$

Nonlinear Parameter Coupling: Both parameters γ and c appear nonlinearly and interact with the integration variable r in complex ways:

- γ appears in both the denominator of the square root and the denominator of the exponential
- c appears quadratically in the exponential with r -dependent coefficients
- The coupling $\gamma + \frac{1}{2r}$ creates parameter-dependent singularity structures

3.3 The Intractability of the Final Stochastic Integral

The analysis of g_0 reveals that even an approximate analytical solution is unattainable. For $g_o = \int_0^1 \mathbb{E} [1 - \exp \{-\gamma(W(r) - c)^2\}] dr$, as the completed square form (Section 3.1) does not yield a simplifiable expression when integrated over r . This suggests that the stochastic integral, which involves the full interaction of $W(r)$ with $dW(r)$, is even more complex.

3.3.1 Steps Attempted to Solve the Stochastic Integral

To explore whether an analytical solution is feasible, several steps were considered:

1. **Itô Formula Application:** The Itô formula was considered to transform the integral. Defining $f(W(r)) = W(r) \cdot G(W(r); \gamma, c)$, the differential is:

$$df(W(r)) = \frac{\partial f}{\partial W} dW(r) + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} dr,$$

where $\frac{\partial f}{\partial W} = G(W(r); \gamma, c) + W(r) \cdot \frac{\partial G}{\partial W}$, and the second derivative $\frac{\partial^2 f}{\partial W^2}$ depends on $\frac{\partial G}{\partial W} = 2\gamma(W(r) - c) \exp \{-\gamma(W(r) - c)^2\}$. The nonlinearity of G results in a complex drift term $\frac{1}{2} \frac{\partial^2 f}{\partial W^2} dr$, which defies closed-form integration. Although Bierens (1997, Lemma 9.6.3) provides solutions for linear stochastic integrals (e.g., Propositions (iii), (iv), and (v) in the main text, yielding tractable forms such as $\sigma \int_0^1 \sin(2\pi k_s^{fr} r) W(r) dr$ and $\sigma \int_0^1 \cos(2\pi k_c^{fr} r) W(r) dr$), the nonlinear structure of the present integral precludes such an approach. As demonstrated in the main article, the Triple Fractional Arctan-augmented Fourier (TFAF) method overcomes this limitation by leveraging $\sin$, $\cos$, and $\arctan$ functions to achieve a solvable form, highlighting the unsolvability of the nonlinear integral in its current form. The relationship between linear and nonlinear stochastic integral approaches reveals a fundamental transition in analytical tractability. The classical Bierens (1997) and Enders Lee (2009) framework establishes that $\sum_{t=2}^T F\left(\frac{t}{T}\right) u_t = F(1)S(1) - \int_0^1 f(r)S_T(r)dr$ where $f(r) = F'(r)$, which becomes particularly elegant for trigonometric functions: when $F\left(\frac{t}{T}\right) = \sin\left(\frac{2\pi kt}{T}\right)$, we obtain $f\left(\frac{t}{T}\right) = (2\pi k) \cos\left(\frac{2\pi kt}{T}\right)$, and when $F\left(\frac{t}{T}\right) = \cos\left(\frac{2\pi kt}{T}\right)$, we derive $f\left(\frac{t}{T}\right) = -(2\pi k) \sin\left(\frac{2\pi kt}{T}\right)$, yielding closed-form solutions such as $\sigma \int_0^1 \sin(2\pi k_s^{fr} r) W(r) dr$ and $\sigma \int_0^1 \cos(2\pi k_c^{fr} r) W(r) dr$. However, when extending to nonlinear integrals of the form $f(W(r)) = W(r) \cdot G(W(r); \gamma, c)$, the Itô formula application $df(W(r)) = \frac{\partial f}{\partial W} dW(r) + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} dr$ creates a complex drift term due to the nonlinear structure of G , where $\frac{\partial G}{\partial W} = 2\gamma(W(r) - c) \exp -\gamma(W(r) - c)^2$, preventing closed-form integration and rendering the classical approach inadequate. The Triple Fractional Arctan-augmented Fourier (TFAF) method resolves this limitation by leveraging $\sin(\cdot)$, $\cos(\cdot)$, and $\arctan(\cdot)$ functions to transform the previously unsolvable nonlinear integral $\int_0^1 W(r) \cdot G(W(r); \gamma, c) dr$ into a tractable form, thus extending analytical capabilities beyond the linear constraints of traditional methods and enabling precise solutions for nonlinear stochastic problems that were previously intractable.

$$\begin{aligned} \text{(iii)} \quad T^{-3/2} \sum_{t=1}^T \sin\left(\frac{2\pi k_s^{fr} t}{T}\right) \bar{\omega}_t &\rightarrow \sigma (2\pi k_s^{fr}) \left[W(1) + (2\pi k_s^{fr}) \int_0^1 \sin(2\pi k_s^{fr} r) W(r) dr \right] \\ &\equiv \sigma f_3 \end{aligned}$$

2. **Taylor Approximation:** A first-order Taylor expansion of $G(W(r); \gamma, c)$ around $\gamma = 0$ was attempted: $G(W(r); \gamma, c) \approx \gamma(W(r) - c)^2$. This simplifies the integral to:

$$\int_0^1 W(r) \cdot \gamma(W(r) - c)^2 dW(r).$$

However, this approximation is valid only for small γ , and with $\gamma = 1.717$ (from simulations), the approximation loses accuracy, rendering the result unreliable.

3. **Expected Value Analysis:** The expected value was computed using Itô isometry:

$$\mathbb{E} \left[\int_0^1 W(r) \cdot G(W(r); \gamma, c) dW(r) \right] = 0,$$

which holds due to the martingale property of Itô integrals. This confirms the mean but provides no insight into the distribution or variance.

4. **Variance Estimation:** The variance was considered via Itô isometry:

$$\text{Var} \left(\int_0^1 W(r) \cdot G(W(r); \gamma, c) dW(r) \right) = \mathbb{E} \left[\int_0^1 (W(r) \cdot G(W(r); \gamma, c))^2 dr \right].$$

This requires evaluating $\mathbb{E} [(W(r) \cdot G(W(r); \gamma, c))^2]$, which, due to the nonlinear G , remains intractable analytically.

These steps highlight that no analytical method yields a closed-form solution due to the nonlinear and stochastic nature of the integrand.

3.3.2 Simulation-Based Evidence of Intractability

The analysis of $g_1 = \int_0^1 r \cdot \mathbb{E} [1 - \exp \{-\gamma(W(r) - c)^2\}] dr$ reveals that even an approximate analytical solution is unattainable. The weighting by r exacerbates the intractability of the inner integral, as the completed square form (Section 3.1) does not simplify when integrated over r , indicating the stochastic integral's complexity.

Despite Monte Carlo simulations with 100,000 replications estimating $\gamma = 1.717$ and $c = 0.283$ (within the observed ranges $\gamma \in [0.00017, 262.990]$ and $c \in [-61.803, 63.120]$, no closed-form solution emerges. The simulation results confirm that the integral's value varies across paths of $W(r)$, with an expected value of zero, but its variance remains dependent on the nonlinear G function. The high $\gamma = 1.717$ induces a moderate transition in G , amplifying stochastic variability, while $c = 0.283$ shifts the transition center, further complicating any attempt at a general analytical form. Thus, the integral remains unsolvable in closed form, necessitating numerical or approximation methods like TFAF.

3.3.3 Comparison with Existing Approaches

This analytical intractability explains why existing methods impose restrictive assumptions:

KSS Test: Linearization and Information Loss Utilizes the Taylor approximation to address the nuisance parameter, thereby avoiding complications associated with nonlinearities during the derivation of the asymptotic distribution.

The KSS test linearizes the ESTAR model using a first-order Taylor approximation around $\gamma = 0$, yielding the auxiliary regression:

$$\Delta y_t = \delta y_{t-1}^3 + \text{error}.$$

The test statistic is the t-ratio for $\delta = 0$. While computationally convenient, this linearization is valid only for small γ , leading to significant information loss when γ is large. Moreover, KSS assumes $c = 0$, which is restrictive in economic applications where non-zero thresholds are common due to factors like transaction costs or equilibrium deviations (e.g., Balassa-Samuelson effects). This assumption results in model misspecification, reducing the test's power, as evidenced by its failure to reject the unit root null in empirical applications to Real Effective Exchange Rates (REER) for Australia, Canada, Japan, and the UK.

Kılıç's Inf-t Test: Partial Solution with Restrictions It is assumed $c = 0$ and grid search is employed over γ . Consequently, an asymptotic distribution has been obtained for a single nonlinear parameter. Kilic (2011) proposes an inf-t test that avoids linearization by optimizing the t-statistic over a grid of γ values:

$$t_{\text{ESTAR}} = \inf_{\gamma \in \Gamma_T} \frac{\hat{\phi}(\gamma)}{\widehat{\text{se}}(\hat{\phi}(\gamma))},$$

where $\Gamma_T = \left[\frac{1}{100s_{zT}}, \frac{100}{s_{zT}} \right]$, and s_{zT} is the sample standard deviation of the transition variable $z_t = \Delta y_{t-d}$. Kılıç's use of the lagged difference (Δy_{t-d}) as the transition variable allows the model to capture nonlinear dynamics driven by past changes, aligning with economic theories of exchange rate adjustments.

Finally, the integrals g_1 and g_2 face identical mathematical barriers, with additional complications from the weighting factor r in g_1 and the squaring of the nonlinear term in g_2 . Consequently, analytical derivation of the asymptotic distribution for the unrestricted ESTAR unit root test requires alternative approaches. This intractability, noted by Leybourne et al. (1998), arises because nonlinear least squares estimation of γ and c does not yield closed-form solutions, making analytical derivation of the asymptotic distribution challenging.

4 The TFAF Method: A Tractable Solution

The Triple Fractional Arctan-augmented Fourier (TFAF) method, introduced in the main text, addresses the analytical intractability of the ESTAR integrals by transforming the problem into a more manageable form. The TFAF model is defined in the first step as follows:

$$y_t = d^{\text{TFAF}}(t; k_s, k_c, k_a) + \omega_t,$$

$$d^{\text{TFAF}}(t; k_s, k_c, k_a) = \alpha_0 + \alpha_1 \sin\left(\frac{2\pi k_s^{fr} t}{T}\right) + \alpha_2 \cos\left(\frac{2\pi k_c^{fr} t}{T}\right) + \alpha_3 \arctan\left(\frac{2\pi k_a^{fr} t}{T}\right).$$

The asymptotic distribution of the TFAF-based unit root test statistic is derived using Proposition 1 of the main text, which provides tractable stochastic integrals:

$$T^{-3/2} \sum_{t=1}^T \sin \left(\frac{2\pi k_s^{fr} t}{T} \right) \omega_t \rightarrow \sigma \int_0^1 \sin (2\pi k_s^{fr} r) W(r) dr, \quad (23)$$

$$T^{-3/2} \sum_{t=1}^T \cos \left(\frac{2\pi k_c^{fr} t}{T} \right) \omega_t \rightarrow \sigma \int_0^1 \cos (2\pi k_c^{fr} r) W(r) dr, \quad (24)$$

$$T^{-3/2} \sum_{t=1}^T \tan^{-1} \left(\frac{2\pi k_a^{fr} t}{T} \right) \omega_t \rightarrow \sigma \int_0^1 \tan^{-1} (2\pi k_a^{fr} r) W(r) dr. \quad (25)$$

These integrals are analytically tractable because the trigonometric functions (sin, cos, arctan) yield well-defined asymptotic behaviors when integrated with Brownian motion, as proven in the main text. The TFAF method results in the asymptotic distribution:

$$t_i^{\text{TFAF}}(k_s^{fr}, k_c^{fr}, k_a^{fr}) \xrightarrow{d} \frac{\int_0^1 W_i(k_s^{fr}, k_c^{fr}, k_a^{fr}, r) dW(r)}{\left(\int_0^1 W_i(k_s^{fr}, k_c^{fr}, k_a^{fr}, r)^2 dr \right)^{1/2}},$$

This distribution is invariant to nuisance parameters beyond the optimized frequencies, providing a general solution.

For the unrestricted ESTAR test, the TFAF method supports both direct and a two-stage estimation approach to handle the nuisance parameter problem, as detailed in the section 4.3.2 and empirical part of the main text. In the first stage, the full ESTAR model is estimated to obtain $\hat{\gamma}$ and $\hat{c}$.

5 Empirical Validation

The practical indirect advantage of the TFAF-based unrestricted ESTAR test is illustrated through an empirical application to bilateral Real Effective Exchange Rates (REER) for Australia, Canada, Japan, and the United Kingdom relative to the US dollar, covering the period from January 2013 to December 2020, as delineated in the empirical section of the main text. The unrestricted ESTAR test rejects the unit root null at the 1% level for Australia (-4.645) and Canada (-4.438), and at the 5% level for the UK (-3.051), while failing to reject for Japan (-2.008). In contrast, both the KSS and Kılıç's inf-t tests fail to reject the null for any country, highlighting their power loss due to restrictive assumptions.

The estimated threshold parameters are significantly non-zero (e.g., $c = 29.045$ for Australia), confirming the misspecification in KSS and Kılıç's tests, which assume $c = 0$. The γ estimates also differ markedly, with Kılıç's grid search yielding $\gamma = 100.00$ for Japan, compared to $\gamma = 0.0012$ for the unrestricted ESTAR test, underscoring the limitations of suboptimal optimization.

6 Why the TFAF Method Is Necessary

The TFAF method is essential for the following reasons:

1. **Analytical Intractability of ESTAR Integrals:** As shown in Section 3, the stochastic integrals in the ESTAR model (e.g., g_0, g_1, g_2) are analytically intractable due to the nonlinear transition function and the stochastic nature of $W(r)$. TFAF transforms these into tractable forms using sin, cos, and arctan functions, enabling a general approximation.
2. **Preservation of Nonlinear Dynamics:** Unlike the KSS test, which linearizes the ESTAR model, TFAF preserves the full nonlinear structure, avoiding information loss.
3. **Flexibility with Non-Zero Thresholds:** TFAF allows for non-zero c , addressing a critical limitation of KSS and Kılıç's tests, as evidenced by empirical results.
4. **Robust Parameter Estimation:** Using genetic optimization, TFAF ensures accurate estimation of γ and c , overcoming the suboptimal grid search in Kılıç's method.
5. **General Applicability:** TFAF successfully derives asymptotic distributions for previously intractable models, such as the Leybourne et al. (1998) test, and improves approximations for the KSS test.

7 Conclusion

The analytical intractability of stochastic integrals in the ESTAR model poses a fundamental barrier to deriving general asymptotic distributions for nonlinear unit root tests. The TFAF method overcomes this by transforming the problem into tractable trigonometric forms, providing a general and robust solution. Its empirical superiority, demonstrated through applications to REER data, and its theoretical advancements over KSS and Kılıç's methods highlight its necessity in nonlinear econometrics.

8 Important Note: Liouville's Theorem and the ESTAR Model Integration Problem

Liouville's Theorem: Theoretical Foundation

While the preceding analysis demonstrates through direct mathematical attempts where specific problems arise in the integration of ESTAR related functions, it is important to note that mathematics literature already contains rigorous theoretical proof that such analytical efforts are fundamentally impossible. **Liouville's Theorem** establishes that no elementary solutions exist for these integrals, confirming that the Triple Fractional Arctan-augmented Fourier (TFAF) method represents the only viable path forward for addressing these analytical challenges.

8.1 Liouville's Theorem: Precise Mathematical Statement

Theorem 1.1 (Liouville's Theorem): Let K be a differential field and $f \in K$. If there exists an elementary extension E of K with $\text{Const}(E) = \text{Const}(K)$, and $g \in E$ such that $D(g) = f$, then there exist $v \in K$, $u_1, \dots, u_m \in K$ with $u_i \neq 0$, and $c_1, \dots, c_m \in \text{Const}(K)$ such that:

$$f = D(v) + \sum_{i=1}^m c_i \frac{D(u_i)}{u_i} \quad (26)$$

Corollary 1.1: Let E be a differential field, and let K be a no-new-constant differential field extension of E generated by adjoining an exponential, that is, $K = E(e^g)$ for some $g \in K$. Suppose that e^g is transcendental over E . For any $f \in E$, the function $fe^g \in K$ has a primitive within some elementary no-new-constant differential field extension of K (i.e., fe^g is elementarily integrable) if and only if there exists some element $a \in E$ such that:

$$f = a' + ag' \quad (27)$$

8.2 Application to the ESTAR Integration Problem

The integrals encountered in the ESTAR model analysis, particularly those involving Gaussian-type exponential functions, fall directly under the scope of Liouville's Theorem. Consider the fundamental integral from equation (11):

$$\int_{-\infty}^{\infty} [1 - \exp\{-\gamma x^2\}] \frac{1}{\sqrt{2\pi r}} \exp\left\{-\frac{(x+c)^2}{2r}\right\} dx \quad (28)$$

This integral contains the term:

$$\int \exp\left\{-\gamma x^2 - \frac{(x+c)^2}{2r}\right\} dx \quad (29)$$

By Liouville's Theorem, exponential functions with quadratic arguments—such as $\exp\left\{-\gamma x^2 - \frac{(x+c)^2}{2r}\right\}$ —cannot be integrated in elementary form. The theorem establishes that no combination of polynomials, exponentials, logarithms, or basic trigonometric functions can express the antiderivative of such functions.

8.3 Theoretical Confirmation of Our Direct Analysis

The mathematical barriers identified in Section 3 including the singular behavior near $r = 0$, the mixed rational algebraic terms, and the double exponential structure are not merely computational obstacles but represent fundamental mathematical impossibilities as proven by Liouville's Theorem.

Specifically, the outer integral:

$$g_0 = \int_0^1 h(r; \gamma, c) dr \quad (30)$$

where $h(r; \gamma, c)$ is defined in equation (17), contains terms that violate the elementary integrability conditions established by Liouville's Theorem. The expression:

$$\sqrt{\frac{\pi}{\gamma + \frac{1}{2r}}} \exp\left\{\frac{c^2}{4r^2\left(\gamma + \frac{1}{2r}\right)} - \frac{c^2}{2r}\right\} \quad (31)$$

involves exponential functions with rational polynomial arguments in r , which cannot be integrated in closed form according to the theorem's corollary.

8.4 Extension to the Stochastic Integral Framework

The stochastic integral:

$$\int_0^1 W(r) \cdot G(W(r); \gamma, c) dW(r) \quad (32)$$

where $G(W(r); \gamma, c) = 1 - \exp \{-\gamma(W(r) - c)^2\}$, compounds the analytical intractability established by Liouville's Theorem. Even if we could solve the deterministic components analytically (which Liouville's Theorem proves impossible), the stochastic nature of the integral introduces additional layers of complexity that transcend elementary mathematical frameworks.

As demonstrated in Section 3.3.1, several approaches were attempted to solve the stochastic integral, including Itô formula application, Taylor approximation, expected value analysis, and variance estimation. Each approach encountered fundamental barriers:

1. **Itô Formula Application:** The nonlinearity of G results in a complex drift term $\frac{1}{2} \frac{\partial^2 f}{\partial W^2} dr$, which defies closed-form integration. Although Bierens (1997) provides solutions for linear stochastic integrals, the nonlinear structure of the present integral precludes such an approach.
2. **Taylor Approximation:** First-order Taylor expansion $G(W(r); \gamma, c) \approx \gamma(W(r) - c)^2$ is valid only for small γ , but with $\gamma = 1.717$ from simulations, the approximation loses accuracy.
3. **Expected Value and Variance Analysis:** While $\mathbb{E} \left[\int_0^1 W(r) \cdot G(W(r); \gamma, c) dW(r) \right] = 0$ due to the martingale property, the variance estimation requires evaluating $\mathbb{E} [(W(r) \cdot G(W(r); \gamma, c))^2]$, which remains intractable due to the nonlinear G .

These steps highlight that no analytical method yields a closed-form solution due to the nonlinear and stochastic nature of the integrand, confirming Liouville's Theorem's.

8.5 The Necessity of TFAF: Transcending Elementary Constraints

Liouville's Theorem establishes that the functions $\sin$, $\cos$, and $\arctan$ while individually elementary can be combined in infinite series to approximate non-elementary integrals. The TFAF method leverages this mathematical property by transforming the analytically intractable ESTAR integrals into tractable trigonometric forms:

$$\int_0^1 W(r) \cdot G(W(r); \gamma, c) dW(r) \approx \int_0^1 W(r) \cdot d^{\text{TFAF}}(r; k_s, k_c, k_a) dW(r) \quad (33)$$

where:

$$d^{\text{TFAF}}(r; k_s, k_c, k_a) = \alpha_0 + \alpha_1 \sin(2\pi k_s^{fr} r) + \alpha_2 \cos(2\pi k_c^{fr} r) + \alpha_3 \arctan(2\pi k_a^{fr} r) \quad (34)$$

This transformation respects the constraints imposed by Liouville's Theorem while providing a practical solution framework.

The relationship between linear and nonlinear stochastic integral approaches reveals a fundamental transition in analytical tractability. The classical Bierens (1997) and Enders & Lee (2012) framework establishes that:

$$\sum_{t=2}^T F\left(\frac{t}{T}\right) u_t = F(1)S(1) - \int_0^1 f(r)S_T(r)dr \quad (35)$$

where $f(r) = F'(r)$, which becomes particularly elegant for trigonometric functions. However, when extending to nonlinear integrals of the form $f(W(r)) = W(r) \cdot G(W(r); \gamma, c)$, the Itô formula application creates a complex drift term due to the nonlinear structure of G , preventing closed-form integration and rendering the classical approach inadequate.

8.6 Theoretical Validation of TFAF's Contribution

The combination of our direct mathematical analysis in Section 3 and the theoretical framework provided by Liouville's Theorem establishes that:

1. **Elementary solutions are mathematically impossible:** No finite combination of elementary functions can integrate the ESTAR exponential terms.
2. **Existing methods' limitations are fundamental:** The restrictive assumptions in KSS and Kılıç's tests are not methodological choices but necessary adaptations to mathematical constraints.
3. **TFAF provides the only viable general solution:** By utilizing trigonometric approximations that transcend elementary limitations, TFAF offers a systematic approach to previously intractable problems.
4. **The method's theoretical foundation is rigorous:** TFAF's approach is grounded in solid mathematical principles that respect the fundamental constraints while providing computational tractability.

8.7 Conclusion

This theoretical validation confirms that the development of the TFAF method constitutes a significant contribution to mathematical methodology in econometrics. By acknowledging the fundamental limitations established by Liouville's Theorem and developing innovative approximation techniques, we provide a practical framework for handling previously intractable problems in nonlinear unit root testing.

The method's success in empirical applications demonstrated through the REER analysis where it rejects unit root nulls that KSS and Kılıç's tests fail to detect further validates its theoretical necessity and practical superiority. The integration of rigorous theoretical analysis with innovative approximation methodology demonstrates that while exact elementary solutions remain impossible, effective approximate solutions can be constructed using advanced trigonometric methods that honor mathematical constraints while providing the computational tractability essential for empirical econometric analysis.

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