Embrittlement of an elasto-plastic medium by an inclusion

Supplementary material

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We present the stress tensor calculated for the geometry of the model. First, we present the stress tensor of the intrusion only. The components of the intrusion stress tensor, $\sigma^I_{\alpha\beta}(\mathbf{x})$, are as follows:

$$\sigma^I_{yy}(\mathbf{x}) = \sigma_0 \left[V_{yy}(\mathbf{x}) - \tan\theta \left(I^+_{yy}(\mathbf{x}) + I^-_{yy}(\mathbf{x})\right)\right] , \tag{1a}$$

$$\sigma^I_{xy}(\mathbf{x}) = \sigma_0 \left[V_{xy}(\mathbf{x}) - \tan\theta \left(I^+_{xy}(\mathbf{x}) + I^-_{xy}(\mathbf{x})\right)\right] , \tag{1b}$$

$$\sigma^I_{xx}(\mathbf{x}) = \sigma_0 \left[V_{xx}(\mathbf{x}) - \tan\theta \left(I^+_{xx}(\mathbf{x}) + I^-_{xx}(\mathbf{x})\right)\right] , \tag{1c}$$

where the functions $V_{yy}(\mathbf{x})$, $V_{xy}(\mathbf{x})$ and $V_{xx}(\mathbf{x})$ are the stress fields generated by the vertical interface of the intrusion, Γ_0. These functions are as follows:

$$V_{yy}(\mathbf{x}) = \frac{2l\tan\theta \left(x^2 - y^2 + l^2\tan^2\theta\right) x}{(x^2 + (l\tan\theta - y)^2)(x^2 + (l\tan\theta + y)^2)} \tag{2a}$$

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$$-2\left(\arctan\left[\frac{l\tan\theta-y}{x}\right]+\arctan\left[\frac{l\tan\theta+y}{x}\right]\right),$$

$$V_{xy}(\mathbf{x})=-\frac{4l\tan\theta\,x^2y}{(x^2+(l\tan\theta-y)^2)\,(x^2+(y+l\tan\theta)^2)}$$
$$+\frac{1}{2}\log\left[\frac{x^2+(l\tan\theta+y)^2}{x^2+(l\tan\theta-y)^2}\right], \tag{2b}$$

$$V_{xx}(\mathbf{x})=-\frac{2l\tan\theta\left(x^2-y^2+l^2\tan^2\theta\right)x}{(x^2+(l\tan\theta-y)^2)\,(x^2+(l\tan\theta+y)^2)}. \tag{2c}$$

The functions $I^{\pm}_{yy}(\mathbf{x})$, $I^{\pm}_{xy}(\mathbf{x})$ and $I^{\pm}_{xx}(\mathbf{x})$ originate from the inclined interfaces of the intrusion, $\Gamma_{\pm}$. These functions are as follows:

$$I^{\pm}_{yy}(\mathbf{x})=\pm\frac{(\tan\theta(x-l)\pm y)\left(l^2(1+\tan^2\theta)\,y-2l\,xy\pm l\tan\theta\left(x^2-y^2-l^2\right)\right)}{\left(1+\tan^2\theta\right)\left((x-l)^2+y^2\right)\left(x^2+(l\tan\theta\mp y)^2\right)}$$
$$+\frac{2\tan^3\theta}{\left(1+\tan^2\theta\right)^2}\left(\arctan\left[\frac{x-l\mp\tan\theta\,y}{\tan\theta(x-l)\pm y}\right]\right. \tag{2c}$$
$$\left.-\arctan\left[\frac{x+l\tan^2\theta\mp\tan\theta\,y}{\tan\theta(x-l)\pm y}\right]\right)$$
$$+\frac{1+3\tan^2\theta}{2\left(1+\tan^2\theta\right)^2}\log\left[\frac{(x-l)^2+y^2}{x^2+(l\tan\theta\mp y)^2}\right],$$

$$I^{\pm}_{xy}(\mathbf{x})=\pm\frac{(\tan\theta(x-l)\pm y)\left(l(l\tan^2\theta-x)(x-l)\mp 2l\tan\theta\,xy+ly^2\right)}{\left(1+\tan^2\theta\right)\left((x-l)^2+y^2\right)\left(x^2+(l\tan\theta\mp y)^2\right)}$$
$$\pm\frac{2\tan\theta^2}{\left(1+\tan^2\theta\right)^2}\left(\arctan\left[\frac{l-x\pm\tan\theta\,y}{\tan\theta(x-l)\pm y}\right]\right. \tag{2b}$$
$$\left.+\arctan\left[\frac{x+l\tan^2\theta\mp\tan\theta\,y}{\tan\theta(x-l)\pm y}\right]\right)$$
$$\pm\frac{\tan\theta(\tan^2\theta-1)}{2\left(1+\tan^2\theta\right)}\log\left[\frac{(x-l)^2+y^2}{x^2+(l\tan\theta\mp y)^2}\right],$$

$$I^{\pm}_{xx}(\mathbf{x})=\mp\frac{(\tan\theta(x-l)\pm y)\left(l^2(1+\tan^2\theta)\,y-2l\,xy\pm l\tan\theta\left(x^2-y^2-l^2\right)\right)}{\left(1+\tan^2\theta\right)\left((x-l)^2+y^2\right)\left(x^2+(l\tan\theta\mp y)^2\right)}$$
$$+\frac{2\tan\theta}{\left(1+\tan^2\theta\right)^2}\left(\arctan\left[\frac{x-l\mp\tan\theta\,y}{\tan\theta(x-l)\pm y}\right]\right. \tag{3a}$$
$$\left.-\arctan\left[\frac{x+l\tan^2\theta\mp\tan\theta\,y}{\tan\theta(x-l)\pm y}\right]\right)$$
$$+\frac{1-\tan^2\theta}{2\left(1+\tan^2\theta\right)^2}\log\left[\frac{(x-l)^2+y^2}{x^2+(l\tan\theta\mp y)^2}\right].$$

Finally, the components of the stress tensor of the model, $\sigma_{\alpha\beta}(\mathbf{x})$, are respectively:

$\sigma_{yy}(x,y)$

$$
\begin{aligned}
&= \Re\mathrm{e}\Bigg[\frac{2\sigma_1}{\pi}\left(\varPsi\left(z,\frac{\pi}{2}\right)-\varPsi_1\left(z,\frac{\pi}{2}\right)\right)-\sigma_0\Bigg\{2l\sin\theta\, e^{i\theta}\,\zeta_{yy}(x-i\,l\tan\theta,y)\\
&\quad+\frac{i\,y\,l\sin^{3/2}\theta\cos\chi}{a(1+\eta^2\tan^2\theta)^{\frac14}}\sqrt{\frac{az}{(z-a)(z-l)}}\times\frac{(z^2-a\,l)(l\tan\theta\tan\chi-z)}{z^2(z^2+l^2\tan^2\theta)}\\
&\quad+\frac{i\,y\,2l\sin^{3/2}\theta\cos\chi}{(1+\eta^2\tan^2\theta)^{\frac14}}\sqrt{\frac{(z-a)(z-l)}{az}}\times\frac{z^2-2l\tan\theta\tan\chi\,z-l^2\tan^2\theta}{(z^2+l^2\tan^2\theta)^2}\\
&\quad-\frac{2l\sin^{3/2}\theta\cos\chi}{(1+\eta^2\tan^2\theta)^{\frac14}}\sqrt{\frac{(z-a)(z-l)}{az}}\times\frac{l\tan\theta\tan\chi-z}{z^2+l^2\tan^2\theta}\\
&\quad+\frac12\left(e^{i4\theta}-4e^{i2\theta}-5\right)\Bigg(\varPsi(z,\phi)+\varPsi(z^*,\phi)-\varPsi_1(z,\phi)-\varPsi_1(z^*,\phi)\\
&\quad+i\Bigg(\frac{y^2}{(x-l)^2+y^2}-\frac{y^2}{(x-i\,l\tan\theta)^2+y^2}\\
&\quad+\frac12\log\left[\frac{(x-i\,l\tan\theta)^2+y^2}{(x-l)^2+y^2}\right]\Bigg)\Bigg)\Bigg\}\Bigg]+\sigma^I_{yy}(x,y)\,,
\end{aligned}
\tag{4a}
$$

$$
\begin{aligned}
&\sigma_{xy}(x,y)\\
&= -\Im\mathrm{m}\Bigg[\frac{2\sigma_1}{\pi}\varPsi_1\left(z,\frac{\pi}{2}\right)+\sigma_0\Bigg\{i\,2l\sin\theta\,e^{i\theta}\,\zeta_{xy}(x-i\,l\tan\theta,y)\\
&\quad+\frac{i\,y\,l\sin^{3/2}\theta\cos\chi}{a(1+\eta^2\tan^2\theta)^{\frac14}}\sqrt{\frac{az}{(z-a)(z-l)}}\times\frac{(z^2-a\,l)(l\tan\theta\tan\chi-z)}{z^2(z^2+l^2\tan^2\theta)}\\
&\quad+\frac{i\,y\,2l\sin^{3/2}\theta\cos\chi}{(1+\eta^2\tan^2\theta)^{\frac14}}\sqrt{\frac{(z-a)(z-l)}{az}}\times\frac{z^2-2l\tan\theta\tan\chi\,z-l^2\tan^2\theta}{(z^2+l^2\tan^2\theta)^2}\\
&\quad-\frac12\left(e^{i4\theta}-4e^{i2\theta}-5\right)\Bigg(\varPsi_1(z,\phi)-\varPsi_1(z^*,\phi)+\frac{(x-l)y}{(x-l)^2+y^2}\\
&\quad-\frac{(x-i\,l\tan\theta)y}{(x-i\,l\tan\theta)^2+y^2}\Bigg)\Bigg\}\Bigg]+\sigma^I_{xy}(x,y)\,,
\end{aligned}
\tag{4b}
$$

$$
\begin{aligned}
&\sigma_{xx}(x,y)\\
&= \Re\mathrm{e}\Bigg[\frac{2\sigma_1}{\pi}\left(\varPsi\left(z,\frac{\pi}{2}\right)+\varPsi_1\left(z,\frac{\pi}{2}\right)\right)-\sigma_0\Bigg\{2l\sin\theta\,e^{i\theta}\,\zeta_{xx}(x-i\,l\tan\theta,y)\\
&\quad-\frac{i\,y\,l\sin^{3/2}\theta\cos\chi}{a(1+\eta^2\tan^2\theta)^{\frac14}}\sqrt{\frac{az}{(z-a)(z-l)}}\times\frac{(z^2-a\,l)(l\tan\theta\tan\chi-z)}{z^2(z^2+l^2\tan^2\theta)}\\
&\quad-\frac{i\,y\,2l\sin^{3/2}\theta\cos\chi}{(1+\eta^2\tan^2\theta)^{\frac14}}\sqrt{\frac{(z-a)(z-l)}{az}}\times\frac{z^2-2l\tan\theta\tan\chi\,z-l^2\tan^2\theta}{(z^2+l^2\tan^2\theta)^2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2l\sin^{3/2}\theta\cos\chi}{(1+\eta^2\tan^2\theta)^{\frac{1}{4}}}\sqrt{\frac{(z-a)(z-l)}{az}} \times \frac{l\tan\theta\tan\chi - z}{z^2+l^2\tan^2\theta} \\
&+ \frac{1}{2}\left(e^{i4\theta} - 4e^{i2\theta} - 5\right)\Bigg(\varPsi(z,\phi) + \varPsi(z^*,\phi) + \varPsi_1(z,\phi) + \varPsi_1(z^*,\phi) \\
&+ i\left(\frac{y^2}{(x-i\,l\tan\theta)^2+y^2} - \frac{y^2}{(x-l)^2+y^2} + \frac{1}{2}\log\left[\frac{(x-i\,l\tan\theta)^2+y^2}{(x-l)^2+y^2}\right]\right) \\
&\qquad \Bigg)\Bigg\}\Bigg] + \sigma^I_{xx}(x,y)\,,
\end{aligned}
\tag{4c}
$$

where $z = x + i\,y$ and $z^* = x - i\,y$ is the complex conjugate of z. The function $\varPsi(z,\varphi)$ is as follows:

$$
\varPsi(z,\varphi) = \eta\sqrt{\frac{a(z-a)}{z(z-l)}}\,\varPi\left(\frac{(1-\eta)z}{z-l};\varphi\middle|1-\eta\right)\,, \tag{5}
$$

where φ is a variable which equals either $\pi/2$ or the amplitude

$$
\phi = \arctan\left[\sqrt{(i\tan\theta - 1)\,/\,(1 - i\,\eta\tan\theta)}\right].
$$

Also, $\varPsi_1(z,\varphi) = i\,y\frac{\partial\varPsi}{\partial z}(z,\varphi)$, where the partial derivative of $\varPsi(z,\varphi)$ with respect to the variable z is:

$$
\begin{aligned}
\frac{\partial\varPsi}{\partial z}(z,\varphi) = \frac{1}{2}\sqrt{\frac{a}{z(z-l)(z-a)}}\Bigg(&\frac{l}{z}\,\mathrm{F}\left(\varphi|1-\eta\right) - \mathrm{E}\left(\varphi|1-\eta\right) \\
&+ \frac{(1-\eta)\sin(2\varphi)z\,\sqrt{1-(1-\eta)\sin^2\varphi}}{2(z-l-(1-\eta)z\sin^2\varphi)}\Bigg).
\end{aligned}
\tag{6}
$$