

In this worksheet we fix some parameters of the system, leaving the others as algebraically independent numbers, to test Property E for Propositions 5 and 6. Note that the system admits the three invariant planes given by $x_i=0$. Inspection shows that the eigenvalues for the Jacobian at every idempotent are algebraically independent. Parameters may be changed in this file but $\gamma_{13}=\gamma_{21}=\gamma_{32}=0$ should be kept to preserve the invariant planes.

```
>
> restart;

> parameters:={gamma[13]=0,gamma[11]=3,gamma[21]=0,gamma[22]=5,
gamma[32]=0,gamma[31]=-1};
      parameters := {γ11 = 3, γ13 = 0, γ21 = 0, γ22 = 5, γ31 = -1, γ32 = 0}
```

```
> with(LinearAlgebra):
> A:=subs(parameters, <<2*gamma[11]*gamma[12],2*gamma[21]*gamma
[22],2*gamma[31]*gamma[32]>>|<2*gamma[12]*gamma[13],2*gamma[22]*
gamma[23],2*gamma[32]*gamma[33]>>|<2*gamma[11]*gamma[13],2*gamma
[21]*gamma[23],2*gamma[31]*gamma[33]>>);
```

$$A := \begin{bmatrix} 6\gamma_{12} & 0 & 0 \\ 0 & 10\gamma_{23} & 0 \\ 0 & 0 & -2\gamma_{33} \end{bmatrix}$$

```
> detA:=simplify(Determinant(A),factor);factor(%);
      detA := -120 γ12 γ23 γ33
      -120 γ12 γ23 γ33
```

```
> PP1 := subs(parameters, x1^2-(gamma[11]*gamma[33]*x2+gamma[12]*
gamma[31]*x3-gamma[12]*x3-gamma[33]*x2)*x1/(gamma[12]*gamma[33]))
;
PP2 := subs(parameters, x2^2-(gamma[11]*gamma[22]*x3+gamma[12]*
gamma[23]*x1-gamma[11]*x3-gamma[23]*x1)*x2/(gamma[11]*gamma[23]))
;
PP3 := subs(parameters, x3^2-(gamma[22]*gamma[33]*x1+gamma[23]*
gamma[31]*x2-gamma[22]*x1-gamma[31]*x2)*x3/(gamma[22]*gamma[31]))
;
```

$$PP1 := x1^2 - \frac{(2\gamma_{33}x2 - 2\gamma_{12}x3)x1}{\gamma_{12}\gamma_{33}}$$

$$PP2 := x2^2 - \frac{(\gamma_{12}\gamma_{23}x1 - \gamma_{23}x1 + 12x3)x2}{3\gamma_{23}}$$

$$PP3 := x^3 + \frac{(5x^1\gamma_{33} - x^2\gamma_{23} - 5x^1 + x^2)x^3}{5} \quad (4)$$

>
>
>

```
> s11:=subs(parameters,-gamma[31]*gamma[11]*(-1+gamma[22]+gamma[23])
)* (gamma[11]*gamma[22]*gamma[33]+gamma[12]*gamma[23]*gamma[31]-
gamma[12]*gamma[23]+gamma[12]*gamma[33]-gamma[22]*gamma[33])/
(gamma[11]^2*gamma[22]^2*gamma[33]^2+2*gamma[11]*gamma[12]*gamma
[22]*gamma[23]*gamma[31]*gamma[33]+gamma[12]^2*gamma[23]^2*gamma
[31]^2-gamma[11]^2*gamma[22]^2*gamma[33]-gamma[11]^2*gamma[22]*
gamma[33]^2-gamma[11]*gamma[12]*gamma[22]*gamma[23]*gamma[31]-
gamma[11]*gamma[12]*gamma[22]*gamma[23]*gamma[33]-gamma[11]*gamma
[12]*gamma[22]*gamma[31]*gamma[33]-gamma[11]*gamma[12]*gamma[23]*
gamma[31]*gamma[33]-gamma[11]*gamma[22]^2*gamma[33]^2-gamma[11]*
gamma[22]*gamma[23]*gamma[31]*gamma[33]-gamma[12]^2*gamma[23]^2*
gamma[31]-gamma[12]^2*gamma[23]*gamma[31]^2-gamma[12]*gamma[22]*
gamma[23]*gamma[31]*gamma[33]-gamma[12]*gamma[23]^2*gamma[31]^2+
gamma[11]^2*gamma[22]*gamma[33]+gamma[11]*gamma[12]*gamma[22]*
gamma[23]+gamma[11]*gamma[12]*gamma[31]*gamma[33]+gamma[11]*gamma
[22]^2*gamma[33]+gamma[11]*gamma[22]*gamma[33]^2+gamma[12]^2*
gamma[23]*gamma[31]+gamma[12]*gamma[23]^2*gamma[31]+gamma[12]*
gamma[23]*gamma[31]^2+gamma[22]*gamma[23]*gamma[31]*gamma[33]-
gamma[11]*gamma[22]*gamma[33]-gamma[12]*gamma[23]*gamma[31])) :
```

```
s22:=subs(parameters,-gamma[22]*(-1+gamma[31]+gamma[33])*gamma
[12]*(gamma[11]*gamma[22]*gamma[33]+gamma[12]*gamma[23]*gamma[31]
+gamma[11]*gamma[23]-gamma[11]*gamma[33]-gamma[23]*gamma[31])/
(gamma[11]^2*gamma[22]^2*gamma[33]^2+2*gamma[11]*gamma[12]*gamma
[22]*gamma[23]*gamma[31]*gamma[33]+gamma[12]^2*gamma[23]^2*gamma
[31]^2-gamma[11]^2*gamma[22]^2*gamma[33]-gamma[11]^2*gamma[22]*
gamma[33]^2-gamma[11]*gamma[12]*gamma[22]*gamma[23]*gamma[31]-
gamma[11]*gamma[12]*gamma[22]*gamma[23]*gamma[33]-gamma[11]*gamma
[12]*gamma[22]*gamma[31]*gamma[33]-gamma[11]*gamma[12]*gamma[23]*
gamma[31]*gamma[33]-gamma[11]*gamma[22]^2*gamma[33]^2-gamma[11]*
gamma[22]*gamma[23]*gamma[31]*gamma[33]-gamma[12]^2*gamma[23]^2*
gamma[31]-gamma[12]^2*gamma[23]*gamma[31]^2-gamma[12]*gamma[22]*
gamma[23]*gamma[31]*gamma[33]-gamma[12]*gamma[23]^2*gamma[31]^2+
gamma[11]^2*gamma[22]*gamma[33]+gamma[11]*gamma[12]*gamma[22]*
gamma[23]+gamma[11]*gamma[12]*gamma[31]*gamma[33]+gamma[11]*gamma
[22]^2*gamma[33]+gamma[11]*gamma[22]*gamma[33]^2+gamma[12]^2*
gamma[23]*gamma[31]+gamma[12]*gamma[23]^2*gamma[31]+gamma[12]*
gamma[23]*gamma[31]^2+gamma[22]*gamma[23]*gamma[31]*gamma[33]-
gamma[11]*gamma[22]*gamma[33]-gamma[12]*gamma[23]*gamma[31])) :
```

```
s33:=subs(parameters,-(-1+gamma[12]+gamma[11])*(gamma[11]*gamma
[22]*gamma[33]+gamma[12]*gamma[23]*gamma[31]-gamma[11]*gamma[22]-
gamma[12]*gamma[31]+gamma[22]*gamma[31])*gamma[33]*gamma[23]/
(gamma[11]^2*gamma[22]^2*gamma[33]^2+2*gamma[11]*gamma[12]*gamma
```

```

[22]*gamma[23]*gamma[31]*gamma[33]+gamma[12]^2*gamma[23]^2*gamma
[31]^2-gamma[11]^2*gamma[22]^2*gamma[33]-gamma[11]^2*gamma[22]*
gamma[33]^2-gamma[11]*gamma[12]*gamma[22]*gamma[23]*gamma[31]-
gamma[11]*gamma[12]*gamma[22]*gamma[23]*gamma[33]-gamma[11]*gamma
[12]*gamma[22]*gamma[31]*gamma[33]-gamma[11]*gamma[12]*gamma[23]*
gamma[31]*gamma[33]-gamma[11]*gamma[22]^2*gamma[33]^2-gamma[11]*
gamma[22]*gamma[23]*gamma[31]*gamma[33]-gamma[12]^2*gamma[23]^2*
gamma[31]-gamma[12]^2*gamma[23]*gamma[31]^2-gamma[12]*gamma[22]*
gamma[23]*gamma[31]*gamma[33]-gamma[12]*gamma[23]^2*gamma[31]^2+
gamma[11]^2*gamma[22]*gamma[33]+gamma[11]*gamma[12]*gamma[22]*
gamma[23]+gamma[11]*gamma[12]*gamma[31]*gamma[33]+gamma[11]*gamma
[22]^2*gamma[33]+gamma[11]*gamma[22]*gamma[33]^2+gamma[12]^2*
gamma[23]*gamma[31]+gamma[12]*gamma[23]^2*gamma[31]+gamma[12]*
gamma[23]*gamma[31]^2+gamma[22]*gamma[23]*gamma[31]*gamma[33]-
gamma[11]*gamma[22]*gamma[33]-gamma[12]*gamma[23]*gamma[31])) :

```

We check again that $v=(s_{11},s_{22},s_{33})$ is an idempotent for these values of the gammas: We consider

```

> tv:=subs(parameters,
simplify(subs({x1=s11,x2=s22,x3=s33},[PP1,PP2,PP3]))):
simplify(tv[2]-s22);
simplify(tv[1]-s11);
simplify(tv[3]-s33);

```

0
0
0

(5)

Now do the corresponding tests for the idempotents

$e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)$

```

> t1:=subs(parameters,
simplify(subs({x1=1,x2=0,x3=0},[PP1,PP2,PP3])));
t2:=subs(parameters,
simplify(subs({x1=0,x2=1,x3=0},[PP1,PP2,PP3])));
t3:=subs(parameters,
simplify(subs({x1=0,x2=0,x3=1},[PP1,PP2,PP3])));

```

$t1 := [1, 0, 0]$
 $t2 := [0, 1, 0]$
 $t3 := [0, 0, 1]$

(6)

Now take $v_1=\langle \text{gamma}[11], \text{gamma}[12], \text{gamma}[13] \rangle$ $v_2=\langle \text{gamma}[21], \text{gamma}[22], \text{gamma}[23] \rangle$, $v_3=\langle \text{gamma}[31], \text{gamma}[32], \text{gamma}[33] \rangle$ the other three idempotents

```

> tv1:=subs(parameters,
simplify(factor(subs(parameters,simplify(subs({x1=gamma[11],x2=
gamma[12],x3=gamma[13]},[PP1,PP2,PP3])))))));

```

```

tv2:=subs(parameters,
simplify(factor(subs(parameters,simplify(subs({x1=gamma[21],x2=
gamma[22],x3=gamma[23]},[PP1,PP2,PP3])))))));
tv3:=subs(parameters,
simplify(factor(subs(parameters,simplify(subs({x1=gamma[31],x2=
gamma[32],x3=gamma[33]},[PP1,PP2,PP3])))))));

```

$$tv1 := [3, \gamma_{12}, 0]$$

$$tv2 := [0, 5, \gamma_{23}]$$

$$tv3 := [-1, 0, \gamma_{33}]$$

(7)

So we have the 7 idempotents, e1,e2,e3,v1,v2,v3,v

```

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Now we calculate DP(ei), DP(vi), their characteristic polynomials and their eigenvalues.

```

> J:=simplify(<<diff(PP1,x1),diff(PP2,x1),diff(PP3,x1)>|<diff(PP1,
x2),diff(PP2,x2),diff(PP3,x2)>|
<diff(PP1,x3),diff(PP2,x3),diff(PP3,x3)>
>,factor):

```

```

> J1:=simplify(subs({x1=1,x2=0,x3=0},J),factor):
Eigenvalues(J1);
cha1:=factor(simplify(CharacteristicPolynomial(J1,x))):

```

$$\begin{bmatrix} 2 \\ \gamma_{33} - 1 \\ -\frac{\gamma_{12}}{3} + \frac{1}{3} \end{bmatrix}$$

(8)

```

> J2:=simplify(subs({x1=0,x2=1,x3=0},J),factor):
Eigenvalues(J2);
cha2:=factor(simplify(CharacteristicPolynomial(J2,x))):

```

$$\begin{bmatrix} 2 \\ \frac{1}{5} - \frac{\gamma_{23}}{5} \\ -\frac{2}{\gamma_{12}} \end{bmatrix}$$

(9)

```

> J3:=simplify(subs({x1=0,x2=0,x3=1},J),factor):
Eigenvalues(J3);
cha3:=factor(simplify(CharacteristicPolynomial(J3,x))):

```

(10)

$$\begin{bmatrix} 2 \\ \frac{2}{\gamma_{33}} \\ -\frac{4}{\gamma_{23}} \end{bmatrix} \quad (10)$$

> Jv1:=simplify(subs(parameters,simplify(subs({x1=gamma[11],x2=gamma[12],x3=gamma[13]},J),factor)),factor):
Eigenvalues(Jv1);
chav1:=factor(simplify(CharacteristicPolynomial(Jv1,x))):

$$\begin{bmatrix} 2 \\ \gamma_{12} + 3 \\ 3\gamma_{33} - 3 - \frac{1}{5}\gamma_{12}\gamma_{23} + \frac{1}{5}\gamma_{12} \end{bmatrix} \quad (11)$$

> Jv2:=simplify(subs(parameters,simplify(subs({x1=gamma[21],x2=gamma[22],x3=gamma[23]},J),factor)),factor):
Eigenvalues(Jv2);
chav2:=factor(simplify(CharacteristicPolynomial(Jv2,x))):

$$\begin{bmatrix} 2 \\ \gamma_{23} + 5 \\ \frac{2(\gamma_{12}\gamma_{23} - 5\gamma_{33})}{\gamma_{12}\gamma_{33}} \end{bmatrix} \quad (12)$$

> Jv3:=simplify(subs(parameters,simplify(subs({x1=gamma[31],x2=gamma[32],x3=gamma[33]},J),factor)),factor):
Eigenvalues(Jv3);
chav3:=factor(simplify(CharacteristicPolynomial(Jv3,x))):

$$\begin{bmatrix} 2 \\ \gamma_{33} - 1 \\ \frac{\gamma_{12}\gamma_{23} - \gamma_{23} - 12\gamma_{33}}{3\gamma_{23}} \end{bmatrix} \quad (13)$$

> Jv4:=simplify(subs(parameters,simplify(subs({x1=s11,x2=s22,x3=s33},J),factor)),factor):
Eigenvalues(Jv4);
chav4:=factor(simplify(CharacteristicPolynomial(Jv4,x))):

$$\begin{bmatrix} 2 \end{bmatrix}, \quad (14)$$

$$\begin{aligned}
& \left[\left(\frac{\gamma_{12}^2 \gamma_{23}^2 \gamma_{33}}{2} + \gamma_{12}^2 \gamma_{23}^2 + 2 \gamma_{12}^2 \gamma_{23} \gamma_{33} + \gamma_{12} \gamma_{23}^2 \gamma_{33} - \frac{15 \gamma_{12} \gamma_{23} \gamma_{33}^2}{2} - 6 \gamma_{12}^2 \gamma_{23} - 4 \gamma_{12} \right. \right. \\
& \gamma_{23}^2 - 18 \gamma_{12} \gamma_{23} \gamma_{33} - 30 \gamma_{12} \gamma_{33}^2 - 15 \gamma_{23} \gamma_{33}^2 + 24 \gamma_{12} \gamma_{23} + 72 \gamma_{12} \gamma_{33} + 40 \gamma_{23} \gamma_{33} + 60 \gamma_{33}^2 \\
& + \frac{1}{2} \left(\gamma_{12}^4 \gamma_{23}^4 \gamma_{33}^2 + 4 \gamma_{12}^4 \gamma_{23}^4 \gamma_{33} + 4 \gamma_{12}^4 \gamma_{23}^3 \gamma_{33}^2 + 4 \gamma_{12}^3 \gamma_{23}^4 \gamma_{33}^2 - 30 \gamma_{12}^3 \gamma_{23}^3 \gamma_{33}^3 - 12 \gamma_{12}^4 \right. \\
& \gamma_{23}^4 + 4 \gamma_{12}^4 \gamma_{23}^2 \gamma_{33}^2 - 32 \gamma_{12}^3 \gamma_{23}^4 \gamma_{33} - 96 \gamma_{12}^3 \gamma_{23}^3 \gamma_{33}^2 - 132 \gamma_{12}^3 \gamma_{23}^2 \gamma_{33}^3 + 4 \gamma_{12}^2 \gamma_{23}^4 \gamma_{33}^2 - 120 \gamma_{12}^2 \\
& \gamma_{23}^3 \gamma_{33}^3 + 225 \gamma_{12}^2 \gamma_{23}^2 \gamma_{33}^4 - 16 \gamma_{12}^4 \gamma_{23}^3 - 72 \gamma_{12}^4 \gamma_{23}^2 \gamma_{33} + 16 \gamma_{12}^4 \gamma_{23} \gamma_{33}^2 + 48 \gamma_{12}^3 \gamma_{23}^4 + 360 \gamma_{12}^3 \\
& \gamma_{23}^3 \gamma_{33}^3 - 136 \gamma_{12}^3 \gamma_{23}^2 \gamma_{33}^2 - 96 \gamma_{12}^3 \gamma_{23}^3 \gamma_{33}^3 - 80 \gamma_{12}^2 \gamma_{23}^4 \gamma_{33} + 672 \gamma_{12}^2 \gamma_{23}^3 \gamma_{33}^2 + 636 \gamma_{12}^2 \gamma_{23}^2 \gamma_{33}^3 \\
& + 1080 \gamma_{12}^2 \gamma_{23} \gamma_{33}^4 - 120 \gamma_{12} \gamma_{23}^3 \gamma_{33}^3 + 900 \gamma_{12} \gamma_{23}^2 \gamma_{33}^4 + 128 \gamma_{12}^4 \gamma_{23}^2 - 32 \gamma_{12}^4 \gamma_{23} \gamma_{33} - 96 \gamma_{12}^3 \\
& \gamma_{23}^3 + 1376 \gamma_{12}^3 \gamma_{23}^2 \gamma_{33} + 1344 \gamma_{12}^3 \gamma_{23} \gamma_{33}^2 - 192 \gamma_{12}^3 \gamma_{33}^3 + 144 \gamma_{12}^2 \gamma_{23}^4 - 1440 \gamma_{12}^2 \gamma_{23}^3 \gamma_{33} - 2028 \\
& \gamma_{12}^2 \gamma_{23}^2 \gamma_{33}^2 + 2688 \gamma_{12}^2 \gamma_{23} \gamma_{33}^3 + 720 \gamma_{12}^2 \gamma_{33}^4 + 1760 \gamma_{12} \gamma_{23}^3 \gamma_{33}^2 - 2400 \gamma_{12} \gamma_{23}^2 \gamma_{33}^3 - 1440 \gamma_{12} \gamma_{23} \\
& \gamma_{33}^4 + 900 \gamma_{23}^2 \gamma_{33}^4 - 1152 \gamma_{12}^3 \gamma_{23}^2 - 2304 \gamma_{12}^3 \gamma_{23} \gamma_{33} + 384 \gamma_{12}^3 \gamma_{33}^2 + 1152 \gamma_{12}^2 \gamma_{23}^3 - 1440 \gamma_{12}^2 \\
& \gamma_{23}^2 \gamma_{33} - 15776 \gamma_{12}^2 \gamma_{23} \gamma_{33}^2 - 7104 \gamma_{12}^2 \gamma_{33}^3 - 4320 \gamma_{12} \gamma_{23}^3 \gamma_{33} + 6320 \gamma_{12} \gamma_{23}^2 \gamma_{33}^2 + 5760 \gamma_{12} \gamma_{23} \\
& \gamma_{33}^3 - 20160 \gamma_{12} \gamma_{33}^4 - 8400 \gamma_{23}^2 \gamma_{33}^3 - 7200 \gamma_{23} \gamma_{33}^4 + 2304 \gamma_{12}^2 \gamma_{23}^2 + 12160 \gamma_{12}^2 \gamma_{23} \gamma_{33} + 11328 \\
& \gamma_{12}^2 \gamma_{33}^2 + 2560 \gamma_{12} \gamma_{23}^3 - 15360 \gamma_{12} \gamma_{23}^2 \gamma_{33} - 9600 \gamma_{12} \gamma_{23} \gamma_{33}^2 + 53760 \gamma_{12} \gamma_{33}^3 + 19600 \gamma_{23}^2 \gamma_{33}^2 \\
& - 43200 \gamma_{33}^4 + 10240 \gamma_{12} \gamma_{23}^2 + 7680 \gamma_{12} \gamma_{23} \gamma_{33} - 26880 \gamma_{12} \gamma_{33}^2 - 12800 \gamma_{23}^2 \gamma_{33} + 54400 \gamma_{23} \\
& \left. \left. \gamma_{33}^2 + 134400 \gamma_{33}^3 - 51200 \gamma_{23} \gamma_{33} - 96000 \gamma_{33}^2 \right)^{1/2} \right) \Bigg/ \left(2 \gamma_{12}^2 \gamma_{23}^2 - 2 \gamma_{12}^2 \gamma_{23} - 2 \gamma_{12} \gamma_{23}^2 \right.
\end{aligned}$$

$$\begin{aligned}
& -37 \gamma_{12} \gamma_{23} \gamma_{33} + 32 \gamma_{12}^2 \gamma_{23} + 12 \gamma_{12}^2 \gamma_{33} + 10 \gamma_{23}^2 \gamma_{33} + 120 \gamma_{33}^2 - 120 \gamma_{33}^3) \Big], \quad , \\
& \left[\left(\frac{\gamma_{12}^2 \gamma_{23}^2 \gamma_{33}}{2} + \gamma_{12}^2 \gamma_{23}^2 + 2 \gamma_{12}^2 \gamma_{23} \gamma_{33} + \gamma_{12}^2 \gamma_{23}^2 \gamma_{33} - \frac{15 \gamma_{12} \gamma_{23} \gamma_{33}^2}{2} - 6 \gamma_{12}^2 \gamma_{23} - 4 \gamma_{12} \right. \right. \\
& \gamma_{23}^2 - 18 \gamma_{12} \gamma_{23} \gamma_{33} - 30 \gamma_{12}^2 \gamma_{33}^2 - 15 \gamma_{23}^2 \gamma_{33}^2 + 24 \gamma_{12} \gamma_{23} + 72 \gamma_{12} \gamma_{33} + 40 \gamma_{23} \gamma_{33} + 60 \gamma_{33}^2 \\
& \left. - \frac{1}{2} \left(\gamma_{12}^4 \gamma_{23}^4 \gamma_{33}^2 + 4 \gamma_{12}^4 \gamma_{23}^4 \gamma_{33} + 4 \gamma_{12}^4 \gamma_{23}^3 \gamma_{33}^2 + 4 \gamma_{12}^3 \gamma_{23}^4 \gamma_{33}^2 - 30 \gamma_{12}^3 \gamma_{23}^3 \gamma_{33}^3 - 12 \gamma_{12}^4 \right. \right. \\
& \gamma_{23}^4 + 4 \gamma_{12}^4 \gamma_{23}^2 \gamma_{33}^2 - 32 \gamma_{12}^3 \gamma_{23}^4 \gamma_{33} - 96 \gamma_{12}^3 \gamma_{23}^3 \gamma_{33}^2 - 132 \gamma_{12}^3 \gamma_{23}^2 \gamma_{33}^3 + 4 \gamma_{12}^2 \gamma_{23}^4 \gamma_{33}^2 - 120 \gamma_{12}^2 \\
& \gamma_{23}^3 \gamma_{33}^3 + 225 \gamma_{12}^2 \gamma_{23}^2 \gamma_{33}^4 - 16 \gamma_{12}^4 \gamma_{23}^3 - 72 \gamma_{12}^4 \gamma_{23}^2 \gamma_{33} + 16 \gamma_{12}^4 \gamma_{23} \gamma_{33}^2 + 48 \gamma_{12}^3 \gamma_{23}^4 + 360 \gamma_{12}^3 \\
& \gamma_{23}^3 \gamma_{33}^3 - 136 \gamma_{12}^3 \gamma_{23}^2 \gamma_{33}^2 - 96 \gamma_{12}^3 \gamma_{23}^3 \gamma_{33}^3 - 80 \gamma_{12}^2 \gamma_{23}^4 \gamma_{33} + 672 \gamma_{12}^2 \gamma_{23}^3 \gamma_{33}^2 + 636 \gamma_{12}^2 \gamma_{23}^2 \gamma_{33}^3 \\
& \left. + 1080 \gamma_{12}^2 \gamma_{23} \gamma_{33}^4 - 120 \gamma_{12} \gamma_{23}^3 \gamma_{33}^3 + 900 \gamma_{12}^2 \gamma_{23}^2 \gamma_{33}^4 + 128 \gamma_{12}^4 \gamma_{23}^2 - 32 \gamma_{12}^4 \gamma_{23} \gamma_{33} - 96 \gamma_{12}^3 \right. \\
& \gamma_{23}^3 + 1376 \gamma_{12}^3 \gamma_{23}^2 \gamma_{33} + 1344 \gamma_{12}^3 \gamma_{23} \gamma_{33}^2 - 192 \gamma_{12}^3 \gamma_{33}^3 + 144 \gamma_{12}^2 \gamma_{23}^4 - 1440 \gamma_{12}^2 \gamma_{23}^3 \gamma_{33} - 2028 \\
& \gamma_{12}^2 \gamma_{23}^2 \gamma_{33}^2 + 2688 \gamma_{12}^2 \gamma_{23} \gamma_{33}^3 + 720 \gamma_{12}^2 \gamma_{33}^4 + 1760 \gamma_{12} \gamma_{23}^3 \gamma_{33}^2 - 2400 \gamma_{12} \gamma_{23}^2 \gamma_{33}^3 - 1440 \gamma_{12} \gamma_{23} \\
& \gamma_{33}^4 + 900 \gamma_{23}^2 \gamma_{33}^4 - 1152 \gamma_{12}^3 \gamma_{23}^2 - 2304 \gamma_{12}^3 \gamma_{23} \gamma_{33} + 384 \gamma_{12}^3 \gamma_{33}^2 + 1152 \gamma_{12}^2 \gamma_{23}^3 - 1440 \gamma_{12}^2 \\
& \gamma_{23}^2 \gamma_{33}^2 - 15776 \gamma_{12}^2 \gamma_{23} \gamma_{33}^2 - 7104 \gamma_{12}^2 \gamma_{33}^3 - 4320 \gamma_{12} \gamma_{23}^3 \gamma_{33} + 6320 \gamma_{12} \gamma_{23}^2 \gamma_{33}^2 + 5760 \gamma_{12} \gamma_{23} \\
& \gamma_{33}^3 - 20160 \gamma_{12} \gamma_{33}^4 - 8400 \gamma_{23}^2 \gamma_{33}^3 - 7200 \gamma_{23} \gamma_{33}^4 + 2304 \gamma_{12}^2 \gamma_{23}^2 + 12160 \gamma_{12}^2 \gamma_{23} \gamma_{33} + 11328 \\
& \gamma_{12}^2 \gamma_{33}^2 + 2560 \gamma_{12} \gamma_{23}^3 - 15360 \gamma_{12} \gamma_{23}^2 \gamma_{33} - 9600 \gamma_{12} \gamma_{23} \gamma_{33}^2 + 53760 \gamma_{12} \gamma_{33}^3 + 19600 \gamma_{23}^2 \gamma_{33}^2 \\
& \left. - 43200 \gamma_{33}^4 + 10240 \gamma_{12} \gamma_{23}^2 + 7680 \gamma_{12} \gamma_{23} \gamma_{33} - 26880 \gamma_{12} \gamma_{33}^2 - 12800 \gamma_{23}^2 \gamma_{33} + 54400 \gamma_{23} \right.
\end{aligned}$$

$$\left[\left(\gamma_{33}^2 + 134400 \gamma_{33}^3 - 51200 \gamma_{23} \gamma_{33} - 96000 \gamma_{33}^2 \right)^{1/2} \right] / \left(2 \gamma_{12}^2 \gamma_{23}^2 - 2 \gamma_{12}^2 \gamma_{23} - 2 \gamma_{12} \gamma_{23}^2 \right. \\ \left. - 37 \gamma_{12} \gamma_{23} \gamma_{33} + 32 \gamma_{12} \gamma_{23} + 12 \gamma_{12} \gamma_{33} + 10 \gamma_{23} \gamma_{33} + 120 \gamma_{33}^2 - 120 \gamma_{33} \right) \Bigg] \quad ,$$

