APPENDIX A: DERIVATION OF THE SENSOR ANGULAR VELOCITY RESPONSE

This appendix is concerned with the complete derivation of the sensor angular velocity response for the Schmidt hammer application. The sensor's cantilever is modeled as an underdamped SDOF system undergoing base-excitation whose equation of motion is a Non-homogenous linear second order differential equation with constant coefficients which have the following form:

$$m\ddot{u} + c\dot{u} + ku = -(f_1 + f_2)$$

where, the first force is a decaying force input as shown in **Figure A.1**:

$$f_{1(t)} = ma_{i(t)}, \qquad (0 < t)$$

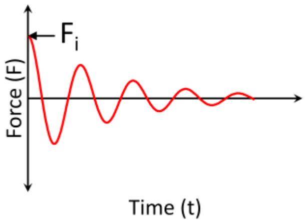

Figure A.1 SH Mass-spring decaying force input

and, the second force is a triangular pulse input as shown in **Figure A.2**:

$$f_{2(t)} = \begin{cases} 0, & (0 < t < t_0) \\ \dfrac{F_p}{0.5\Delta t}(t - t_0), & (t_0 < t < t_1) \\ -\dfrac{F_p}{0.5\Delta t}(t - t_2), & (t_1 < t < t_2) \\ 0, & (t_2 < t) \end{cases}$$

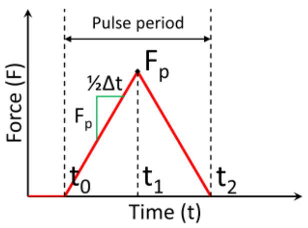

Figure A.2 Triangular pulse input

where, $t_0 = \pi/2\omega_i,\ t_1 = t_0 + 0.5\Delta t,\ t_2 = t_0 + \Delta t$

The displacement response can be obtained by the superposition of the solutions of the individual inputs:

$$u_{(y=L,t)} = \begin{cases} u_{1(L,t)}, & (0 < t < t_0) \\ u_{1(L,t)} + u_{2(L,t>t_0)}\Big|_{t_0}^{t}, & (t_0 < t < t_1) \\ u_{1(L,t)} + u_{2(L,t>t_0)}\Big|_{t_0}^{t} - u_{2(L,t>t_0)}\Big|_{t_1}^{t} + u_{2(L,t>t_1)}\Big|_{t_1}^{t}, & (t_1 < t < t_2) \\ u_{1(L,t)} + u_{2(L,t>t_0)}\Big|_{t_0}^{t} - u_{2(L,t>t_0)}\Big|_{t_1}^{t} + u_{2(L,t>t_1)}\Big|_{t_1}^{t} - u_{2(L,t>t_1)}\Big|_{t_2}^{t}, & (t_2 < t) \end{cases}$$

A.1 Solving for the Decaying Force Input

The solution by the method of undetermined coefficients is the summation of the homogeneous (general) and the non-homogeneous (particular) solutions:

$$u_{(t)} = u_{(t)h} + u_{(t)p}$$

A.1.1 Solving the Homogeneous Part (General Solution):

$$m\ddot{u} + c\dot{u} + ku = 0$$

Assume a general solution:

$$u_{(t)h} = e^{rt}$$

$$\dot{u}_{(t)} = re^{rt}$$

$$\ddot{u}_{(t)} = r^2e^{rt}$$

Substitute:

$$m(r^2e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

$$e^{rt}(mr^2 + cr + k) = 0$$

The auxiliary equation or characteristic equation is:

$$mr^2 + cr + k = 0$$

$$A = m, B = c, C = k$$

Its roots are:

$$r_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Since:

$$\beta = \frac{\omega}{\omega_n}, \omega_n = \sqrt{\frac{k}{m}}, k = m{\omega_n}^2, \zeta = \frac{c}{c_{critical}}, c_{critical} = 2\sqrt{km} = 2m\omega_n, c = \zeta(2m\omega_n)$$

Therefore:

$$r_{1,2} = \frac{-2\zeta m\omega_n \pm \sqrt{(2\zeta m\omega_n)^2 - 4m(m{\omega_n}^2)}}{2m}$$

$$r_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Since $\zeta < 1$, then $\zeta^2 - 1 < 0$, therefore the roots are complex roots

$$r_{1,2} = -\zeta\omega_n \pm i\sqrt{1-\zeta^2}\omega_n$$

If put in the form:

$$r_{1,2} = a \pm ib$$

$$a = -\zeta\omega_n, \qquad b = \sqrt{1-\zeta^2}\omega_n$$

Then the general solution is in the form:

$$u_{(t)} = e^{at}(c_1 \sin bt + c_2 \cos bt)$$

Therefore:

$$u_{(t)} = e^{-\zeta\omega_n t}\left(c_1 \sin\sqrt{1-\zeta^2}\omega_n t + c_2 \cos\sqrt{1-\zeta^2}\omega_n t\right)$$

$$\dot{u}_{(t)} = \omega_n e^{-\zeta\omega_n t}\left[\left(-\zeta c_1 - \sqrt{1-\zeta^2}c_2\right)\sin\sqrt{1-\zeta^2}\omega_n t + \left(\sqrt{1-\zeta^2}c_1 - \zeta c_2\right)\cos\sqrt{1-\zeta^2}\omega_n t\right]$$

$$\ddot{u}_{(t)} = {\omega_n}^2 e^{-\zeta\omega_n t}\left[\left((2\zeta^2-1)c_1 + 2\zeta\sqrt{1-\zeta^2}c_2\right)\sin\sqrt{1-\zeta^2}\omega_n t - \left(2\zeta\sqrt{1-\zeta^2}c_1 - (2\zeta^2-1)c_2\right)\cos\sqrt{1-\zeta^2}\omega_n t\right]$$

The previous solution is called the general solution because it is the same homogeneous solution for all types of input force (always putting the R.H.S. = 0). The evaluation of the constants of the general solution is carried out after putting both the homogeneous and particular solutions together in the full equation.

A.1.2 Solving the Non-homogeneous Part (Particular Solution):

$$m\ddot{u} + c\dot{u} + ku = f_{1(t)}$$

$$f_{1(t)} = ma_{i(t)}$$

To obtain the output acceleration of the mass-spring system as an input acceleration ($a_{i(t)}$) to the sensor's cantilever system, it must be noted that the mass-spring system is under free-vibration (free = not governed by any external force input), therefore the decaying force input (with respect to the cantilever) is a transient output (with respect to the mass-spring system) which is solved only by the homogeneous part without any particular part. Once the output acceleration is determined, it will be re-used as the input (with respect to the cantilever) during solving the non-homogeneous part.

Evaluating the constants in the general solution using the initial conditions of the mass-spring transient vibration:

At $t = 0$, then:

$$x_{(t=0)} = X_0 \qquad \dot{x}_{(t=0)} = \dot{X}_0$$

$$X_0 = c_2 \qquad \dot{X}_0 = \left(\sqrt{1-{\zeta_i}^2}c_1 - \zeta c_2\right)\omega_n$$

$$c_2 = X_0 \qquad c_1 = \frac{\dot{X}_0 + \omega_n \zeta_i X_0}{\omega_n\sqrt{1-{\zeta_i}^2}}$$

Substituting in the acceleration solution:

$$\ddot{x}_{(t)} = \omega_n e^{-\zeta_i \omega_n t}\left(\frac{(2{\zeta_i}^2 - 1)\dot{X}_0 + \zeta_i \omega_n X_0}{\sqrt{1-{\zeta_i}^2}} \sin\sqrt{1-{\zeta_i}^2}\omega_n t - (2\zeta_i \dot{X}_0 + \omega_n X_0)\cos\sqrt{1-{\zeta_i}^2}\omega_n t\right)$$

In current case $\dot{X}_0 = 0$ since the SH mass starts from rest, and $\omega_n = \omega_i$, therefore:

$$a_{i(t)} = (X_0 {\omega_i}^2) e^{-\zeta_i \omega_i t}\left(\frac{\zeta_i}{\sqrt{1-{\zeta_i}^2}} \sin\sqrt{1-{\zeta_i}^2}\omega_i t - \cos\sqrt{1-{\zeta_i}^2}\omega_i t\right)$$

Therefore:

$$f_{1(t)} = \frac{\zeta_i}{\sqrt{1-{\zeta_i}^2}} m(X_0 {\omega_i}^2) e^{-\zeta_i \omega_i t}\sin\sqrt{1-{\zeta_i}^2}\omega_i t - m(X_0 {\omega_i}^2)e^{-\zeta_i \omega_i t}\cos\sqrt{1-{\zeta_i}^2}\omega_i t$$

Now, the solution of the non-homogenous part is continued. Since $f_{1(t)}$ is a sum combination, then the particular solution for each function can be determined separately and then take their sum to find the final particular solution of the given equation.

Let:

$$\sqrt{1-{\zeta_i}^2}\omega_i = \omega_d, \qquad \frac{m X_0 {\omega_i}^2 \zeta_i}{\sqrt{1-{\zeta_i}^2}} = F_{d1}, \qquad -m X_0 {\omega_i}^2 = F_{d2}$$

Therefore,

$$f_{(t)1} = e^{-\zeta_i \omega_i t}.F_{d1}\sin\omega_d t\,, \qquad f_{(t)2} = e^{-\zeta_i \omega_i t}.F_{d2}\cos\omega_d t$$

Since $f_{(t)1}$ is a product of two functions, then take the product of the guesses for the individual pieces of the given function:

$$f_{(t)1} = e^{-\zeta_i \omega_i t}.F_{d1}\sin\omega_d t$$

Assume a general solution:

$$u_{(t)p} = Ce^{-\zeta_i\omega_i t}(A\ \cos\omega_d t + B\ \sin\omega_d t) \equiv e^{-\zeta_i\omega_i t}(A\ \cos\omega_d t + B\ \sin\omega_d t)$$
$$\dot{u}_{(t)} = e^{-\zeta_i\omega_i t}[(-A\omega_d - \zeta_i\omega_i B)\ \sin\omega_d t + (B\omega_d - \zeta_i\omega_i A)\ \cos\omega_d t]$$
$$\ddot{u}_{(t)} = e^{-\zeta_i\omega_i t}\Big[\big(-\omega_d(B\omega_d - \zeta_i\omega_i A) - \zeta_i\omega_i(-A\omega_d - \zeta_i\omega_i B)\big)\ \sin\omega_d t + \big(\omega_d(-A\omega_d - \zeta_i\omega_i B) - \zeta_i\omega_i(B\omega_d - \zeta_i\omega_i A)\big)\ \cos\omega_d t\Big]$$

Substitute:

$$\big[m\big(-\omega_d(B\omega_d - \zeta_i\omega_i A) - \zeta_i\omega_i(-A\omega_d - \zeta_i\omega_i B)\big) + c(-A\omega_d - \zeta_i\omega_i B) + Bk\big]\ \sin\omega_d t + \big[m\big(\omega_d(-A\omega_d - \zeta_i\omega_i B) - \zeta_i\omega_i(B\omega_d - \zeta_i\omega_i A)\big) + c(B\omega_d - \zeta_i\omega_i A) + Ak\big]\cos\omega_d t = F_{d1}\sin\omega_d t$$

Since $m = \frac{k}{{\omega_n}^2}$, $c = \frac{k2\zeta}{\omega_n}$, $\beta = \frac{\omega_i}{\omega_n}$, therefore:

$$\left[\frac{2\zeta_i\omega_d\omega_i}{{\omega_n}^2}A - \frac{2\zeta\omega_d}{\omega_n}A + \frac{{\zeta_i}^2{\omega_i}^2}{{\omega_n}^2}B - \frac{2\zeta\zeta_i\omega_i}{\omega_n}B - \frac{{\omega_d}^2}{{\omega_n}^2}B + B\right]\ \sin\omega_d t + \left[-A\frac{{\omega_d}^2}{{\omega_n}^2} - \frac{2\zeta_i\omega_d\omega_i}{{\omega_n}^2}B + \frac{{\zeta_i}^2{\omega_i}^2}{{\omega_n}^2}A + \frac{2\zeta\omega_d}{\omega_n}B - \frac{2\zeta\zeta_i\omega_i}{\omega_n}A + A\right]\cos\omega_d t = \frac{F_{d1}}{k}\sin\omega_d t$$

By equating two sides:

$$\left(\frac{2\zeta_i\omega_d\omega_i}{{\omega_n}^2} - \frac{2\zeta\omega_d}{\omega_n}\right)A_1 + \left(\frac{{\zeta_i}^2{\omega_i}^2}{{\omega_n}^2} - \frac{2\zeta\zeta_i\omega_i}{\omega_n} - \frac{{\omega_d}^2}{{\omega_n}^2} + 1\right)B_1 = \frac{F_{d1}}{k}$$
$$-\left(\frac{2\zeta_i\omega_d\omega_i}{{\omega_n}^2} - \frac{2\zeta\omega_d}{\omega_n}\right)B_1 + \left(\frac{{\zeta_i}^2{\omega_i}^2}{{\omega_n}^2} - \frac{2\zeta\zeta_i\omega_i}{\omega_n} - \frac{{\omega_d}^2}{{\omega_n}^2} + 1\right)A_1 = 0$$

Similarly, F_{d2} equations will be formed. Therefore for F_{d1} and F_{d2}, let:

$$C_1A_1 + C_2B_1 = \frac{F_{d1}}{k} \qquad C_1A_2 + C_2B_2 = 0$$

$$-C_1B_1 + C_2A_1 = 0 \qquad -C_1B_2 + C_2A_2 = \frac{F_{d2}}{k}$$

$$A_1 = \frac{C_1(F_{d1}/k)}{{C_1}^2 + {C_2}^2} = A(F_{d1}/k), \qquad A_2 = \frac{C_2(F_{d2}/k)}{{C_1}^2 + {C_2}^2} = B(F_{d2}/k),$$

$$B_1 = \frac{C_2(F_{d1}/k)}{{C_1}^2 + {C_2}^2} = B(F_{d1}/k) \qquad B_2 = \frac{-C_1(F_{d2}/k)}{{C_1}^2 + {C_2}^2} = -A(F_{d2}/k)$$

Therefore:

$$A = \frac{C_1}{{C_1}^2 + {C_2}^2} = \frac{2\beta(\zeta - \zeta_i\beta)\sqrt{1 - {\zeta_i}^2}}{(2\beta)^2(\zeta - \zeta_i\beta)^2\left(1 - {\zeta_i}^2\right) + \left(1 - 2\zeta\zeta_i\beta + \left(2{\zeta_i}^2 - 1\right)\beta^2\right)^2}$$

$$B = \frac{C_2}{{C_1}^2 + {C_2}^2} = \frac{1 - 2\zeta\zeta_i\beta + \left(2{\zeta_i}^2 - 1\right)\beta^2}{(2\beta)^2(\zeta - \zeta_i\beta)^2\left(1 - {\zeta_i}^2\right) + \left(1 - 2\zeta\zeta_i\beta + \left(2{\zeta_i}^2 - 1\right)\beta^2\right)^2}$$

Therefore:

$$u_{(t)p} = (F_{d1}/k)e^{-\zeta_i\omega_i t}(A\ \cos\omega_d t + B\ \sin\omega_d t) + (F_{d2}/k)e^{-\zeta_i\omega_i t}(B\ \cos\omega_d t - A\ \sin\omega_d t)$$

Therefore:

$$u_{(t)p} = \frac{m(X_0{\omega_i}^2)}{k}e^{-\zeta_i\omega_i t}\left[\left(\frac{\zeta_i}{\sqrt{1 - {\zeta_i}^2}}B + A\right)\sin\sqrt{1 - {\zeta_i}^2}\omega_i t + \left(\frac{\zeta_i}{\sqrt{1 - {\zeta_i}^2}}\ A - B\right)\cos\sqrt{1 - {\zeta_i}^2}\omega_i t\right]$$

A.1.3 The Total Solution (Sum of the homogenous and particular solutions)

$$u_{(t)} = u_{(t)h} + u_{(t)p}$$

$$u_{(t)} = e^{-\zeta\omega_n t}\left(c_1 \sin\sqrt{1 - \zeta^2}\omega_n t + c_2\cos\sqrt{1 - \zeta^2}\omega_n t\right) + \frac{m(u_{i0}{\omega_i}^2)}{k}e^{-\zeta_i\omega_i t}\left[\left(\frac{\zeta_i}{\sqrt{1 - {\zeta_i}^2}}B + A\right)\sin\sqrt{1 - {\zeta_i}^2}\omega_i t + \left(\frac{\zeta_i}{\sqrt{1 - {\zeta_i}^2}}\ A - B\right)\cos\sqrt{1 - {\zeta_i}^2}\omega_i t\right]$$

$$\dot{u}_{(t)} = \omega_n e^{-\zeta\omega_n t}\left[\left(-\zeta c_1 - \sqrt{1 - \zeta^2}c_2\right)\sin\sqrt{1 - \zeta^2}\omega_n t + \left(\sqrt{1 - \zeta^2}c_1 - \zeta c_2\right)\cos\sqrt{1 - \zeta^2}\omega_n t\right] + \frac{m(u_{i0}{\omega_i}^2)\omega_i}{k}e^{-\zeta_i\omega_i t}\left[\left(\frac{1 - 2{\zeta_i}^2}{\sqrt{1 - {\zeta_i}^2}}B - 2\zeta_i A\right)\sin\sqrt{1 - {\zeta_i}^2}\omega_i t + \left(\frac{1 - 2{\zeta_i}^2}{\sqrt{1 - {\zeta_i}^2}}\ A + 2\zeta_i B\right)\cos\sqrt{1 - {\zeta_i}^2}\omega_i t\right]$$

Evaluating the constants for decaying forced vibrations in the general form:

At $t = 0$, then $u_{(t=0)} = u_0$:

$$u_0 = c_2 + \frac{m(X_0{\omega_i}^2)}{k}\left(\frac{\zeta_i}{\sqrt{1 - {\zeta_i}^2}}\ A - B\right)$$

$$d = \frac{m(X_0 {\omega_i}^2)}{k}\left(\frac{\zeta_i}{\sqrt{1-{\zeta_i}^2}}\,A - B\right)$$

$$c_2 = u_0 - d$$

At $t = 0$, then $\dot{u}_{(t=0)} = \dot{u}_0$:

$$\dot{u}_0 = \left(\sqrt{1-\zeta^2}c_1 - \zeta c_2\right)\omega_n + \frac{m(X_0 {\omega_i}^2)\omega_i}{k}\left(\frac{1-2{\zeta_i}^2}{\sqrt{1-{\zeta_i}^2}}\,A + 2\zeta_i B\right)$$

$$v = \frac{m(X_0 {\omega_i}^2)\omega_i}{k}\left(\frac{1-2{\zeta_i}^2}{\sqrt{1-{\zeta_i}^2}}\,A + 2\zeta_i B\right)$$

$$c_1 = \frac{(\dot{u}_0 - v) + \omega_n\zeta(u_0 - d)}{\omega_n\sqrt{1-\zeta^2}}$$

Therefore:

$$u_{(t)} = e^{-\zeta\omega_n t}\left[\frac{(\dot{u}_0 - v) + \omega_n\zeta(u_0 - d)}{\omega_n\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n t + (u_0 - d)\cos\sqrt{1-\zeta^2}\omega_n t\right]$$
$$+\frac{m(X_0 {\omega_i}^2)}{k}e^{-\zeta_i\omega_i t}\left[\left(\frac{\zeta_i}{\sqrt{1-{\zeta_i}^2}}B + A\right)\sin\sqrt{1-{\zeta_i}^2}\omega_i t\right.$$
$$\left.+\left(\frac{\zeta_i}{\sqrt{1-{\zeta_i}^2}}\,A - B\right)\cos\sqrt{1-{\zeta_i}^2}\omega_i t\right]$$

$$A = \frac{2\beta(\zeta - \zeta_i\beta)\sqrt{1-{\zeta_i}^2}}{(2\beta)^2(\zeta-\zeta_i\beta)^2\left(1-{\zeta_i}^2\right) + \left(1 - 2\zeta\zeta_i\beta + \left(2{\zeta_i}^2 - 1\right)\beta^2\right)^2}$$

$$B = \frac{1 - 2\zeta\zeta_i\beta + \left(2{\zeta_i}^2 - 1\right)\beta^2}{(2\beta)^2(\zeta-\zeta_i\beta)^2\left(1-{\zeta_i}^2\right) + \left(1 - 2\zeta\zeta_i\beta + \left(2{\zeta_i}^2 - 1\right)\beta^2\right)^2}$$

A.2 Solving for the Triangular Pulse Input

Such phenomena of an impulsive nature and problems can be modeled by “Dirac’s delta function,” and solve them very effectively by using the Laplace transform method. The solution by the Laplace transform method is the convolution integral of the Laplace solution to the Dirac's delta function.

A.2.1 Solving for the Dirac's Delta Function

An impulse input – constant force for an infinitesimal instant of time – is given on the R.H.S as:

$$m\ddot{u} + c\dot{u} + ku = I\delta_{(t-t_0)}$$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + {\omega_n}^2 u = \frac{{\omega_n}^2}{k} I\delta_{(t-t_0)}$$

Assume boundary conditions:

$$u_{(t=0)} = u_0, \qquad \dot{u}_{(t=0)} = \dot{u}_0$$

Assume a Laplace transform:

$$(s^2U - su_0 - \dot{u}_0) + 2\zeta\omega_n(sU - u_0) + {\omega_n}^2 U = \frac{{\omega_n}^2}{k} I(e^{-t_0 s})$$

$$U(s^2 + 2\zeta\omega_n s + {\omega_n}^2) - u_0 s - 2\zeta\omega_n u_0 - \dot{u}_0 = \frac{{\omega_n}^2}{k} I e^{-t_0 s}$$

$$U_{(s)} = \frac{\frac{{\omega_n}^2}{k} I e^{-t_0 s} + u_0 s + 2\zeta\omega_n u_0 + \dot{u}_0}{(s^2 + 2\zeta\omega_n s + {\omega_n}^2)}$$

$$U_{(s)} = \frac{\frac{{\omega_n}^2}{k} I e^{-t_0 s}}{(s^2 + 2\zeta\omega_n s + {\omega_n}^2)} + \frac{u_0 s}{(s^2 + 2\zeta\omega_n s + {\omega_n}^2)} + \frac{2\zeta\omega_n u_0 + \dot{u}_0}{(s^2 + 2\zeta\omega_n s + {\omega_n}^2)}$$

Since the function in terms of the roots is:

$$s^2 + 2\zeta\omega_n s + {\omega_n}^2 = (s-a)(s-b)$$

$$a, b = -\zeta\omega_n \pm i\sqrt{1-\zeta^2}\,\omega_n$$

Substitute:

$$U_{(s)} = \frac{e^{-t_0 s}}{(s-a)(s-b)}\left(\frac{{\omega_n}^2}{k} I\right) + \frac{s}{(s-a)(s-b)}(u_0) + \frac{1}{(s-a)(s-b)}(2\zeta\omega_n u_0 + \dot{u}_0)$$

Since (using partial fractions)

$$F_{(s)} = \frac{1}{(s-a)(s-b)} = \frac{\frac{1}{a-b}}{(s-a)} - \frac{\frac{1}{a-b}}{(s-b)}$$

The Laplace inverse is:

$$f_{(t)} = \mathcal{L}^{-1}\big(F_{(s)}\big) = \frac{1}{a-b}e^{at} - \frac{1}{a-b}e^{bt} = \frac{e^{at} - e^{bt}}{a-b}$$

Substitute:

$$U_{(s)} = \frac{e^{-t_0 s}}{(s-a)(s-b)}\left(\frac{{\omega_n}^2}{k}I\right) + \frac{s}{(s-a)(s-b)}(u_0) + \frac{1}{(s-a)(s-b)}(2\zeta\omega_n u_0 + \dot{u}_0)$$

$$u_{(t)} = \mathcal{L}^{-1}\left(F_{(s)}e^{-t_0 s}\left(\frac{{\omega_n}^2}{k}I\right) + F_{(s)}s(u_0) + F_{(s)}(2\zeta\omega_n u_0 + \dot{u}_0)\right)$$

$$u_{(t)} = f_{(t-t_0)}\cup_{(t-t_0)}\left(\frac{{\omega_n}^2}{k}I\right) + \left(\frac{ae^{at}-be^{bt}}{a-b}\right)(u_0) + \left(\frac{e^{at}-e^{bt}}{a-b}\right)(2\zeta\omega_n u_0 + \dot{u}_0)$$

$$u_{(t)} = \begin{cases} (u_0)\left(\frac{ae^{at}-be^{bt}}{a-b}\right) + (2\zeta\omega_n u_0 + \dot{u}_0)\left(\frac{e^{at}-e^{bt}}{a-b}\right), & (0 < t < t_0) \\ \left(\frac{{\omega_n}^2}{k}I\right)\left(\frac{e^{a(t-t_0)}-e^{b(t-t_0)}}{a-b}\right) + (u_0)\left(\frac{ae^{at}-be^{bt}}{a-b}\right) + (2\zeta\omega_n u_0 + \dot{u}_0)\left(\frac{e^{at}-e^{bt}}{a-b}\right), & (t_0 < t) \end{cases}$$

where:

$$a, b = -\zeta\omega_n \pm i\sqrt{1-\zeta^2}\omega_n$$

$$a - b = 2i\sqrt{1-\zeta^2}\omega_n$$

$$a + b = -2\zeta\omega_n$$

Converting the exponential forms to trigonometry forms using Euler's formulas:

$$\begin{aligned} e^{at} &= e^{\left(-\zeta\omega_n + i\sqrt{1-\zeta^2}\omega_n\right)t} = e^{-\zeta\omega_n t} \times e^{i\sqrt{1-\zeta^2}\omega_n t} \\ &= e^{-\zeta\omega_n t}\left(\cos\sqrt{1-\zeta^2}\omega_n t + i\sin\sqrt{1-\zeta^2}\omega_n t\right) \\ e^{bt} &= e^{\left(-\zeta\omega_n - i\sqrt{1-\zeta^2}\omega_n\right)t} = e^{-\zeta\omega_n t} \times e^{-i\sqrt{1-\zeta^2}\omega_n t} \\ &= e^{-\zeta\omega_n t}\left(\cos\sqrt{1-\zeta^2}\omega_n t - i\sin\sqrt{1-\zeta^2}\omega_n t\right) \end{aligned}$$

Therefore:

$$e^{at} - e^{bt} = e^{-\zeta\omega_n t}\left(2i\sin\sqrt{1-\zeta^2}\omega_n t\right)$$

And:

$$\begin{aligned} ae^{at} - be^{bt} &= ae^{\left(-\zeta\omega_n + i\sqrt{1-\zeta^2}\omega_n\right)t} - be^{\left(-\zeta\omega_n - i\sqrt{1-\zeta^2}\omega_n\right)t} \\ &= e^{-\zeta\omega_n t}\left(ae^{i\sqrt{1-\zeta^2}\omega_n t} - be^{-i\sqrt{1-\zeta^2}\omega_n t}\right) \\ &= e^{-\zeta\omega_n t}\left((a-b)\cos\sqrt{1-\zeta^2}\omega_n t + (a+b)i\sin\sqrt{1-\zeta^2}\omega_n t\right) \end{aligned}$$

Therefore:

$$u_{1(t)} = (u_0)\frac{ae^{at} - be^{bt}}{a-b} + (2\zeta\omega_n u_0 + \dot{u}_0)\frac{e^{at} - e^{bt}}{a-b}$$

$$u_{1(t)} = e^{-\zeta\omega_n t}\left[(u_0)\cos\sqrt{1-\zeta^2}\omega_n t + \left(\frac{\dot{u}_0 + \omega_n\zeta u_0}{\omega_n\sqrt{1-\zeta^2}}\right)\sin\sqrt{1-\zeta^2}\omega_n t\right]$$

Therefore:

$$u_{2(t-t_0)} = \left(\frac{{\omega_n}^2}{k}I\right)\left(\frac{e^{a(t-t_0)} - e^{b(t-t_0)}}{a-b}\right)$$

$$u_{2(t-t_0)} = \frac{I}{k}e^{-\zeta\omega_n(t-t_0)}\left[\frac{\omega_n}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_0)\right]$$

$$\dot{u}_{2(t-t_0)} = \frac{I}{k}{\omega_n}^2 e^{-\zeta\omega_n(t-t_0)}\left[\cos\sqrt{1-\zeta^2}\omega_n(t-t_0) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_0)\right]$$

The product of the impulse by delta Dirac function is a force; because delta Dirac is just a unit impulse force ($f = (1)/\Delta t$) multiplied by the unit step function ($u_{(t-t_0)} - u_{(t-(t_0+\Delta t))}$). If the force is acting for a long period of time ($\Delta t \gg 0$), then the deflection solution by Laplace method is considered an infinitesimal deflection and should be integrated over time as if summing up the deflection from a series of short successive impulses (convolution integral).

A.2.2 Solving for the Triangular Pulse Input

Using the convolution integral method, the solution of the infinitesimal impulse is integrated at start:

$$du_{2(t>t_0)} = \frac{I}{k}e^{-\zeta\omega_n(t-t_0)}\left[\frac{\omega_n}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_0)\right]$$

$$u_{2(t>t_0)} = \frac{1}{k}\int_{\mathrm{T}=t_0}^{\mathrm{T}=t} f_{2(t=\mathrm{T})}d\mathrm{T}.\, e^{-\zeta\omega_n(t-\mathrm{T})}\left[\frac{\omega_n}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right]$$

For $(t_0 < t < t_1),\ t_1 = t_0 + 0.5\Delta t$:

$$f_{2(t>t_0)} = \frac{F_p}{0.5\Delta t}(t-t_0)$$

$$u_{2(t>t_0)} = \frac{F_p/k}{0.5\Delta t}\frac{\omega_n}{\sqrt{1-\zeta^2}}\int_{\mathrm{T}=t_0}^{\mathrm{T}=t}(\mathrm{T}-t_0)\left(\sin\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)d\mathrm{T}$$

Using integration by parts for integrating the product of three functions:

$$\int_{\mathrm{T}=t_0}^{\mathrm{T}=t}(\mathrm{T}-t_0)\left(\sin\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)d\mathrm{T}$$

$$=\Bigg[\left(\frac{\zeta}{\omega_n}\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)\left((\mathrm{T}-t_0)+\frac{1-2\zeta^2}{\zeta\omega_n}\right)\left(\sin\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)$$

$$+\left(\frac{\sqrt{1-\zeta^2}}{\omega_n}\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)\left((\mathrm{T}-t_0)-\frac{2\zeta}{\omega_n}\right)\left(\cos\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)\Bigg]_{\mathrm{T}=t_0}^{\mathrm{T}=t}$$

For $(t_1 < t < t_2),\ t_2 = t_0 + \Delta t$:

$$f_{2(t>t_1)} = -\frac{F_p}{0.5\Delta t}(t-t_2)$$

The solution is the same as before but with changing the force equation constant which has no effect on the integration, therefore:

$$u_{2(t>t_1)} = -\frac{F_p/k}{0.5\Delta t}\frac{\omega_n}{\sqrt{1-\zeta^2}}\int_{\mathrm{T}=t_1}^{\mathrm{T}=t}(\mathrm{T}-t_2)\left(\sin\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)d\mathrm{T}$$

Using integration by parts:

$$\int_{\mathrm{T}=t_1}^{\mathrm{T}=t}(\mathrm{T}-t_2)\left(\sin\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)d\mathrm{T}$$

$$=\Bigg[\left(\frac{\zeta}{\omega_n}\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)\left((\mathrm{T}-t_2)+\frac{1-2\zeta^2}{\zeta\omega_n}\right)\left(\sin\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)$$

$$+\left(\frac{\sqrt{1-\zeta^2}}{\omega_n}\right)\left(e^{-\zeta\omega_n(t-\mathrm{T})}\right)\left((\mathrm{T}-t_2)-\frac{2\zeta}{\omega_n}\right)\left(\cos\sqrt{1-\zeta^2}\omega_n(t-\mathrm{T})\right)\Bigg]_{\mathrm{T}=t_1}^{\mathrm{T}=t}$$

A.2.3 The Total Solution:

$$u_{(y=L,t)} = \begin{cases} u_{1(L,t)}, & (0<t<t_0) \\ u_{1(L,t)} + u_{2(L,t>t_0)}\Big|_{t_0}^{t}, & (t_0<t<t_1) \\ u_{1(L,t)} + u_{2(L,t>t_0)}\Big|_{t_0}^{t} - u_{2(L,t>t_0)}\Big|_{t_1}^{t} + u_{2(L,t>t_1)}\Big|_{t_1}^{t}, & (t_1<t<t_2) \\ u_{1(L,t)} + u_{2(L,t>t_0)}\Big|_{t_0}^{t} - u_{2(L,t>t_0)}\Big|_{t_1}^{t} + u_{2(L,t>t_1)}\Big|_{t_1}^{t} - u_{2(L,t>t_1)}\Big|_{t_2}^{t}, & (t_2<t) \end{cases}$$

$$u_{1(t)} = e^{-\zeta\omega_n t}\left[(u_0)\cos\sqrt{1-\zeta^2}\omega_n t + \left(\frac{\dot{u}_0+\omega_n\zeta u_0}{\omega_n\sqrt{1-\zeta^2}}\right)\sin\sqrt{1-\zeta^2}\omega_n t\right]$$

Using the superposition concept which sums new responses to initial responses, the solution to the initial conditions ($u_{1(t)}$) is replaced by the solution to the decaying input force (as it is considered the solution that represents the cantilever's response state before the triangular pulse takes place).

$$u_{2(L,t>t_0)}\Big|_{t_0}^{t} = \frac{F_p/k}{0.5\Delta t}\left((t-t_0)-\frac{2\zeta}{\omega_n}\right) - \frac{F_p/k}{0.5\Delta t}\left(e^{-\zeta\omega_n(t-t_0)}\right)\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}\omega_n}\sin\sqrt{1-\zeta^2}\omega_n(t-t_0)-\frac{2\zeta}{\omega_n}\cos\sqrt{1-\zeta^2}\omega_n(t-t_0)\right]$$

$$u_{2(L,t>t_0)}\Big|_{t_1}^{t} = \frac{F_p/k}{0.5\Delta t}\left((t-t_0)-\frac{2\zeta}{\omega_n}\right) - \frac{F_p/k}{0.5\Delta t}\left(e^{-\zeta\omega_n(t-t_1)}\right)\left[\left(0.5\Delta t\frac{\zeta}{\sqrt{1-\zeta^2}}+\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}\omega_n}\right)\sin\sqrt{1-\zeta^2}\omega_n(t-t_1) + \left(0.5\Delta t-\frac{2\zeta}{\omega_n}\right)\cos\sqrt{1-\zeta^2}\omega_n(t-t_1)\right]$$

$$u_{2(L,t>t_1)}\Big|_{t_1}^{t} = -\frac{F_p/k}{0.5\Delta t}\left((t-t_2)-\frac{2\zeta}{\omega_n}\right) + \frac{F_p/k}{0.5\Delta t}\left(e^{-\zeta\omega_n(t-t_1)}\right)\left[\left(-0.5\Delta t\frac{\zeta}{\sqrt{1-\zeta^2}}+\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}\omega_n}\right)\sin\sqrt{1-\zeta^2}\omega_n(t-t_1) + \left(-0.5\Delta t-\frac{2\zeta}{\omega_n}\right)\cos\sqrt{1-\zeta^2}\omega_n(t-t_1)\right]$$

$$u_{2(L,t>t_1)}\Big|_{t_2}^{t} = -\frac{F_p/k}{0.5\Delta t}\left((t-t_2)-\frac{2\zeta}{\omega_n}\right) + \frac{F_p/k}{0.5\Delta t}\left(e^{-\zeta\omega_n(t-t_2)}\right)\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}\omega_n}\sin\sqrt{1-\zeta^2}\omega_n(t-t_2)-\frac{2\zeta}{\omega_n}\cos\sqrt{1-\zeta^2}\omega_n(t-t_2)\right]$$

Since the impact duration is too small and less than quarter the periodic time of the cantilever oscillation $\Delta t < 0.25T$, therefore the max deflection response (amplitude) will occur after the impact duration has passed when $t > t_2$ and should be calculated from the final branch equation. Consequently, it is not needed to state all the 4 equations of superposition, just their sum is needed to construct the final branch equation.

$$u_{(L,t>t_2)} = \frac{F_p/k}{0.5\Delta t.\omega_n}\Bigg[-e^{-\zeta\omega_n(t-t_0)}\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_0)-2\zeta\cos\sqrt{1-\zeta^2}\omega_n(t-t_0)\right] + 2e^{-\zeta\omega_n(t-t_1)}\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_1)-2\zeta\cos\sqrt{1-\zeta^2}\omega_n(t-t_1)\right] - e^{-\zeta\omega_n(t-t_2)}\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_2)-2\zeta\cos\sqrt{1-\zeta^2}\omega_n(t-t_2)\right]\Bigg]$$

where:

$$\frac{F_p}{k} = \delta_{tip}.a_p$$

Therefore, the angle of deflection:

$$\theta_{(y=L,t>t_2)} = \left.\frac{du_{(y,t>t_2)}}{dy}\right|_{y=L}$$
$$= \frac{\bar{\theta}_{tip}a_p}{0.5\Delta t.\omega_n}\left\{-\left(e^{-\zeta\omega_n(t-t_0)}\right)\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_0) - 2\zeta\cos\sqrt{1-\zeta^2}\omega_n(t-t_0)\right]\right.$$
$$+ 2\left(e^{-\zeta\omega_n(t-t_1)}\right)\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_1) - 2\zeta\cos\sqrt{1-\zeta^2}\omega_n(t-t_1)\right]$$
$$\left.- \left(e^{-\zeta\omega_n(t-t_2)}\right)\left[\frac{1-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_2) - 2\zeta\cos\sqrt{1-\zeta^2}\omega_n(t-t_2)\right]\right\}$$

Therefore, the angular velocity response:

$$\Omega_{(y=L,t>t_2)} = \frac{d\theta_{(y=L,t>t_2)}}{dt}$$
$$= \frac{\bar{\theta}_{tip}a_p}{0.5\Delta t}\left\{-\left(e^{-\zeta\omega_n(t-t_0)}\right)\left[\cos\sqrt{1-\zeta^2}\omega_n(t-t_0) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_0)\right]\right.$$
$$+ 2\left(e^{-\zeta\omega_n(t-t_1)}\right)\left[\cos\sqrt{1-\zeta^2}\omega_n(t-t_1) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_1)\right]$$
$$\left.- \left(e^{-\zeta\omega_n(t-t_2)}\right)\left[\cos\sqrt{1-\zeta^2}\omega_n(t-t_2) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\omega_n(t-t_2)\right]\right\}$$

A.3 Applying the SDOF Solution to the Cantilever system

Consider a cantilever beam as shown in **Figure A.3**; when the beam is statically analyzed under constant acceleration ($a = gravitational\ acceleration$), the beam self-mass and the attached mass at the cantilever tip induce a static force that causes a static deflection (dy).

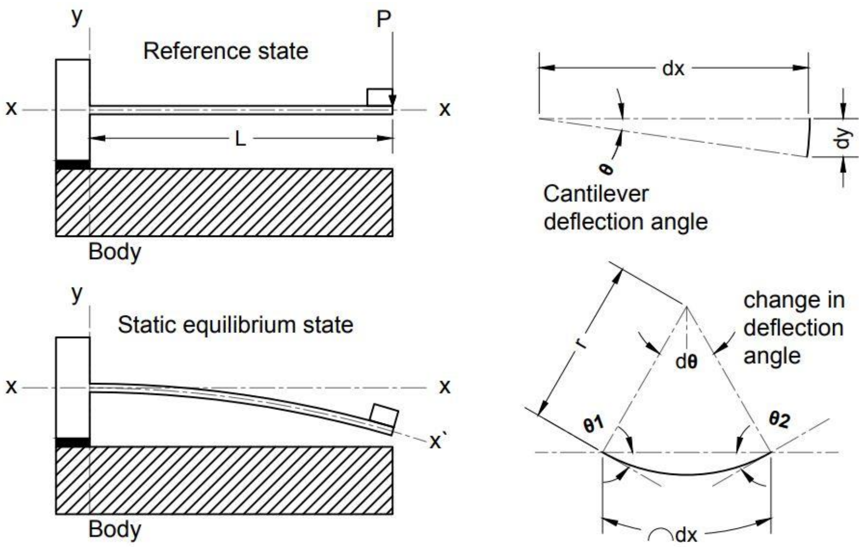

Figure A.3 Schematic diagram showing the change in deflection angle along the length of a cantilever beam

From solid mechanics, as the moment (M) on the beam increases, the radius of curvature (r) decreases; thus, the relationship is inversely proportional. The constant can be determined based on the geometry of the cross-section (I) and the beam's material elasticity modulus (E):

$$M_1 r_1 = M_2 r_2 = const. = EI$$

$$\frac{1}{r} = \frac{M}{EI}$$

where M_1 and M_2 are arbitrary moment values causing curvature radii (r_1) and (r_2), respectively. By referring to **Figure A.3**, the change in the slope of deflection is $d\theta = dx/r$; thus, the slope of deflection ($\theta_{(x)}$) can be related to the moment:

$$\theta_{(x)} = \int \frac{1}{r} dx = \int \frac{M_{(x)}}{EI} dx$$

Therefore, the deflection ($y_{(x)}$) can be obtained:

$$\frac{dy_{(x)}}{dx} = \tan \theta_{(x)} \approx \theta_{(x)} \ [rad]$$

$$y_{(x)} = \int \theta_{(x)} \, dx = \int \int \frac{M_{(x)}}{EI} dx^2$$

where $M_{(x)}$ is the total moment as a function of x and can be calculated using the superposition method by summing up the moment from the gyroscope mass ($M_{1(x)}$) and the moment from the cantilever self-mass ($M_{2(x)}$). Assuming a constant acceleration (a) is acting on the masses perpendicular to the x-axis, the resulting moment ($M_{(x)}$) can be calculated as:

$$M_{(x)} = M_{1(x)} + M_{2(x)}$$

For the moment resulting from the gyroscope mass ($M_{1(x)}$):

$$M_{1(x)} = -m_1 aL + m_1 ax$$

$$M_{1(x)} = m_1 a(x - L)$$

Thus, the angle of deflection due to the gyroscope mass ($\theta_{1(x)}$) can be calculated as the first integral with respect to dx:

$$\theta_{1(x)} = \int \frac{M_{1(x)}}{EI} dx = \frac{m_1 a}{EI} \int (x - L)\, dx$$

$$\theta_{1(x)} = \frac{m_1 a}{EI} \left(\frac{x^2}{2} - Lx + c_1 \right)$$

Note that at $x = 0$, due to fixed support, $\theta_{1(x)} = 0$; therefore, $c_1 = 0$. For the moment resulting from the cantilever self-mass ($M_{2(x)}$):

$$M_{2(x)} = \frac{-m_2 aL^2}{2} + m_2 aLx - \frac{m_2 ax^2}{2}$$

$$M_{2(x)} = -\frac{m_2 a}{2} (\, x^2 - 2Lx + L^2)$$

Thus, the angle of deflection due to the cantilever self-mass ($\theta_{2(x)}$) can be calculated as the first integral with respect to dx:

$$\theta_{2(x)} = \int \frac{M_{2(x)}}{EI} dx = -\frac{m_2 a}{2EI} \int (x^2 - 2Lx + L^2) dx$$

$$\theta_{2(x)} = -\frac{m_2 a}{2EI} \left(\frac{x^3}{3} - Lx^2 + L^2 x + c_2 \right)$$

Note that at $x = 0$, due to fixed support, $\theta_{2(x)} = 0$; therefore, $c_2 = 0$. Therefore, the total cumulative angle of deflection ($\theta_{(x)}$) measured from the fixed end towards any position along the cantilever length is given as the sum:

$$\theta_{(x)} = \theta_{1(x)} + \theta_{2(x)}$$

$$\theta_{(x)} = \frac{m_1 a}{EI} \left(\frac{x^2}{2} - Lx \right) - \frac{m_2 a}{2EI} \left(\frac{x^3}{3} - Lx^2 + L^2 x \right)$$

$$\theta_{(x)} = \frac{a}{EI} \left(m_1 \left(\frac{x^2}{2} - Lx \right) - \frac{m_2}{2} \left(\frac{x^3}{3} - Lx^2 + L^2 x \right) \right)$$

From Equation (4), the deflection ($y_{(x)}$) at any position can be calculated by integrating the total angle of deflection ($\theta_{(x)}$):

$$y_{(x)} = \int \theta_{(x)}\, dx = \frac{a}{EI} \int \left(m_1 \left(\frac{x^2}{2} - Lx \right) - \frac{m_2}{2} \left(\frac{x^3}{3} - Lx^2 + L^2 x \right) \right) dx$$

$$y_{(x)} = \frac{a}{EI}\left(m_1\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) - \frac{m_2}{2}\left(\frac{x^4}{12} - \frac{Lx^3}{3} + \frac{L^2x^2}{2}\right) + c_3\right)$$

Note that at $x = 0$, due to fixed support, $y = 0$; therefore, $c_3 = 0$. Therefore, the angle of deflection and the deflection at the cantilever tip ($x = L$) can be evaluated:

$$\theta_{\mathrm{tip}} = \theta_{(x=L)} = -\frac{L^2}{EI}\left(\frac{m_1}{2} + \frac{\overline{m}_2 L}{6}\right) a$$

$$y_{\mathrm{tip}} = y_{(x=L)} = -\frac{L^3}{EI}\left(\frac{m_1}{3} + \frac{\overline{m}_2 L}{8}\right) a = -\delta_{tip}.a$$

Then, δ_{tip} becomes the deflection due to a unit acceleration. So, the spring stiffness (k) (the force required to generate a unit deflection) at the cantilever tip can be calculated:

$$u_{tip} = \frac{P_{tip}}{k_{tip}} = \frac{m_{total}.a}{k_{tip}} = y_{\mathrm{tip}} = \delta_{tip}[\frac{mm}{mm/s^2}].a[\frac{mm}{s^2}] = [mm]$$

$$\frac{m_{total}}{k_{tip}} = \delta_{tip} = \frac{m_1}{\frac{3EI}{L^3}} + \frac{\overline{m}_2 L}{\frac{8EI}{L^3}}$$

Therefore, to apply the deflection response equation of the mass-spring system to the cantilever system, we replace:

$$\frac{P}{k} = \delta_{tip}.a_i$$

Notation

The following symbols are used in this appendix:

a_i	Acceleration amplitude of the SH mass-spring system;
a_p	Peak acceleration due to the input triangular pulse;
c	Viscous damping coefficient;
E	Elasticity modulus of the cantilever material;
F_i	Force amplitude of the hammer mass-spring system;
F_p, P	Peak force of the impact;
f_n	Natural frequency;
I	Moment of inertia of the cantilever cross-section;
j	Input jerk;
k	Cantilever spring constant;
L	Cantilever length;
M	Moment at the cantilever fixed end;
m_1	Gyroscope mass;
$\bar{m}_2$	Cantilever mass per unit length;
m_3	SH mass including cantilever mass;
t	Time;
t_0	Impact instant;
Δt	Impact duration;
$u_{(t)}$	Deflection response of the cantilever tip;
u_0	Initial cantilever deflection;
$\dot{u}_0$	Initial cantilever velocity;
β	Ratio of the mass-spring frequency to the cantilever frequency;
δ_{tip}	Deflection of the cantilever tip per unit acceleration;
ζ	Damping coefficient of the cantilever material;
ζ_i	Damping coefficient of SH mass-spring system;
θ_{tip}	Angle of deflection of the cantilever tip;
ω_i	Natural frequency of the mass spring system in rad;
ω_{nc}	Natural frequency of the cantilever in rad;