

An incomplete closed form of the gamma-Pareto convolution

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This is a proof of Eq. (25) of the text of *Comparison of the gamma-Pareto convolution with conventional methods of characterising metformin pharmacokinetics in dogs*. All of the parameters are non-negative reals, and some of them are definite positive (e.g., β). n is a non-negative integer index and not a parameter. Regions of convergence are not presented and are left to the reader.

Given a gamma-Pareto type I convolution, Eq. (21) of the text,

$$\begin{aligned} \text{GPC} \left(\begin{array}{c} a \quad b \\ \alpha \quad \beta \end{array} \middle| t \right) &= \text{GD}(a, b; x) * \text{PD}(\alpha, \beta; x) (t) \\ &= \theta(t - \beta) \frac{\alpha b^a \beta^\alpha}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} t^{a-\alpha+n-1} B_{1-\frac{\beta}{t}}(a+n, -\alpha) \end{aligned} \quad (1)$$

For the following identity

$$B_z(p, q) = \frac{z^p}{p} {}_2F_1(p, 1 - q; p + 1; z) , \quad (2)$$

we substitute $z = 1 - \frac{\beta}{t}$, $p = a + n$, $q = -\alpha$ leading to

$$B_{1-\frac{\beta}{t}}(a+n, -\alpha) = \frac{\left(1 - \frac{\beta}{t}\right)^{a+n}}{a+n} {}_2F_1\left(a+n, \alpha+1; a+n+1; 1 - \frac{\beta}{t}\right) . \quad (3)$$

Substituting this into Eq. (1)

$$\begin{aligned} \theta(t - \beta) \frac{\alpha b^a \beta^\alpha}{\Gamma(a)} \\ \sum_{n=0}^{\infty} \frac{(-b)^n t^{a-\alpha+n-1} \left(1 - \frac{\beta}{t}\right)^{a+n}}{n!(a+n)} {}_2F_1\left(\alpha+1, a+n; a+n+1; 1 - \frac{\beta}{t}\right) \end{aligned} \quad (4)$$

The Taylor series expansion around ∞ of the summand of which leads to

$$\begin{aligned} \theta(t - \beta) \frac{\alpha b^a \beta^\alpha}{\Gamma(a)} \\ \sum_{n=0}^{\infty} t^{a-\alpha+n} \left[-\frac{\pi(-b)^n \csc(\pi\alpha) \Gamma(a+n)}{t \Gamma(\alpha+1) n! \Gamma(a+n-\alpha)} - \frac{(-b)^n \beta^{-\alpha-1} t^\alpha}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\beta}{t}\right)^{m+1} (a-m+n)_m}{m! (\alpha-m)} \right] , \end{aligned} \quad (5)$$

which is the infinite sum of a single term plus a second infinite series, where the second series can be expressed in closed form as a single term

$$\sum_{m=0}^{\infty} -\frac{\left(-\frac{\beta}{t}\right)^{m+1} (a-m+n)_m}{m!(\alpha-m)} = \frac{\beta}{\alpha t} {}_2F_1\left(-a-n+1, -\alpha; 1-\alpha; \frac{\beta}{t}\right),$$

Moreover, from (3), ${}_2F_1(p, 1-q; p+1; z) = \frac{p}{z^p} B_z(p, q)$, if we let $z = \frac{\beta}{t}$, $p = -\alpha$, and $q = a+n$, then

$${}_2F_1\left(-a-n+1, -\alpha; 1-\alpha; \frac{\beta}{t}\right) = \alpha \left(\frac{\beta}{t}\right)^{\alpha} B_{\frac{\beta}{t}}(-\alpha, a+n),$$

which allows to rewrite Eq. (5) as

$$\begin{aligned} \theta(t-\beta) \frac{\alpha b^a \beta^{\alpha}}{\Gamma(a)} \\ \sum_{n=0}^{\infty} t^{a-\alpha+n} \left[-\frac{\pi(-b)^n \csc(\pi\alpha) \Gamma(a+n)}{t \Gamma(\alpha+1) n! \Gamma(a+n-\alpha)} + \frac{(-b)^n}{t n!} B_{\frac{\beta}{t}}(-\alpha, a+n) \right] \end{aligned} \quad (6)$$

Finally, note that the left hand term of this equation simplifies as

$$\begin{aligned} \theta(t-\beta) \frac{\alpha b^a \beta^{\alpha}}{\Gamma(a)} \sum_{n=0}^{\infty} t^{a-\alpha+n} \left[-\frac{\pi(-b)^n \csc(\pi\alpha) \Gamma(a+n)}{t \Gamma(\alpha+1) n! \Gamma(a+n-\alpha)} \right] \\ = -\theta(t-\beta) \pi \csc(\pi\alpha) \frac{b^a \beta^{\alpha}}{\Gamma(\alpha)} t^{a-\alpha-1} {}_1\tilde{F}_1(a, a-\alpha; -bt) \\ = \theta(t-\beta) \alpha b^a \beta^{\alpha} \Gamma(-\alpha) t^{a-\alpha-1} {}_1\tilde{F}_1(a; a-\alpha; -bt), \end{aligned} \quad (7)$$

where the latter simplification is from $-\pi \csc(\pi\alpha) = \Gamma(-\alpha) \Gamma(\alpha+1) = \alpha \Gamma(-\alpha) \Gamma(\alpha)$. Substituting the Eq. (7) result into Eq. (6) yields

$$\begin{aligned} \text{GPC} \left(\begin{matrix} a & b \\ \alpha & \beta \end{matrix} \middle| t \right) = -\theta(t-\beta) \frac{\alpha b^a \beta^{\alpha}}{\Gamma(a)} t^{a-\alpha-1} \sum_{n=0}^{\infty} \frac{(-bt)^n}{n!} B_{\frac{\beta}{t}}(-\alpha, a+n) \\ + \theta(t-\beta) \alpha b^a \beta^{\alpha} \Gamma(-\alpha) t^{a-\alpha-1} {}_1\tilde{F}_1(a; a-\alpha; -bt), \end{aligned} \quad (8)$$

which is Eq. (25) of the main text.