

Price Competition in a Vertizontally Differentiated Duopoly

Symmetric situation: centralized and decentralized pricing

Equilibrium (analysis) when firms price qualities “centralized”

```
In[1]:= Clear["Global`*"]
$Assumptions =
  { $\beta > 0$ ,  $\delta > 0$ ,  $\tau_h > 0$ ,  $\tau_l > 0$ ,  $\mu_h > 0$ ,  $\mu_l > 0$ ,  $\mu > 0$ ,  $ch > 0$ ,  $cl > 0$ ,  $ch > cl$ ,  $\delta > ch - cl$ };
AssumptionOne = { $ch + \tau_h > cl + \tau_l$ ,  $ch + \tau_h < cl + \tau_l + \delta$ };
AssumptionTwo = { $ch + 2 * \tau_h > cl + 2 * \tau_l$ ,  $ch + 2 * \tau_h < cl + 2 * \tau_l + \delta$ };
```

For given prices, compute the marginal agents on the segments (x and y), the demand for each brand-quality (q), and the resulting profit for each brand-quality (π)

```
In[5]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_h * xx + p1h == \tau_h * (1 - xx) + p2h$ , xx][[1, 1, 2]];
xl[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_l * xx + p1l == \tau_l * (1 - xx) + p2l$ , xx][[1, 1, 2]];
y1[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p1h == \beta - p1l$ , yy][[1, 1, 2]];
y2[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p2h == \beta - p2l$ , yy][[1, 1, 2]];
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] *  $\mu_h + (1 - y1[p1h, p1l, p2h, p2l]) * \mu$ ;
q1l[p1h_, p1l_, p2h_, p2l_] = xl[p1h, p1l, p2h, p2l] *  $\mu_l + y1[p1h, p1l, p2h, p2l] * \mu$ ;
q2h[p1h_, p1l_, p2h_, p2l_] =
  (1 - xh[p1h, p1l, p2h, p2l]) *  $\mu_h + (1 - y2[p1h, p1l, p2h, p2l]) * \mu$ ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - xl[p1h, p1l, p2h, p2l]) *  $\mu_l + y2[p1h, p1l, p2h, p2l] * \mu$ ;
 $\pi_{1h}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1h - ch) * q1h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{1l}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1l - cl) * q1l[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2h}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2h - ch) * q2h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2l}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2l - cl) * q2l[p1h, p1l, p2h, p2l]]$ ;
```

Solve for equilibrium prices (second-order conditions are dealt with in the paper)

```
In[17]:= FullSimplify@Solve[{D[ $\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ ,  $p_{1h}] == 0$ ,
  D[ $\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ ,  $p_{1l}] == 0$ ,
  D[ $\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ ,  $p_{2h}] == 0$ ,
  D[ $\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ ,  $p_{2l}] == 0$ }, { $p_{1h}, p_{1l}, p_{2h}, p_{2l}$ }]

Out[17]= {{ $p_{1h} \rightarrow \frac{(\delta \mu_h \mu_l + (2 (ch + cl + \delta) \mu + \delta \mu_h) \mu_l \tau_h + 4 \mu (ch \mu_h + (2 \mu + \mu_h + \mu_l) \tau_h) \tau_l)}{(\delta \mu_h \mu_l + 4 \mu (\mu_l \tau_h + \mu_h \tau_l))}$ ,
   $p_{1l} \rightarrow \frac{(cl \mu_l (\delta \mu_h + 4 \mu \tau_h) + (2 (ch + cl) \mu \mu_h + \delta \mu_h \mu_l + 4 \mu (2 \mu + \mu_h + \mu_l) \tau_h) \tau_l)}{(\delta \mu_h \mu_l + 4 \mu (\mu_l \tau_h + \mu_h \tau_l))}$ ,
   $p_{2h} \rightarrow \frac{(ch \delta \mu_h \mu_l + (2 (ch + cl + \delta) \mu + \delta \mu_h) \mu_l \tau_h + 4 \mu (ch \mu_h + (2 \mu + \mu_h + \mu_l) \tau_h) \tau_l)}{(\delta \mu_h \mu_l + 4 \mu (\mu_l \tau_h + \mu_h \tau_l))}$ ,
   $p_{2l} \rightarrow \frac{(cl \mu_l (\delta \mu_h + 4 \mu \tau_h) + (2 (ch + cl) \mu \mu_h + \delta \mu_h \mu_l + 4 \mu (2 \mu + \mu_h + \mu_l) \tau_h) \tau_l)}{(\delta \mu_h \mu_l + 4 \mu (\mu_l \tau_h + \mu_h \tau_l))}}$ }}
```

(Symmetric) equilibrium prices (to be used for further analysis)

```
In[18]:= ph = Simplify[( $ch \delta \mu_h \mu_l + (2 (ch + cl + \delta) \mu + \delta \mu_h) \mu_l \tau_h + 4 \mu (ch \mu_h + (2 \mu + \mu_h + \mu_l) \tau_h) \tau_l$ ) /
  ( $\delta \mu_h \mu_l + 4 \mu (\mu_l \tau_h + \mu_h \tau_l)$ )] ;
pl = Simplify[( $cl \mu_l (\delta \mu_h + 4 \mu \tau_h) + (2 (ch + cl) \mu \mu_h + \delta \mu_h \mu_l + 4 \mu (2 \mu + \mu_h + \mu_l) \tau_h) \tau_l$ ) /
  ( $\delta \mu_h \mu_l + 4 \mu (\mu_l \tau_h + \mu_h \tau_l)$ )] ;
```

Static properties of the prices

```
In[20]:= FullSimplify@Reduce[p1 > cl && And @@ $Assumptions]
FullSimplify@Reduce[ph > ch && And @@ $Assumptions]
Assuming[And @@ AssumptionOne, FullSimplify@Reduce[ph - pl > 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionOne, FullSimplify@Reduce[ph - pl < 0 && And @@ $Assumptions]]
FullSimplify@Limit[ph,  $\mu \rightarrow 0$ ]
FullSimplify@Limit[pl,  $\mu \rightarrow 0$ ]
```

Out[20]= True

Out[21]= True

Out[22]= True

Out[23]= True

Out[24]= $ch + \tau_h$

Out[25]= $cl + \tau_l$

Comparative statics of prices

```

In[26]:= FullSimplify@Reduce[D[ph, ch] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[pl, ch] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ph, cl] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[pl, cl] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ph, ch] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[pl, ch] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ph, cl] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[pl, cl] > 0 && And @@ $Assumptions]
Assuming[And @@ AssumptionTwo,
  FullSimplify@Reduce[D[ph,  $\mu$ h] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[pl,  $\mu$ h] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[ph,  $\mu$ l] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[pl,  $\mu$ l] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[ph,  $\delta$ ] > 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[pl,  $\delta$ ] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[ph,  $\mu$ ] > 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[pl,  $\mu$ ] > 0 && And @@ $Assumptions]]

Out[26]= True
Out[27]= True
Out[28]= True
Out[29]= True
Out[30]= True
Out[31]= True
Out[32]= True
Out[33]= True
Out[34]= True
Out[35]= True
Out[36]= True
Out[37]= True
Out[38]= True
Out[39]= True
Out[40]= True
Out[41]= True

```

Comparative statics of demand

```
In[42]:= y = Simplify[y1[ph, pl, ph, pl]] ;
Assuming[And @@ AssumptionOne, FullSimplify@Reduce[D[y, ch] > 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionOne, FullSimplify@Reduce[D[y, cl] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionOne, FullSimplify@Reduce[D[y, th] > 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionOne, FullSimplify@Reduce[D[y, tl] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[y, mh] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[y, ml] > 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionOne, FullSimplify@Reduce[D[y, d] < 0 && And @@ $Assumptions]]
Assuming[And @@ AssumptionTwo, FullSimplify@Reduce[D[y, mu] > 0 && And @@ $Assumptions]]
```

Out[43]= True

Out[44]= True

Out[45]= True

Out[46]= True

Out[47]= True

Out[48]= True

Out[49]= True

Out[50]=
$$\mu h < \frac{\mu l \tau h (-ch + cl + \delta - 2 \tau h + 2 \tau l)}{(ch - cl + 2 \tau h - 2 \tau l) \tau l}$$

Comparative statics of profits (using parameter restriction)

```
In[51]:=  $\mu_1 = \mu_h$  ;
 $\tau_1 = \tau_h$  ;
 $\pi_h = \text{Simplify}[\pi_{1h}[p_h, p_1, p_h, p_1]]$  ;
 $\pi_1 = \text{Simplify}[\pi_{11}[p_h, p_1, p_h, p_1]]$  ;
 $\pi_t = \text{Simplify}[\pi_h + \pi_1]$  ;
FullSimplify@Reduce[D[ $\pi_h$ , c1] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ $\pi_h$ , ch] < 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ $\pi_1$ , c1] < 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ $\pi_1$ , ch] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ $\pi_t$ , c1] > 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ $\pi_t$ , ch] < 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ $\pi_t$ , c1] < 0 && And @@ $Assumptions]
FullSimplify@Reduce[D[ $\pi_t$ , ch] > 0 && And @@ $Assumptions]
```

Out[56]= True

Out[57]= True

Out[58]= True

Out[59]= True

Out[60]= $2 c_1 + \delta > 2 c_h$

Out[61]= $2 c_1 + \delta > 2 c_h$

Out[62]= $2 c_h > 2 c_1 + \delta$

Out[63]= $2 c_h > 2 c_1 + \delta$

Equilibrium (analysis) when firms price qualities “decentralized”

```
In[64]:= Clear["Global`*"]
$Assumptions =
  { $\beta > 0$ ,  $\delta > 0$ ,  $\tau_h > 0$ ,  $\tau_1 > 0$ ,  $\mu_h > 0$ ,  $\mu_1 > 0$ ,  $\mu > 0$ ,  $c_h > 0$ ,  $c_1 > 0$ ,  $c_h > c_1$ ,  $\delta > c_h - c_1$ } ;
AssumptionOne = { $c_h + \tau_h > c_1 + \tau_1$ ,  $c_h + \tau_h < c_1 + \tau_1 + \delta$ } ;

In[67]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_h * xx + p1h == \tau_h * (1 - xx) + p2h$ , xx][[1, 1, 2]] ;
x1[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_1 * xx + p1l == \tau_1 * (1 - xx) + p2l$ , xx][[1, 1, 2]] ;
y1[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p1h == \beta - p1l$ , yy][[1, 1, 2]] ;
y2[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p2h == \beta - p2l$ , yy][[1, 1, 2]] ;
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] *  $\mu_h + (1 - y1[p1h, p1l, p2h, p2l]) * \mu$  ;
q1l[p1h_, p1l_, p2h_, p2l_] = x1[p1h, p1l, p2h, p2l] *  $\mu_1 + y1[p1h, p1l, p2h, p2l] * \mu$  ;
q2h[p1h_, p1l_, p2h_, p2l_] =
  (1 - xh[p1h, p1l, p2h, p2l]) *  $\mu_h + (1 - y2[p1h, p1l, p2h, p2l]) * \mu$  ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - x1[p1h, p1l, p2h, p2l]) *  $\mu_1 + y2[p1h, p1l, p2h, p2l] * \mu$  ;
 $\pi_{1h}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p1h - c_h) * q1h[p1h, p1l, p2h, p2l]]$  ;
 $\pi_{1l}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p1l - c_1) * q1l[p1h, p1l, p2h, p2l]]$  ;
 $\pi_{2h}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p2h - c_h) * q2h[p1h, p1l, p2h, p2l]]$  ;
 $\pi_{2l}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p2l - c_1) * q2l[p1h, p1l, p2h, p2l]]$  ;
```

Solve for equilibrium prices (second-order conditions are dealt with in the paper)

```
In[79]:= FullSimplify@Solve[{D[ $\pi_1 h[p_1 h, p_1 l, p_2 h, p_2 l]$ ,  $p_1 h$ ] == 0,
  D[ $\pi_1 l[p_1 h, p_1 l, p_2 h, p_2 l]$ ,  $p_1 l$ ] == 0, D[ $\pi_2 h[p_1 h, p_1 l, p_2 h, p_2 l]$ ,  $p_2 h$ ] == 0,
  D[ $\pi_2 l[p_1 h, p_1 l, p_2 h, p_2 l]$ ,  $p_2 l$ ] == 0}, { $p_1 h, p_1 l, p_2 h, p_2 l$ }]
```

```
Out[79]:= {{ $p_1 h \rightarrow (\delta \mu l (ch \delta \mu h + 2 (ch + cl + \delta) \mu \tau h + \delta \mu h \tau h) +$ 
   $2 \mu (2 ch \delta \mu h + (4 ch \mu + 2 cl \mu + \delta (4 \mu + 2 \mu h + \mu l)) \tau h) \tau l) /$ 
   $(\delta \mu l (\delta \mu h + 4 \mu \tau h) + 4 \mu (\delta \mu h + 3 \mu \tau h) \tau l),$ 
   $p_1 l \rightarrow (cl \delta \mu l (\delta \mu h + 4 \mu \tau h) + \delta \mu h (2 (ch + cl) \mu + \delta \mu l) \tau l +$ 
   $2 \mu (2 ch \mu + 4 cl \mu + \delta (2 \mu + \mu h + 2 \mu l)) \tau h \tau l) /$ 
   $(\delta \mu l (\delta \mu h + 4 \mu \tau h) + 4 \mu (\delta \mu h + 3 \mu \tau h) \tau l),$ 
   $p_2 h \rightarrow (\delta \mu l (ch \delta \mu h + 2 (ch + cl + \delta) \mu \tau h + \delta \mu h \tau h) +$ 
   $2 \mu (2 ch \delta \mu h + (4 ch \mu + 2 cl \mu + \delta (4 \mu + 2 \mu h + \mu l)) \tau h) \tau l) /$ 
   $(\delta \mu l (\delta \mu h + 4 \mu \tau h) + 4 \mu (\delta \mu h + 3 \mu \tau h) \tau l),$ 
   $p_2 l \rightarrow (cl \delta \mu l (\delta \mu h + 4 \mu \tau h) + \delta \mu h (2 (ch + cl) \mu + \delta \mu l) \tau l +$ 
   $2 \mu (2 ch \mu + 4 cl \mu + \delta (2 \mu + \mu h + 2 \mu l)) \tau h \tau l) /$ 
   $(\delta \mu l (\delta \mu h + 4 \mu \tau h) + 4 \mu (\delta \mu h + 3 \mu \tau h) \tau l)}}$ 
```

(Symmetric) equilibrium prices (to be used for further analysis)

```
In[80]:= phd = Simplify[( $\delta \mu l (ch \delta \mu h + 2 (ch + cl + \delta) \mu \tau h + \delta \mu h \tau h) +$ 
   $2 \mu (2 ch \delta \mu h + (4 ch \mu + 2 cl \mu + \delta (4 \mu + 2 \mu h + \mu l)) \tau h) \tau l) /$ 
   $(\delta \mu l (\delta \mu h + 4 \mu \tau h) + 4 \mu (\delta \mu h + 3 \mu \tau h) \tau l)] ;$ 
  pld = Simplify[( $cl \delta \mu l (\delta \mu h + 4 \mu \tau h) + \delta \mu h (2 (ch + cl) \mu + \delta \mu l) \tau l +$ 
   $2 \mu (2 ch \mu + 4 cl \mu + \delta (2 \mu + \mu h + 2 \mu l)) \tau h \tau l) /$ 
   $(\delta \mu l (\delta \mu h + 4 \mu \tau h) + 4 \mu (\delta \mu h + 3 \mu \tau h) \tau l)] ;$ 
```

Static properties of the prices

```
In[82]:= Assuming[And@@AssumptionOne, FullSimplify@Reduce[pld > cl && And@@$Assumptions]]
Assuming[And@@AssumptionOne, FullSimplify@Reduce[phd > ch && And@@$Assumptions]]
Assuming[And@@AssumptionOne &&  $\delta \geq \tau h$  &&  $\delta \geq \tau l$ ,
  FullSimplify@Reduce[phd > pld && And@@$Assumptions]]
Assuming[And@@AssumptionOne &&  $\delta \geq \tau h$  &&  $\delta \geq \tau l$ ,
  FullSimplify@Reduce[phd - pld <  $\delta$  && And@@$Assumptions]]
```

```
Out[82]= True
```

```
Out[83]= True
```

```
Out[84]= True
```

```
Out[85]= True
```

Comparing centralized equilibrium with decentralized pricing equilibrium

```
In[86]:= Clear["Global`*"]
$Assumptions =
  { $\beta > 0, \delta > 0, \tau h > 0, \tau l > 0, \mu h > 0, \mu l > 0, \mu > 0, ch > 0, cl > 0, ch > cl, \delta > ch - cl$ };
```

```

In[88]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[τh * xx + p1h == τh * (1 - xx) + p2h, xx][[1, 1, 2]];
xl[p1h_, p1l_, p2h_, p2l_] = Solve[τl * xx + p1l == τl * (1 - xx) + p2l, xx][[1, 1, 2]];
y1[p1h_, p1l_, p2h_, p2l_] = Solve[β + δ * yy - p1h == β - p1l, yy][[1, 1, 2]];
y2[p1h_, p1l_, p2h_, p2l_] = Solve[β + δ * yy - p2h == β - p2l, yy][[1, 1, 2]];
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] * μh + (1 - y1[p1h, p1l, p2h, p2l]) * μ;
q1l[p1h_, p1l_, p2h_, p2l_] = xl[p1h, p1l, p2h, p2l] * μl + y1[p1h, p1l, p2h, p2l] * μ;
q2h[p1h_, p1l_, p2h_, p2l_] =
    (1 - xh[p1h, p1l, p2h, p2l]) * μh + (1 - y2[p1h, p1l, p2h, p2l]) * μ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - xl[p1h, p1l, p2h, p2l]) * μl + y2[p1h, p1l, p2h, p2l] * μ;
π1h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1h - ch) * q1h[p1h, p1l, p2h, p2l]];
π1l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1l - cl) * q1l[p1h, p1l, p2h, p2l]];
π2h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2h - ch) * q2h[p1h, p1l, p2h, p2l]];
π2l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2l - cl) * q2l[p1h, p1l, p2h, p2l]];

```

```

In[100]:= μl = 1;
μh = 1;
μ = 1;
τl = τh;
phc =
    Simplify[(ch δ μh μl + (2 (ch + cl + δ) μ + δ μh) μl τh + 4 μ (ch μh + (2 μ + μh + μl) τh) τl) /
        (δ μh μl + 4 μ (μl τh + μh τl))];
plc = Simplify[(cl μl (δ μh + 4 μ τh) + (2 (ch + cl) μ μh + δ μh μl + 4 μ (2 μ + μh + μl) τh) τl) /
        (δ μh μl + 4 μ (μl τh + μh τl))];
phd = Simplify[(δ μl (ch δ μh + 2 (ch + cl + δ) μ τh + δ μh τh) +
        2 μ (2 ch δ μh + (4 ch μ + 2 cl μ + δ (4 μ + 2 μh + μl)) τh) τl) /
        (δ μl (δ μh + 4 μ τh) + 4 μ (δ μh + 3 μ τh) τl)];
pld = Simplify[(cl δ μl (δ μh + 4 μ τh) + δ μh (2 (ch + cl) μ + δ μl) τl +
        2 μ (2 ch μ + 4 cl μ + δ (2 μ + μh + 2 μl)) τh τl) /
        (δ μl (δ μh + 4 μ τh) + 4 μ (δ μh + 3 μ τh) τl)];
yc = Simplify[y1[phc, plc, phc, plc]];
πhc = Simplify[π1h[phc, plc, phc, plc]];
πlc = Simplify[π1l[phc, plc, phc, plc]];
πtc = Simplify[πhc + πlc];
yd = Simplify[y1[phd, pld, phd, pld]];
πhd = Simplify[π1h[phd, pld, phd, pld]];
πld = Simplify[π1l[phd, pld, phd, pld]];
πtd = Simplify[πhd + πld];

```

```

In[116]:= FullSimplify@Reduce[phd < phc && And @@ $Assumptions]
FullSimplify@Reduce[pld < plc && And @@ $Assumptions]
FullSimplify@Reduce[πhd < πhc && And @@ $Assumptions]
FullSimplify@Reduce[πld < πlc && And @@ $Assumptions]
FullSimplify@Reduce[πtd < πtc && And @@ $Assumptions]

```

Out[116]= True

Out[117]= True

Out[118]= True

Out[119]= True

Out[120]= True

```

In[121]:= FullSimplify@Reduce[yd < yc && And @@ $Assumptions]
FullSimplify@Reduce[yd > yc && And @@ $Assumptions]

Out[121]= 2 ch > 2 c1 +  $\delta$ 

Out[122]= 2 c1 +  $\delta$  > 2 ch

In[123]:= welldiff = Simplify[( $\delta * (yd + yc) - 2 * (ch - c1) * (yc - yd)$ )] ;
FullSimplify@Reduce[welldiff < 0 && And @@ $Assumptions]
FullSimplify@Reduce[welldiff == 0 && And @@ $Assumptions]

Out[124]= 2 ch  $\neq$  2 c1 +  $\delta$ 

Out[125]= 2 ch == 2 c1 +  $\delta$ 

```

Equilibrium when one firm prices centralized and the other decentralized

```

In[126]:= Clear["Global`*"]
$Assumptions =
  { $\beta > 0$ ,  $\delta > 0$ ,  $\tau_h > 0$ ,  $\tau_l > 0$ ,  $\mu_h > 0$ ,  $\mu_l > 0$ ,  $\mu > 0$ ,  $ch > 0$ ,  $c1 \geq 0$ ,  $ch > c1$ ,  $\delta > ch - c1$ };

In[128]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_h * xx + p1h == \tau_h * (1 - xx) + p2h$ , xx][[1, 1, 2]];
xl[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_l * xx + p1l == \tau_l * (1 - xx) + p2l$ , xx][[1, 1, 2]];
y1[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p1h == \beta - p1l$ , yy][[1, 1, 2]];
y2[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p2h == \beta - p2l$ , yy][[1, 1, 2]];
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] *  $\mu_h + (1 - y1[p1h, p1l, p2h, p2l]) * \mu$ ;
q1l[p1h_, p1l_, p2h_, p2l_] = xl[p1h, p1l, p2h, p2l] *  $\mu_l + y1[p1h, p1l, p2h, p2l] * \mu$ ;
q2h[p1h_, p1l_, p2h_, p2l_] =
  (1 - xh[p1h, p1l, p2h, p2l]) *  $\mu_h + (1 - y2[p1h, p1l, p2h, p2l]) * \mu$ ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - xl[p1h, p1l, p2h, p2l]) *  $\mu_l + y2[p1h, p1l, p2h, p2l] * \mu$ ;
 $\pi1h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1h - ch) * q1h[p1h, p1l, p2h, p2l]]$ ;
 $\pi1l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1l - c1) * q1l[p1h, p1l, p2h, p2l]]$ ;
 $\pi2h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2h - ch) * q2h[p1h, p1l, p2h, p2l]]$ ;
 $\pi2l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2l - c1) * q2l[p1h, p1l, p2h, p2l]]$ ;

```

In[140]:= FullSimplify@Solve[{D[$\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, $p_{1h}] = 0$,
 $D[\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, $p_{1l}] = 0$,
 $D[\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, $p_{2h}] = 0$,
 $D[\pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, $p_{2l}] = 0$ }, { $p_{1h}, p_{1l}, p_{2h}, p_{2l}$ }

Out[140]= { { $p_{1h} \rightarrow \frac{(3 \delta \mu l^2 (3 \delta \mu h + 4 \mu \tau h) (ch \delta \mu h + 2 (ch + cl + \delta) \mu \tau h + \delta \mu h \tau h) + 2 \mu \mu l (24 ch \delta^2 \mu h^2 + \delta \mu h (62 ch \mu + 18 cl \mu + 3 \delta (16 \mu + 8 \mu h + 5 \mu l)) \tau h + 8 \mu (4 ch \mu + 2 cl \mu + \delta (8 \mu + 4 \mu h + 3 \mu l)) \tau h^2) \tau l + 8 \mu^2 (2 ch \mu h (3 \delta \mu h + 5 \mu \tau h) + \tau h (2 cl \mu \mu h + \delta \mu h (12 \mu + 6 \mu h + 5 \mu l) + 6 \mu (2 \mu + \mu h + \mu l) \tau h)) \tau l^2) / (9 \delta^3 \mu h^2 \mu l^2 + 48 \delta^2 \mu \mu h \mu l (\mu l \tau h + \mu h \tau l) + 96 \mu^3 \tau h \tau l (\mu l \tau h + \mu h \tau l) + 16 \delta \mu^2 (3 \mu l \tau h + \mu h \tau l) (\mu l \tau h + 3 \mu h \tau l))$,
 $p_{1l} \rightarrow \frac{(3 cl \delta \mu l^2 (\delta \mu h + 4 \mu \tau h) (3 \delta \mu h + 4 \mu \tau h) + \mu l (2 ch \mu (3 \delta \mu h + 2 \mu \tau h) (3 \delta \mu h + 4 \mu \tau h) + 2 cl \mu (5 \delta \mu h + 4 \mu \tau h) (3 \delta \mu h + 10 \mu \tau h) + \delta (3 \delta \mu h + 4 \mu \tau h) (3 \delta \mu h \mu l + 2 \mu (10 \mu + 5 \mu h + 6 \mu l) \tau h)) \tau l + 4 \mu (3 \delta \mu h^2 (2 (ch + cl) \mu + \delta \mu l) + 4 \mu \mu h (2 ch \mu + 4 cl \mu + \delta (6 \mu + 3 \mu h + 4 \mu l)) \tau h + 12 \mu^2 (2 \mu + \mu h + \mu l) \tau h^2) \tau l^2) / (9 \delta^3 \mu h^2 \mu l^2 + 48 \delta^2 \mu \mu h \mu l (\mu l \tau h + \mu h \tau l) + 96 \mu^3 \tau h \tau l (\mu l \tau h + \mu h \tau l) + 16 \delta \mu^2 (3 \mu l \tau h + \mu h \tau l) (\mu l \tau h + 3 \mu h \tau l))$,
 $p_{2h} \rightarrow \frac{(3 \delta \mu l^2 (3 \delta \mu h + 4 \mu \tau h) (ch \delta \mu h + 2 (ch + cl + \delta) \mu \tau h + \delta \mu h \tau h) + 8 \mu \mu l (6 ch \delta^2 \mu h^2 + \delta \mu h (14 ch \mu + 6 cl \mu + 3 \delta (4 \mu + 2 \mu h + \mu l)) \tau h + \mu (8 ch \mu + 4 cl \mu + \delta (10 \mu + 5 \mu h + 3 \mu l)) \tau h^2) \tau l + 16 \mu^2 \mu h (3 ch \delta \mu h + (4 ch \mu + 2 cl \mu + \delta (6 \mu + 3 \mu h + 2 \mu l)) \tau h) \tau l^2) / (9 \delta^3 \mu h^2 \mu l^2 + 48 \delta^2 \mu \mu h \mu l (\mu l \tau h + \mu h \tau l) + 96 \mu^3 \tau h \tau l (\mu l \tau h + \mu h \tau l) + 16 \delta \mu^2 (3 \mu l \tau h + \mu h \tau l) (\mu l \tau h + 3 \mu h \tau l))$,
 $p_{2l} \rightarrow \frac{(cl (3 \delta \mu l + 4 \mu \tau l) (\mu l (\delta \mu h + 4 \mu \tau h) (3 \delta \mu h + 4 \mu \tau h) + 2 \mu \mu h (3 \delta \mu h + 8 \mu \tau h) \tau l) + \tau l (\mu l (3 \delta \mu h + 4 \mu \tau h) (3 \delta \mu h (2 ch \mu + \delta \mu l) + 4 \mu (2 ch \mu + \delta (4 \mu + 2 \mu h + 3 \mu l)) \tau h) + 4 \mu \mu h (3 \delta \mu h (2 ch \mu + \delta \mu l) + 2 \mu (4 ch \mu + \delta (6 \mu + 3 \mu h + 5 \mu l)) \tau h) \tau l)) / (9 \delta^3 \mu h^2 \mu l^2 + 48 \delta^2 \mu \mu h \mu l (\mu l \tau h + \mu h \tau l) + 96 \mu^3 \tau h \tau l (\mu l \tau h + \mu h \tau l) + 16 \delta \mu^2 (3 \mu l \tau h + \mu h \tau l) (\mu l \tau h + 3 \mu h \tau l))$ }

```

In[141]:= phmc = Simplify[(3 δ μ12 (3 δ μh + 4 μ τh) (ch δ μh + 2 (ch + c1 + δ) μ τh + δ μh τh) +
  2 μ μ1 (24 ch δ2 μh2 + δ μh (62 ch μ + 18 c1 μ + 3 δ (16 μ + 8 μh + 5 μ1)) τh +
  8 μ (4 ch μ + 2 c1 μ + δ (8 μ + 4 μh + 3 μ1)) τh2) τ1 + 8 μ2 (2 ch μh (3 δ μh + 5 μ τh) +
  τh (2 c1 μ μh + δ μh (12 μ + 6 μh + 5 μ1) + 6 μ (2 μ + μh + μ1) τh)) τ12) /
  (9 δ3 μh2 μ12 + 48 δ2 μ μh μ1 (μ1 τh + μh τ1) + 96 μ3 τh τ1 (μ1 τh + μh τ1) +
  16 δ μ2 (3 μ1 τh + μh τ1) (μ1 τh + 3 μh τ1))] ;
plmc = Simplify[(3 c1 δ μ12 (δ μh + 4 μ τh) (3 δ μh + 4 μ τh) +
  μ1 (2 ch μ (3 δ μh + 2 μ τh) (3 δ μh + 4 μ τh) + 2 c1 μ (5 δ μh + 4 μ τh) (3 δ μh + 10 μ τh) +
  δ (3 δ μh + 4 μ τh) (3 δ μh μ1 + 2 μ (10 μ + 5 μh + 6 μ1) τh)) τ1 +
  4 μ (3 δ μh2 (2 (ch + c1) μ + δ μ1) + 4 μ μh (2 ch μ + 4 c1 μ + δ (6 μ + 3 μh + 4 μ1)) τh +
  12 μ2 (2 μ + μh + μ1) τh2) τ12) /
  (9 δ3 μh2 μ12 + 48 δ2 μ μh μ1 (μ1 τh + μh τ1) + 96 μ3 τh τ1 (μ1 τh + μh τ1) +
  16 δ μ2 (3 μ1 τh + μh τ1) (μ1 τh + 3 μh τ1))] ;
phmd = Simplify[(3 δ μ12 (3 δ μh + 4 μ τh) (ch δ μh + 2 (ch + c1 + δ) μ τh + δ μh τh) +
  8 μ μ1 (6 ch δ2 μh2 + δ μh (14 ch μ + 6 c1 μ + 3 δ (4 μ + 2 μh + μ1)) τh +
  μ (8 ch μ + 4 c1 μ + δ (10 μ + 5 μh + 3 μ1)) τh2) τ1 +
  16 μ2 μh (3 ch δ μh + (4 ch μ + 2 c1 μ + δ (6 μ + 3 μh + 2 μ1)) τh) τ12) /
  (9 δ3 μh2 μ12 + 48 δ2 μ μh μ1 (μ1 τh + μh τ1) + 96 μ3 τh τ1 (μ1 τh + μh τ1) +
  16 δ μ2 (3 μ1 τh + μh τ1) (μ1 τh + 3 μh τ1))] ;
plmd = Simplify[(c1 (3 δ μ1 + 4 μ τ1) (μ1 (δ μh + 4 μ τh) (3 δ μh + 4 μ τh) +
  2 μ μh (3 δ μh + 8 μ τh) τ1) +
  τ1 (μ1 (3 δ μh + 4 μ τh) (3 δ μh (2 ch μ + δ μ1) + 4 μ (2 ch μ + δ (4 μ + 2 μh + 3 μ1)) τh) +
  4 μ μh (3 δ μh (2 ch μ + δ μ1) + 2 μ (4 ch μ + δ (6 μ + 3 μh + 5 μ1)) τh) τ1)) /
  (9 δ3 μh2 μ12 + 48 δ2 μ μh μ1 (μ1 τh + μh τ1) + 96 μ3 τh τ1 (μ1 τh + μh τ1) +
  16 δ μ2 (3 μ1 τh + μh τ1) (μ1 τh + 3 μh τ1))] ;

In[145]:= μ1 = μh ;
τ1 = τh ;
c1 = 0 ;
πmc = Simplify[π1h[phmc, plmc, phmd, plmd] + π1l[phmc, plmc, phmd, plmd]] ;
πmd = Simplify[π2h[phmc, plmc, phmd, plmd] + π2l[phmc, plmc, phmd, plmd]] ;
FullSimplify@Reduce[phmc > phmd && And @@ $Assumptions]
FullSimplify@Reduce[plmc > plmd && And @@ $Assumptions]
Assuming[μh == 1 && μ == 1 && τh == 1 && δ == 1,
  FullSimplify@Reduce[πmc < πmd && And @@ $Assumptions]]

Out[150]= True

Out[151]= True

Out[152]= True

In[153]:= cutofffxh = Simplify[xh[phmc, plmc, phmd, plmd]] ;
cutofffxl = Simplify[xl[phmc, plmc, phmd, plmd]] ;
cutofffyc = Simplify[y1[phmc, plmc, phmd, plmd]] ;
cutofffyd = Simplify[y2[phmc, plmc, phmd, plmd]] ;
FullSimplify@Reduce[cutofffxh < 1/2 && And @@ $Assumptions]
FullSimplify@Reduce[cutofffxl < 1/2 && And @@ $Assumptions]
FullSimplify@Reduce[cutofffyc < cutofffyd && And @@ $Assumptions]

Out[157]= True

Out[158]= True

Out[159]= 2 ch < δ

```

Endogenous organizational structure

Pricing centralized is a best response to pricing decentralized

```

In[160]:= Clear["Global`*"]
$Assumptions =
  {β > 0, δ > 0, τh > 0, τl > 0, μh > 0, μl > 0, μ > 0, ch > 0, c1 >= 0, ch > c1, δ > ch - c1};

In[162]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[τh * xx + p1h == τh * (1 - xx) + p2h, xx][[1, 1, 2]];
xl[p1h_, p1l_, p2h_, p2l_] = Solve[τl * xx + p1l == τl * (1 - xx) + p2l, xx][[1, 1, 2]];
y1[p1h_, p1l_, p2h_, p2l_] = Solve[β + δ * yy - p1h == β - p1l, yy][[1, 1, 2]];
y2[p1h_, p1l_, p2h_, p2l_] = Solve[β + δ * yy - p2h == β - p2l, yy][[1, 1, 2]];
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] * μh + (1 - y1[p1h, p1l, p2h, p2l]) * μ;
q1l[p1h_, p1l_, p2h_, p2l_] = xl[p1h, p1l, p2h, p2l] * μl + y1[p1h, p1l, p2h, p2l] * μ;
q2h[p1h_, p1l_, p2h_, p2l_] =
  (1 - xh[p1h, p1l, p2h, p2l]) * μh + (1 - y2[p1h, p1l, p2h, p2l]) * μ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - xl[p1h, p1l, p2h, p2l]) * μl + y2[p1h, p1l, p2h, p2l] * μ;
π1h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1h - ch) * q1h[p1h, p1l, p2h, p2l]];
π1l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1l - c1) * q1l[p1h, p1l, p2h, p2l]];
π2h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2h - ch) * q2h[p1h, p1l, p2h, p2l]];
π2l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2l - c1) * q2l[p1h, p1l, p2h, p2l]];

In[174]:= (* pricing centralized against rival pricing decentralized *)
phmc = Simplify[(3 δ μl^2 (3 δ μh + 4 μ τh) (ch δ μh + 2 (ch + c1 + δ) μ τh + δ μh τh) +
  2 μ μl (24 ch δ^2 μh^2 + δ μh (62 ch μ + 18 c1 μ + 3 δ (16 μ + 8 μh + 5 μl)) τh +
  8 μ (4 ch μ + 2 c1 μ + δ (8 μ + 4 μh + 3 μl)) τh^2) τl + 8 μ^2 (2 ch μh (3 δ μh + 5 μ τh) +
  τh (2 c1 μ μh + δ μh (12 μ + 6 μh + 5 μl) + 6 μ (2 μ + μh + μl) τh)) τl^2) /
  (9 δ^3 μh^2 μl^2 + 48 δ^2 μ μh μl (μl τh + μh τl) + 96 μ^3 τh τl (μl τh + μh τl) +
  16 δ μ^2 (3 μl τh + μh τl) (μl τh + 3 μh τl))] ;
plmc = Simplify[(3 c1 δ μl^2 (δ μh + 4 μ τh) (3 δ μh + 4 μ τh) +
  μl (2 ch μ (3 δ μh + 2 μ τh) (3 δ μh + 4 μ τh) + 2 c1 μ (5 δ μh + 4 μ τh) (3 δ μh + 10 μ τh) +
  δ (3 δ μh + 4 μ τh) (3 δ μh μl + 2 μ (10 μ + 5 μh + 6 μl) τh)) τl +
  4 μ (3 δ μh^2 (2 (ch + c1) μ + δ μl) + 4 μ μh (2 ch μ + 4 c1 μ + δ (6 μ + 3 μh + 4 μl)) τh +
  12 μ^2 (2 μ + μh + μl) τh^2) τl^2) /
  (9 δ^3 μh^2 μl^2 + 48 δ^2 μ μh μl (μl τh + μh τl) + 96 μ^3 τh τl (μl τh + μh τl) +
  16 δ μ^2 (3 μl τh + μh τl) (μl τh + 3 μh τl))] ;
phmd = Simplify[(3 δ μl^2 (3 δ μh + 4 μ τh) (ch δ μh + 2 (ch + c1 + δ) μ τh + δ μh τh) +
  8 μ μl (6 ch δ^2 μh^2 + δ μh (14 ch μ + 6 c1 μ + 3 δ (4 μ + 2 μh + μl)) τh +
  μ (8 ch μ + 4 c1 μ + δ (10 μ + 5 μh + 3 μl)) τh^2) τl +
  16 μ^2 μh (3 ch δ μh + (4 ch μ + 2 c1 μ + δ (6 μ + 3 μh + 2 μl)) τh) τl^2) /
  (9 δ^3 μh^2 μl^2 + 48 δ^2 μ μh μl (μl τh + μh τl) + 96 μ^3 τh τl (μl τh + μh τl) +
  16 δ μ^2 (3 μl τh + μh τl) (μl τh + 3 μh τl))] ;
plmd = Simplify[(c1 (3 δ μl + 4 μ τl) (μl (δ μh + 4 μ τh) (3 δ μh + 4 μ τh) +
  2 μ μh (3 δ μh + 8 μ τh) τl) +
  τl (μl (3 δ μh + 4 μ τh) (3 δ μh (2 ch μ + δ μl) + 4 μ (2 ch μ + δ (4 μ + 2 μh + 3 μl)) τh) +
  4 μ μh (3 δ μh (2 ch μ + δ μl) + 2 μ (4 ch μ + δ (6 μ + 3 μh + 5 μl)) τh) τl)) /
  (9 δ^3 μh^2 μl^2 + 48 δ^2 μ μh μl (μl τh + μh τl) + 96 μ^3 τh τl (μl τh + μh τl) +
  16 δ μ^2 (3 μl τh + μh τl) (μl τh + 3 μh τl))] ;
πcd = Simplify[π1h[phmc, plmc, phmd, plmd] + π1l[phmc, plmc, phmd, plmd]] ;

```

```

In[179]:= (* pricing decentralized as well yields the following outcome *)
phd = Simplify[( $\delta \mu_1 (ch \delta \mu_h + 2 (ch + c_1 + \delta) \mu \tau_h + \delta \mu_h \tau_h) +$ 
 $2 \mu (2 ch \delta \mu_h + (4 ch \mu + 2 c_1 \mu + \delta (4 \mu + 2 \mu_h + \mu_1)) \tau_h) \tau_1) /$ 
 $(\delta \mu_1 (\delta \mu_h + 4 \mu \tau_h) + 4 \mu (\delta \mu_h + 3 \mu \tau_h) \tau_1)]$ ;
pld = Simplify[( $c_1 \delta \mu_1 (\delta \mu_h + 4 \mu \tau_h) + \delta \mu_h (2 (ch + c_1) \mu + \delta \mu_1) \tau_1 +$ 
 $2 \mu (2 ch \mu + 4 c_1 \mu + \delta (2 \mu + \mu_h + 2 \mu_1)) \tau_h \tau_1) /$ 
 $(\delta \mu_1 (\delta \mu_h + 4 \mu \tau_h) + 4 \mu (\delta \mu_h + 3 \mu \tau_h) \tau_1)]$ ;
 $\pi dd = \text{Simplify}[\pi_{1h}[phd, pld, phd, pld] + \pi_{1l}[phd, pld, phd, pld]]$ ;

In[182]:=  $\mu_1 = \mu_h$ ;
 $\tau_1 = \tau_h$ ;
 $c_1 = 0$ ;
Simplify[ $\pi dd$ ];
Simplify[ $\pi cd$ ];
FullSimplify@Reduce[ $\pi cd > \pi dd \ \&\& \ \&\& \ \$Assumptions$ ]

Out[187]= True

```

Pricing centralized is a best response to pricing centralized

```

In[188]:= Clear["Global`*"]
$Assumptions =
  { $\beta > 0, \delta > 0, \tau_h > 0, \tau_l > 0, \mu_h > 0, \mu_l > 0, \mu > 0, ch > 0, c_1 \geq 0, ch > c_1, \delta > ch - c_1$ };

In[190]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_h * xx + p1h == \tau_h * (1 - xx) + p2h, xx][[1, 1, 2]]$ ;
xl[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_l * xx + p1l == \tau_l * (1 - xx) + p2l, xx][[1, 1, 2]]$ ;
y1[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p1h == \beta - p1l, yy][[1, 1, 2]]$ ;
y2[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p2h == \beta - p2l, yy][[1, 1, 2]]$ ;
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] *  $\mu_h + (1 - y1[p1h, p1l, p2h, p2l]) * \mu$ ;
q1l[p1h_, p1l_, p2h_, p2l_] = xl[p1h, p1l, p2h, p2l] *  $\mu_l + y1[p1h, p1l, p2h, p2l] * \mu$ ;
q2h[p1h_, p1l_, p2h_, p2l_] =
  (1 - xh[p1h, p1l, p2h, p2l]) *  $\mu_h + (1 - y2[p1h, p1l, p2h, p2l]) * \mu$ ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - xl[p1h, p1l, p2h, p2l]) *  $\mu_l + y2[p1h, p1l, p2h, p2l] * \mu$ ;
 $\pi_{1h}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p1h - ch) * q1h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{1l}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p1l - c_1) * q1l[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2h}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p2h - ch) * q2h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2l}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p2l - c_1) * q2l[p1h, p1l, p2h, p2l]]$ ;

```

```

In[202]:= (* pricing decentralized against rival pricing centralized *)
phmc = Simplify[(3 δ μ l2 (3 δ μ h + 4 μ τ h) (ch δ μ h + 2 (ch + cl + δ) μ τ h + δ μ h τ h) +
  2 μ μ l (24 ch δ2 μ h2 + δ μ h (62 ch μ + 18 cl μ + 3 δ (16 μ + 8 μ h + 5 μ l)) τ h +
  8 μ (4 ch μ + 2 cl μ + δ (8 μ + 4 μ h + 3 μ l)) τ h2) τ l + 8 μ2 (2 ch μ h (3 δ μ h + 5 μ τ h) +
  τ h (2 cl μ μ h + δ μ h (12 μ + 6 μ h + 5 μ l) + 6 μ (2 μ + μ h + μ l) τ h)) τ l2) /
  (9 δ3 μ h2 μ l2 + 48 δ2 μ μ h μ l (μ l τ h + μ h τ l) + 96 μ3 τ h τ l (μ l τ h + μ h τ l) +
  16 δ μ2 (3 μ l τ h + μ h τ l) (μ l τ h + 3 μ h τ l))] ;
plmc = Simplify[(3 cl δ μ l2 (δ μ h + 4 μ τ h) (3 δ μ h + 4 μ τ h) +
  μ l (2 ch μ (3 δ μ h + 2 μ τ h) (3 δ μ h + 4 μ τ h) + 2 cl μ (5 δ μ h + 4 μ τ h) (3 δ μ h + 10 μ τ h) +
  δ (3 δ μ h + 4 μ τ h) (3 δ μ h μ l + 2 μ (10 μ + 5 μ h + 6 μ l) τ h)) τ l +
  4 μ (3 δ μ h2 (2 (ch + cl) μ + δ μ l) + 4 μ μ h (2 ch μ + 4 cl μ + δ (6 μ + 3 μ h + 4 μ l)) τ h +
  12 μ2 (2 μ + μ h + μ l) τ h2) τ l2) /
  (9 δ3 μ h2 μ l2 + 48 δ2 μ μ h μ l (μ l τ h + μ h τ l) + 96 μ3 τ h τ l (μ l τ h + μ h τ l) +
  16 δ μ2 (3 μ l τ h + μ h τ l) (μ l τ h + 3 μ h τ l))] ;
phmd = Simplify[(3 δ μ l2 (3 δ μ h + 4 μ τ h) (ch δ μ h + 2 (ch + cl + δ) μ τ h + δ μ h τ h) +
  8 μ μ l (6 ch δ2 μ h2 + δ μ h (14 ch μ + 6 cl μ + 3 δ (4 μ + 2 μ h + μ l)) τ h +
  μ (8 ch μ + 4 cl μ + δ (10 μ + 5 μ h + 3 μ l)) τ h2) τ l +
  16 μ2 μ h (3 ch δ μ h + (4 ch μ + 2 cl μ + δ (6 μ + 3 μ h + 2 μ l)) τ h) τ l2) /
  (9 δ3 μ h2 μ l2 + 48 δ2 μ μ h μ l (μ l τ h + μ h τ l) + 96 μ3 τ h τ l (μ l τ h + μ h τ l) +
  16 δ μ2 (3 μ l τ h + μ h τ l) (μ l τ h + 3 μ h τ l))] ;
plmd = Simplify[(cl (3 δ μ l + 4 μ τ l) (μ l (δ μ h + 4 μ τ h) (3 δ μ h + 4 μ τ h) +
  2 μ μ h (3 δ μ h + 8 μ τ h) τ l) +
  τ l (μ l (3 δ μ h + 4 μ τ h) (3 δ μ h (2 ch μ + δ μ l) + 4 μ (2 ch μ + δ (4 μ + 2 μ h + 3 μ l)) τ h) +
  4 μ μ h (3 δ μ h (2 ch μ + δ μ l) + 2 μ (4 ch μ + δ (6 μ + 3 μ h + 5 μ l)) τ h) τ l)) /
  (9 δ3 μ h2 μ l2 + 48 δ2 μ μ h μ l (μ l τ h + μ h τ l) + 96 μ3 τ h τ l (μ l τ h + μ h τ l) +
  16 δ μ2 (3 μ l τ h + μ h τ l) (μ l τ h + 3 μ h τ l))] ;
πdc = Simplify[π1h[phmd, plmd, phmc, plmc] + π1l[phmd, plmd, phmc, plmc]] ;

In[207]:= (* pricing centralized against rival pricing centralized *)
phc =
  Simplify[(ch δ μ h μ l + (2 (ch + cl + δ) μ + δ μ h) μ l τ h + 4 μ (ch μ h + (2 μ + μ h + μ l) τ h) τ l) /
    (δ μ h μ l + 4 μ (μ l τ h + μ h τ l))] ;
plc = Simplify[(cl μ l (δ μ h + 4 μ τ h) + (2 (ch + cl) μ μ h + δ μ h μ l + 4 μ (2 μ + μ h + μ l) τ h) τ l) /
  (δ μ h μ l + 4 μ (μ l τ h + μ h τ l))] ;
πcc = Simplify[π1h[phc, plc, phc, plc] + π1l[phc, plc, phc, plc]] ;

In[209]:= μ l = μ h ;
τ l = τ h ;
cl = 0 ;
Simplify[πdc] ;
Simplify[πcc] ;
FullSimplify@Reduce[πcc > πdc && And @@ $Assumptions]

Out[214]= True

```

Three asymmetric cases

Asymmetric brand-loyal base (μ)

```

In[215]:= Clear["Global`*"]
$Assumptions = { $\beta > 0$ ,  $\delta > 0$ ,  $\tau_h > 0$ ,  $\tau_l > 0$ ,
 $\mu_h > 0$ ,  $\mu_l > 0$ ,  $\mu_1 > 0$ ,  $\mu_2 > 0$ ,  $ch > 0$ ,  $c_l \geq 0$ ,  $ch > c_l$ ,  $\delta > ch - c_l$ } ;

In[217]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_h * xx + p1h == \tau_h * (1 - xx) + p2h$ , xx][[1, 1, 2]];
xl[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_l * xx + p1l == \tau_l * (1 - xx) + p2l$ , xx][[1, 1, 2]];
y1[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p1h == \beta - p1l$ , yy][[1, 1, 2]];
y2[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p2h == \beta - p2l$ , yy][[1, 1, 2]];
q1h[p1h_, p1l_, p2h_, p2l_] =
  xh[p1h, p1l, p2h, p2l] *  $\mu_h + (1 - y1[p1h, p1l, p2h, p2l]) * \mu_1$ ;
q1l[p1h_, p1l_, p2h_, p2l_] = xl[p1h, p1l, p2h, p2l] *  $\mu_l + y1[p1h, p1l, p2h, p2l] * \mu_1$ ;
q2h[p1h_, p1l_, p2h_, p2l_] =
  (1 - xh[p1h, p1l, p2h, p2l]) *  $\mu_h + (1 - y2[p1h, p1l, p2h, p2l]) * \mu_2$ ;
q2l[p1h_, p1l_, p2h_, p2l_] =
  (1 - xl[p1h, p1l, p2h, p2l]) *  $\mu_l + y2[p1h, p1l, p2h, p2l] * \mu_2$ ;
 $\pi_{1h}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1h - ch) * q1h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{1l}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1l - c_l) * q1l[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2h}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2h - ch) * q2h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2l}[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2l - c_l) * q2l[p1h, p1l, p2h, p2l]]$ ;

```

In[229]:= FullSimplify@Solve[{D[$\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, p_{1h}] == 0,

D[$\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, p_{1l}] == 0,

D[$\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, p_{2h}] == 0,

D[$\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$, p_{2l}] == 0}, { $p_{1h}, p_{1l}, p_{2h}, p_{2l}$ }]

Out[229]= { { $p_{1h} \rightarrow \frac{(3 \mu_1^2 (ch (3 \delta \mu h + 4 \mu_1 \tau h) (\delta \mu h + 2 \mu_2 \tau h) + \tau h (\delta \mu h (2 (cl + \delta) (2 \mu_1 + \mu_2) + 3 \delta \mu h) + 4 \mu_2 (2 (cl + \delta) \mu_1 + \delta \mu h) \tau h)) + 4 \mu_1 (6 ch \delta (\mu_1 + \mu_2) \mu h^2 + \mu h (6 (3 ch + cl) \mu_1 \mu_2 + 2 \delta (\mu_1 + \mu_2) (4 \mu_1 + 2 \mu_2 + 3 \mu h) + 3 \delta (2 \mu_1 + \mu_2) \mu_1) \tau h + 4 \mu_1 \mu_2 (4 \mu_1 + 2 \mu_2 + 3 (\mu h + \mu_1)) \tau h^2) \tau_1 + 16 \mu_1 \mu_2 \mu h (3 ch \mu h + (4 \mu_1 + 2 \mu_2 + 3 (\mu h + \mu_1)) \tau h) \tau_1^2)}{(9 \delta^2 \mu h^2 \mu_1^2 + 24 \delta (\mu_1 + \mu_2) \mu h \mu_1 (\mu_1 \tau h + \mu h \tau_1) + 48 \mu_1 \mu_2 (\mu_1 \tau h + \mu h \tau_1)^2)},$

$p_{1l} \rightarrow \frac{(3 cl \mu_1^2 (3 \delta^2 \mu h^2 + 8 \delta (\mu_1 + \mu_2) \mu h \tau h + 16 \mu_1 \mu_2 \tau h^2) + \mu_1 (3 \delta \mu h^2 (4 (ch + cl) \mu_1 + 2 (ch + 3 cl) \mu_2 + 3 \delta \mu_1) + 4 \mu h (6 (ch + 3 cl) \mu_1 \mu_2 + \delta (8 \mu_1^2 + 6 \mu_1 (\mu_2 + \mu h + \mu_1) + \mu_2 (4 \mu_2 + 3 \mu h + 6 \mu_1))) \tau h + 16 \mu_1 \mu_2 (4 \mu_1 + 2 \mu_2 + 3 (\mu h + \mu_1)) \tau h^2) \tau_1 + 4 \mu_2 \mu h (6 (ch + cl) \mu_1 \mu h + 3 \delta \mu h \mu_1 + 4 \mu_1 (4 \mu_1 + 2 \mu_2 + 3 (\mu h + \mu_1)) \tau h) \tau_1^2)}{(9 \delta^2 \mu h^2 \mu_1^2 + 24 \delta (\mu_1 + \mu_2) \mu h \mu_1 (\mu_1 \tau h + \mu h \tau_1) + 48 \mu_1 \mu_2 (\mu_1 \tau h + \mu h \tau_1)^2)},$

$p_{2h} \rightarrow \frac{(3 \mu_1^2 (ch (\delta \mu h + 2 \mu_1 \tau h) (3 \delta \mu h + 4 \mu_2 \tau h) + \tau h (\delta \mu h (2 (cl + \delta) (\mu_1 + 2 \mu_2) + 3 \delta \mu h) + 4 \mu_1 (2 (cl + \delta) \mu_2 + \delta \mu h) \tau h)) + 4 \mu_1 (6 ch \delta (\mu_1 + \mu_2) \mu h^2 + \mu h (6 (3 ch + cl) \mu_1 \mu_2 + 2 \delta (\mu_1 + \mu_2) (2 \mu_1 + 4 \mu_2 + 3 \mu h) + 3 \delta (\mu_1 + 2 \mu_2) \mu_1) \tau h + 4 \mu_1 \mu_2 (2 \mu_1 + 4 \mu_2 + 3 (\mu h + \mu_1)) \tau h^2) \tau_1 + 16 \mu_1 \mu_2 \mu h (3 ch \mu h + (2 \mu_1 + 4 \mu_2 + 3 (\mu h + \mu_1)) \tau h) \tau_1^2)}{(9 \delta^2 \mu h^2 \mu_1^2 + 24 \delta (\mu_1 + \mu_2) \mu h \mu_1 (\mu_1 \tau h + \mu h \tau_1) + 48 \mu_1 \mu_2 (\mu_1 \tau h + \mu h \tau_1)^2)},$

$p_{2l} \rightarrow \frac{(3 cl \mu_1^2 (3 \delta^2 \mu h^2 + 8 \delta (\mu_1 + \mu_2) \mu h \tau h + 16 \mu_1 \mu_2 \tau h^2) + \mu_1 (3 \delta \mu h^2 (2 (ch + 3 cl) \mu_1 + 4 (ch + cl) \mu_2 + 3 \delta \mu_1) + 4 \mu h (6 (ch + 3 cl) \mu_1 \mu_2 + \delta (4 \mu_1^2 + 8 \mu_2^2 + 6 \mu_2 (\mu h + \mu_1) + 3 \mu_1 (2 \mu_2 + \mu h + 2 \mu_1))) \tau h + 16 \mu_1 \mu_2 (2 \mu_1 + 4 \mu_2 + 3 (\mu h + \mu_1)) \tau h^2) \tau_1 + 4 \mu_1 \mu h (6 (ch + cl) \mu_2 \mu h + 3 \delta \mu h \mu_1 + 4 \mu_2 (2 \mu_1 + 4 \mu_2 + 3 (\mu h + \mu_1)) \tau h) \tau_1^2)}{(9 \delta^2 \mu h^2 \mu_1^2 + 24 \delta (\mu_1 + \mu_2) \mu h \mu_1 (\mu_1 \tau h + \mu h \tau_1) + 48 \mu_1 \mu_2 (\mu_1 \tau h + \mu h \tau_1)^2)} \}$

```

In[230]:= p1h = Simplify[(3 μ1^2 (ch (3 δ μh + 4 μ1 τh) (δ μh + 2 μ2 τh) +
    τh (δ μh (2 (c1 + δ) (2 μ1 + μ2) + 3 δ μh) + 4 μ2 (2 (c1 + δ) μ1 + δ μh) τh)) +
    4 μ1 (6 ch δ (μ1 + μ2) μh^2 + μh (6 (3 ch + c1) μ1 μ2 + 2 δ (μ1 + μ2) (4 μ1 + 2 μ2 + 3 μh) +
    3 δ (2 μ1 + μ2) μ1) τh + 4 μ1 μ2 (4 μ1 + 2 μ2 + 3 (μh + μ1)) τh^2) τ1 +
    16 μ1 μ2 μh (3 ch μh + (4 μ1 + 2 μ2 + 3 (μh + μ1)) τh) τ1^2) /
    (9 δ^2 μh^2 μ1^2 + 24 δ (μ1 + μ2) μh μ1 (μ1 τh + μh τ1) + 48 μ1 μ2 (μ1 τh + μh τ1)^2)] ;
p1l = Simplify[(3 c1 μ1^2 (3 δ^2 μh^2 + 8 δ (μ1 + μ2) μh τh + 16 μ1 μ2 τh^2) +
    μ1 (3 δ μh^2 (4 (ch + c1) μ1 + 2 (ch + 3 c1) μ2 + 3 δ μ1) +
    4 μh (6 (ch + 3 c1) μ1 μ2 + δ (8 μ1^2 + 6 μ1 (μ2 + μh + μ1) + μ2 (4 μ2 + 3 μh + 6 μ1))) τh +
    16 μ1 μ2 (4 μ1 + 2 μ2 + 3 (μh + μ1)) τh^2) τ1 +
    4 μ2 μh (6 (ch + c1) μ1 μh + 3 δ μh μ1 + 4 μ1 (4 μ1 + 2 μ2 + 3 (μh + μ1)) τh) τ1^2) /
    (9 δ^2 μh^2 μ1^2 + 24 δ (μ1 + μ2) μh μ1 (μ1 τh + μh τ1) + 48 μ1 μ2 (μ1 τh + μh τ1)^2)] ;
p2h = Simplify[(3 μ1^2 (ch (δ μh + 2 μ1 τh) (3 δ μh + 4 μ2 τh) +
    τh (δ μh (2 (c1 + δ) (μ1 + 2 μ2) + 3 δ μh) + 4 μ1 (2 (c1 + δ) μ2 + δ μh) τh)) +
    4 μ1 (6 ch δ (μ1 + μ2) μh^2 + μh (6 (3 ch + c1) μ1 μ2 + 2 δ (μ1 + μ2) (2 μ1 + 4 μ2 + 3 μh) +
    3 δ (μ1 + 2 μ2) μ1) τh + 4 μ1 μ2 (2 μ1 + 4 μ2 + 3 (μh + μ1)) τh^2) τ1 +
    16 μ1 μ2 μh (3 ch μh + (2 μ1 + 4 μ2 + 3 (μh + μ1)) τh) τ1^2) /
    (9 δ^2 μh^2 μ1^2 + 24 δ (μ1 + μ2) μh μ1 (μ1 τh + μh τ1) + 48 μ1 μ2 (μ1 τh + μh τ1)^2)] ;
p2l = Simplify[(3 c1 μ1^2 (3 δ^2 μh^2 + 8 δ (μ1 + μ2) μh τh + 16 μ1 μ2 τh^2) +
    μ1 (3 δ μh^2 (2 (ch + 3 c1) μ1 + 4 (ch + c1) μ2 + 3 δ μ1) +
    4 μh (6 (ch + 3 c1) μ1 μ2 + δ (4 μ1^2 + 8 μ2^2 + 6 μ2 (μh + μ1) + 3 μ1 (2 μ2 + μh + 2 μ1))) τh +
    16 μ1 μ2 (2 μ1 + 4 μ2 + 3 (μh + μ1)) τh^2) τ1 +
    4 μ1 μh (6 (ch + c1) μ2 μh + 3 δ μh μ1 + 4 μ2 (2 μ1 + 4 μ2 + 3 (μh + μ1)) τh) τ1^2) /
    (9 δ^2 μh^2 μ1^2 + 24 δ (μ1 + μ2) μh μ1 (μ1 τh + μh τ1) + 48 μ1 μ2 (μ1 τh + μh τ1)^2)] ;

In[234]:= μ1 = μh ;
τ1 = τh ;
c1 = 0 ;
Assuming[μ1 > μ2, FullSimplify@Reduce[p1h > p2h && And @@ $Assumptions]]
Assuming[μ1 > μ2, FullSimplify@Reduce[p1l > p2l && And @@ $Assumptions]]

Out[237]= True

Out[238]= True

In[239]:= cutofffxh = Simplify[xh[p1h, p1l, p2h, p2l]] ;
cutoffx1 = Simplify[x1[p1h, p1l, p2h, p2l]] ;
cutoffy1 = Simplify[y1[p1h, p1l, p2h, p2l]] ;
cutoffy2 = Simplify[y2[p1h, p1l, p2h, p2l]] ;
Assuming[μ1 > μ2, FullSimplify@Reduce[cutofffxh < 1/2 && And @@ $Assumptions]]
Assuming[μ1 > μ2, FullSimplify@Reduce[cutoffx1 < 1/2 && And @@ $Assumptions]]
Assuming[μ1 > μ2, FullSimplify@Reduce[cutoffy1 > cutoffy2 && And @@ $Assumptions]]

Out[243]= True

Out[244]= True

Out[245]= 2 ch < δ

```

```
In[246]:=  $\pi_1 = \text{Simplify}[\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]] ;$ 
 $\pi_2 = \text{Simplify}[\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]] ;$ 
Assuming[ $\mu_1 > \mu_2$ , FullSimplify@Reduce[ $\pi_1 > \pi_2 \ \&\& \text{And} @@ \$Assumptions$ ]]

Out[248]:= True
```

Asymmetric transportation costs (τ_h and τ_l)

```
In[249]:= Clear["Global`*"]
$Assumptions = { $\beta > 0$ ,  $\delta > 0$ ,  $\tau_{1h} > 0$ ,  $\tau_{1l} > 0$ ,  $\tau_{2h} > 0$ ,
 $\tau_{2l} > 0$ ,  $\mu_h > 0$ ,  $\mu_l > 0$ ,  $\mu > 0$ ,  $ch > 0$ ,  $c_l \geq 0$ ,  $ch > c_l$ ,  $\delta > ch - c_l$ };

In[251]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_{1h} * xx + p_{1h} = \tau_{2h} * (1 - xx) + p_{2h}$ , xx][[1, 1, 2]];
xl[p1h_, p1l_, p2h_, p2l_] = Solve[ $\tau_{1l} * xx + p_{1l} = \tau_{2l} * (1 - xx) + p_{2l}$ , xx][[1, 1, 2]];
y1[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p_{1h} = \beta - p_{1l}$ , yy][[1, 1, 2]];
y2[p1h_, p1l_, p2h_, p2l_] = Solve[ $\beta + \delta * yy - p_{2h} = \beta - p_{2l}$ , yy][[1, 1, 2]];
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] *  $\mu_h + (1 - y1[p1h, p1l, p2h, p2l]) * \mu$ ;
q1l[p1h_, p1l_, p2h_, p2l_] = xl[p1h, p1l, p2h, p2l] *  $\mu_l + y1[p1h, p1l, p2h, p2l] * \mu$ ;
q2h[p1h_, p1l_, p2h_, p2l_] =
(1 - xh[p1h, p1l, p2h, p2l]) *  $\mu_h + (1 - y2[p1h, p1l, p2h, p2l]) * \mu$ ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - xl[p1h, p1l, p2h, p2l]) *  $\mu_l + y2[p1h, p1l, p2h, p2l] * \mu$ ;
 $\pi_{1h}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p_{1h} - ch) * q1h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{1l}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p_{1l} - c_l) * q1l[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2h}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p_{2h} - ch) * q2h[p1h, p1l, p2h, p2l]]$ ;
 $\pi_{2l}[p1h_, p1l_, p2h_, p2l_] = \text{Simplify}[(p_{2l} - c_l) * q2l[p1h, p1l, p2h, p2l]]$ ;

In[263]:= FullSimplify@Solve[{D[ $\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ , p1h] == 0,
D[ $\pi_{1h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{1l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ , p1l] == 0,
D[ $\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ , p2h] == 0,
D[ $\pi_{2h}[p_{1h}, p_{1l}, p_{2h}, p_{2l}] + \pi_{2l}[p_{1h}, p_{1l}, p_{2h}, p_{2l}]$ , p2l] == 0}, {p1h, p1l, p2h, p2l}]

Out[263]:= { {p1h  $\rightarrow$ 
- (( $\delta \mu (5 \delta \mu_h \mu_l + 4 \mu (\mu_l (\tau_{1h} + \tau_{2h}) + \mu_h (\tau_{1l} + \tau_{2l}))) (2 \mu_l (\delta \mu_h + \mu (\tau_{1h} + \tau_{2h})) (ch \delta$ 
 $\mu_h + \delta \mu_h \tau_{2h} + ch \mu (\tau_{1h} + \tau_{2h}) - c_l \mu (\tau_{1h} + \tau_{2h}) + \delta \mu (\tau_{1h} + \tau_{2h})) +$ 
 $\mu_h (\delta \mu_l (ch \delta \mu_h + \delta \mu_h \tau_{1h} + ch \mu (\tau_{1h} + \tau_{2h}) - c_l \mu (\tau_{1h} + \tau_{2h}) + \delta \mu (\tau_{1h} + \tau_{2h})) -$ 
 $2 \mu (\tau_{1h} + \tau_{2h}) (c_l \delta \mu_l + \delta \mu_l \tau_{2l} - ch \mu (\tau_{1l} + \tau_{2l}) + c_l \mu (\tau_{1l} + \tau_{2l}))) +$ 
 $4 \mu (2 \delta \mu_h \mu_l + \mu \mu_l (\tau_{1h} + \tau_{2h}) + \mu \mu_h (\tau_{1l} + \tau_{2l}))$ 
 $(-2 \delta (ch \delta \mu_h + \delta \mu_h \tau_{2h} + ch \mu (\tau_{1h} + \tau_{2h}) - c_l \mu (\tau_{1h} + \tau_{2h}) + \delta \mu (\tau_{1h} + \tau_{2h}))$ 
 $(\delta \mu_h \mu_l + \mu \mu_l (\tau_{1h} + \tau_{2h}) + \mu \mu_h (\tau_{1l} + \tau_{2l})) + \delta \mu_h (-ch \delta (\delta \mu_h \mu_l +$ 
 $\mu \mu_l (\tau_{1h} + \tau_{2h}) + \mu \mu_h (\tau_{1l} + \tau_{2l})) - \delta (\delta \mu_l (\mu_h \tau_{1h} + \mu (\tau_{1h} + \tau_{2h})) + \mu$ 
 $(\mu_l \tau_{1l} (\tau_{1h} + \tau_{2h}) + \mu_h \tau_{1h} (\tau_{1l} + \tau_{2l}) + \mu (\tau_{1h} + \tau_{2h}) (\tau_{1l} + \tau_{2l})))))) /$ 
 $(3 \delta^2 \mu \mu_h (3 \delta^2 \mu_h^2 \mu_l^2 + 8 \delta \mu \mu_h \mu_l (\mu_l (\tau_{1h} + \tau_{2h}) + \mu_h (\tau_{1l} + \tau_{2l})) +$ 
 $4 \mu^2 (\mu_l (\tau_{1h} + \tau_{2h}) + \mu_h (\tau_{1l} + \tau_{2l}))^2))$ ),
p1l  $\rightarrow$ 
 $(9 ch \delta \mu \mu_h^2 \mu_l \tau_{1l} + 3 \delta^2 \mu_h^2 \mu_l^2 \tau_{1l} + 6 ch \mu^2 \mu_h \mu_l \tau_{1h} \tau_{1l} +$ 
 $18 \delta \mu^2 \mu_h \mu_l \tau_{1h} \tau_{1l} +$ 
 $8 \delta \mu \mu_h^2 \mu_l \tau_{1h} \tau_{1l} +$ 
 $8 \delta \mu \mu_h \mu_l^2 \tau_{1h} \tau_{1l} +$ 
 $12 \mu^3 \mu_l \tau_{1h}^2 \tau_{1l} +$ 
 $4 \mu^2 \mu_h \mu_l \tau_{1h}^2 \tau_{1l} +$ 
 $4 \mu^2 \mu_l^2 \tau_{1h}^2 \tau_{1l} + 6 ch \mu^2 \mu_h^2 \tau_{1l}^2 +$ 
 $12 \mu^3 \mu_h \tau_{1h} \tau_{1l}^2 + 4 \mu^2 \mu_h^2 \tau_{1h} \tau_{1l}^2 +$ 
 $4 \mu^2 \mu_h \mu_l \tau_{1h} \tau_{1l}^2 + 6 ch \mu^2 \mu_h \mu_l \tau_{1l} \tau_{2h} +$ 
```

$$\begin{aligned}
& 18 \delta \mu^2 \mu h \mu l \tau_{11} \tau_{2h} + 10 \delta \mu \mu h^2 \mu l \tau_{11} \tau_{2h} + \\
& 8 \delta \mu \mu h \mu l^2 \tau_{11} \tau_{2h} + 24 \mu^3 \mu l \tau_{1h} \tau_{11} \tau_{2h} + \\
& 12 \mu^2 \mu h \mu l \tau_{1h} \tau_{11} \tau_{2h} + 8 \mu^2 \mu l^2 \tau_{1h} \tau_{11} \tau_{2h} + \\
& 12 \mu^3 \mu h \tau_{11}^2 \tau_{2h} + 8 \mu^2 \mu h^2 \tau_{11}^2 \tau_{2h} + \\
& 4 \mu^2 \mu h \mu l \tau_{11}^2 \tau_{2h} + 12 \mu^3 \mu l \tau_{11} \tau_{2h}^2 + \\
& 8 \mu^2 \mu h \mu l \tau_{11} \tau_{2h}^2 + 4 \mu^2 \mu l^2 \tau_{11} \tau_{2h}^2 + \\
& \left(6 \delta^2 \mu h^2 \mu l^2 + 3 ch \mu \mu h \left(3 \delta \mu h \mu l + 4 \mu \mu h \tau_{11} + 2 \mu \mu l \left(\tau_{1h} + \tau_{2h} \right) \right) + \right. \\
& \quad 2 \delta \mu \mu h \mu l \left(9 \mu \left(\tau_{1h} + \tau_{2h} \right) + 8 \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(4 \tau_{1h} + 3 \tau_{11} + 5 \tau_{2h} \right) \right) + \\
& \quad 4 \mu^2 \left(2 \mu l^2 \left(\tau_{1h} + \tau_{2h} \right)^2 + 2 \mu h^2 \tau_{11} \left(\tau_{1h} + 2 \tau_{2h} \right) + \mu h \mu l \left(\tau_{1h} + \tau_{2h} \right) \right. \\
& \quad \left. \left(\tau_{1h} + 3 \tau_{11} + 2 \tau_{2h} \right) + 3 \mu \left(\tau_{1h} + \tau_{2h} \right) \left(2 \mu h \tau_{11} + \mu l \left(\tau_{1h} + \tau_{2h} \right) \right) \right) \tau_{21} + \\
& \quad 2 \mu \mu h \left(3 ch \mu \mu h + 3 \delta \mu h \mu l + 2 \mu \left(\left(\mu h + 2 \mu l \right) \tau_{1h} + 2 \left(\mu h + \mu l \right) \tau_{2h} + 3 \mu \left(\tau_{1h} + \tau_{2h} \right) \right) \right) \\
& \quad \tau_{21}^2 + 3 cl \left(\delta \mu h \mu l + 2 \mu \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu \mu h \left(\tau_{11} + \tau_{21} \right) \right) \\
& \quad \left. \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right) / \\
& \left(3 \left(\delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right. \\
& \quad \left. \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right), \\
p2h \rightarrow & \left(9 \delta^2 \mu \mu h \mu l^2 \tau_{1h} + 6 \delta^2 \mu h^2 \mu l^2 \tau_{1h} + 6 \delta \mu^2 \mu l^2 \tau_{1h}^2 + 6 \delta \mu \mu h \mu l^2 \tau_{1h}^2 + \right. \\
& 24 \delta \mu^2 \mu h \mu l \tau_{1h} \tau_{11} + 16 \delta \mu \mu h^2 \mu l \tau_{1h} \tau_{11} + \\
& 10 \delta \mu \mu h \mu l^2 \tau_{1h} \tau_{11} + 12 \mu^3 \mu l \tau_{1h}^2 \tau_{11} + 8 \mu^2 \mu h \mu l \tau_{1h}^2 \tau_{11} + \\
& 8 \mu^2 \mu l^2 \tau_{1h}^2 \tau_{11} + 12 \mu^3 \mu h \tau_{1h} \tau_{11}^2 + 8 \mu^2 \mu h^2 \tau_{1h} \tau_{11}^2 + \\
& 8 \mu^2 \mu h \mu l \tau_{1h} \tau_{11}^2 + 9 \delta^2 \mu \mu h \mu l^2 \tau_{2h} + 3 \delta^2 \mu h^2 \mu l^2 \tau_{2h} + \\
& 12 \delta \mu^2 \mu l^2 \tau_{1h} \tau_{2h} + 6 \delta \mu \mu h \mu l^2 \tau_{1h} \tau_{2h} + 24 \delta \mu^2 \mu h \mu l \tau_{11} \tau_{2h} + \\
& 8 \delta \mu \mu h^2 \mu l \tau_{11} \tau_{2h} + 10 \delta \mu \mu h \mu l^2 \tau_{11} \tau_{2h} + 24 \mu^3 \mu l \tau_{1h} \tau_{11} \tau_{2h} + \\
& 12 \mu^2 \mu h \mu l \tau_{1h} \tau_{11} \tau_{2h} + 16 \mu^2 \mu l^2 \tau_{1h} \tau_{11} \tau_{2h} + 12 \mu^3 \mu h \tau_{11}^2 \tau_{2h} + \\
& 4 \mu^2 \mu h^2 \tau_{11}^2 \tau_{2h} + 8 \mu^2 \mu h \mu l \tau_{11}^2 \tau_{2h} + 6 \delta \mu^2 \mu l^2 \tau_{2h}^2 + \\
& 12 \mu^3 \mu l \tau_{11} \tau_{2h}^2 + 4 \mu^2 \mu h \mu l \tau_{11} \tau_{2h}^2 + 8 \mu^2 \mu l^2 \tau_{11} \tau_{2h}^2 + \\
& 4 \mu \left(\mu l \left(3 \mu + 2 \mu h + \mu l \right) \tau_{1h} \left(2 \delta \mu h + \mu \tau_{1h} \right) + \mu \mu h \left(6 \mu + 4 \mu h + 3 \mu l \right) \tau_{1h} \tau_{11} + \right. \\
& \quad \left(2 \delta \mu h \mu l \left(3 \mu + \mu h + \mu l \right) + \mu \mu l \left(6 \mu + 3 \mu h + 2 \mu l \right) \tau_{1h} + \mu \mu h \left(6 \mu + 2 \mu h + 3 \mu l \right) \tau_{11} \right) \\
& \quad \left. \tau_{2h} + \mu \mu l \left(3 \mu + \mu h + \mu l \right) \tau_{2h}^2 \right) \tau_{21} + \\
& 4 \mu^2 \mu h \left(\left(2 \mu h + \mu l \right) \tau_{1h} + \left(\mu h + \mu l \right) \tau_{2h} + 3 \mu \left(\tau_{1h} + \tau_{2h} \right) \right) \tau_{21}^2 + \\
& 3 cl \mu \mu l \left(\tau_{1h} + \tau_{2h} \right) \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) + \\
& 3 ch \left(\delta \mu h \mu l + \mu \mu l \left(\tau_{1h} + \tau_{2h} \right) + 2 \mu \mu h \left(\tau_{11} + \tau_{21} \right) \right) \\
& \quad \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) / \\
& \left(3 \left(\delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right. \\
& \quad \left. \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right), \\
p21 \rightarrow & \left(6 \delta^2 \mu h^2 \mu l^2 \tau_{11} + 18 \delta \mu^2 \mu h \mu l \tau_{1h} \tau_{11} + 10 \delta \mu \mu h^2 \mu l \tau_{1h} \tau_{11} + \right. \\
& 16 \delta \mu \mu h \mu l^2 \tau_{1h} \tau_{11} + 12 \mu^3 \mu l \tau_{1h}^2 \tau_{11} + 8 \mu^2 \mu h \mu l \tau_{1h}^2 \tau_{11} + \\
& 8 \mu^2 \mu l^2 \tau_{1h}^2 \tau_{11} + 6 \delta \mu \mu h^2 \mu l \tau_{11}^2 + 12 \mu^3 \mu h \tau_{1h} \tau_{11}^2 + 8 \mu^2 \mu h^2 \tau_{1h} \tau_{11}^2 + \\
& 8 \mu^2 \mu h \mu l \tau_{1h} \tau_{11}^2 + 18 \delta \mu^2 \mu h \mu l \tau_{11} \tau_{2h} + 8 \delta \mu \mu h^2 \mu l \tau_{11} \tau_{2h} + \\
& 16 \delta \mu \mu h \mu l^2 \tau_{11} \tau_{2h} + 24 \mu^3 \mu l \tau_{1h} \tau_{11} \tau_{2h} + 12 \mu^2 \mu h \mu l \tau_{1h} \tau_{11} \tau_{2h} + \\
& 16 \mu^2 \mu l^2 \tau_{1h} \tau_{11} \tau_{2h} + 12 \mu^3 \mu h \tau_{11}^2 \tau_{2h} + 4 \mu^2 \mu h^2 \tau_{11}^2 \tau_{2h} + \\
& 8 \mu^2 \mu h \mu l \tau_{11}^2 \tau_{2h} + 12 \mu^3 \mu l \tau_{11} \tau_{2h}^2 + 4 \mu^2 \mu h \mu l \tau_{11} \tau_{2h}^2 + 8 \mu^2 \mu l^2 \tau_{11} \tau_{2h}^2 + \\
& \left(3 \delta^2 \mu h^2 \mu l^2 + 2 \delta \mu \mu h \mu l \left(9 \mu \left(\tau_{1h} + \tau_{2h} \right) + 4 \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(5 \tau_{1h} + 3 \tau_{11} + 4 \tau_{2h} \right) \right) \right) + \\
& \quad 4 \mu^2 \left(\mu l^2 \left(\tau_{1h} + \tau_{2h} \right)^2 + 2 \mu h^2 \tau_{11} \left(2 \tau_{1h} + \tau_{2h} \right) + \mu h \mu l \left(\tau_{1h} + \tau_{2h} \right) \right. \\
& \quad \left. \left(2 \tau_{1h} + 3 \tau_{11} + \tau_{2h} \right) + 3 \mu \left(\tau_{1h} + \tau_{2h} \right) \left(2 \mu h \tau_{11} + \mu l \left(\tau_{1h} + \tau_{2h} \right) \right) \right) \tau_{21} + \\
& 4 \mu^2 \mu h \left(3 \mu \left(\tau_{1h} + \tau_{2h} \right) + \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(2 \tau_{1h} + \tau_{2h} \right) \right) \tau_{21}^2 + \\
& 3 ch \mu \mu h \left(\tau_{11} + \tau_{21} \right) \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) + \\
& 3 cl \left(\delta \mu h \mu l + 2 \mu \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu \mu h \left(\tau_{11} + \tau_{21} \right) \right) \\
& \quad \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) / \\
& \left(3 \left(\delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right. \\
& \quad \left. \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right) \} \}
\end{aligned}$$

In[264]:= p1h = Simplify[

$$\begin{aligned}
 & - \left((\delta \mu (5 \delta \mu h \mu 1 + 4 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21))) (2 \mu 1 (\delta \mu h + \mu (\tau 1h + \tau 2h)) \right. \\
 & \quad (\text{ch } \delta \mu h + \delta \mu h \tau 2h + \text{ch } \mu (\tau 1h + \tau 2h) - \text{cl } \mu (\tau 1h + \tau 2h) + \delta \mu (\tau 1h + \tau 2h)) + \\
 & \quad \mu h (\delta \mu 1 (\text{ch } \delta \mu h + \delta \mu h \tau 1h + \text{ch } \mu (\tau 1h + \tau 2h) - \text{cl } \mu (\tau 1h + \tau 2h) + \delta \mu (\tau 1h + \tau 2h)) - \\
 & \quad \quad 2 \mu (\tau 1h + \tau 2h) (\text{cl } \delta \mu 1 + \delta \mu 1 \tau 21 - \text{ch } \mu (\tau 11 + \tau 21) + \text{cl } \mu (\tau 11 + \tau 21))) + \\
 & \quad 4 \mu (2 \delta \mu h \mu 1 + \mu \mu 1 (\tau 1h + \tau 2h) + \mu \mu h (\tau 11 + \tau 21)) \\
 & \quad (-2 \delta (\text{ch } \delta \mu h + \delta \mu h \tau 2h + \text{ch } \mu (\tau 1h + \tau 2h) - \text{cl } \mu (\tau 1h + \tau 2h) + \delta \mu (\tau 1h + \tau 2h)) \\
 & \quad (\delta \mu h \mu 1 + \mu \mu 1 (\tau 1h + \tau 2h) + \mu \mu h (\tau 11 + \tau 21)) + \\
 & \quad \delta \mu h (-\text{ch } \delta (\delta \mu h \mu 1 + \mu \mu 1 (\tau 1h + \tau 2h) + \mu \mu h (\tau 11 + \tau 21)) - \\
 & \quad \quad \delta (\delta \mu 1 (\mu h \tau 1h + \mu (\tau 1h + \tau 2h)) + \mu (\mu 1 \tau 11 (\tau 1h + \tau 2h) + \\
 & \quad \quad \quad \mu h \tau 1h (\tau 11 + \tau 21) + \mu (\tau 1h + \tau 2h) (\tau 11 + \tau 21)))) \Big) / \\
 & \left(3 \delta^2 \mu \mu h (3 \delta^2 \mu h^2 \mu 1^2 + 8 \delta \mu \mu h \mu 1 (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21)) + \right. \\
 & \quad \left. 4 \mu^2 (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21))^2 \right) \Big) ;
 \end{aligned}$$

$$\begin{aligned}
 \text{p11} = \text{Simplify} \Big[& (9 \text{ch } \delta \mu \mu h^2 \mu 1 \tau 11 + 3 \delta^2 \mu h^2 \mu 1^2 \tau 11 + 6 \text{ch } \mu^2 \mu h \mu 1 \tau 1h \tau 11 + \\
 & 18 \delta \mu^2 \mu h \mu 1 \tau 1h \tau 11 + 8 \delta \mu \mu h^2 \mu 1 \tau 1h \tau 11 + 8 \delta \mu \mu h \mu 1^2 \tau 1h \tau 11 + 12 \mu^3 \mu 1 \tau 1h^2 \tau 11 + \\
 & 4 \mu^2 \mu h \mu 1 \tau 1h^2 \tau 11 + 4 \mu^2 \mu 1^2 \tau 1h^2 \tau 11 + 6 \text{ch } \mu^2 \mu h^2 \tau 11^2 + 12 \mu^3 \mu h \tau 1h \tau 11^2 + \\
 & 4 \mu^2 \mu h^2 \tau 1h \tau 11^2 + 4 \mu^2 \mu h \mu 1 \tau 1h \tau 11^2 + 6 \text{ch } \mu^2 \mu h \mu 1 \tau 11 \tau 2h + 18 \delta \mu^2 \mu h \mu 1 \tau 11 \tau 2h + \\
 & 10 \delta \mu \mu h^2 \mu 1 \tau 11 \tau 2h + 8 \delta \mu \mu h \mu 1^2 \tau 11 \tau 2h + 24 \mu^3 \mu 1 \tau 1h \tau 11 \tau 2h + \\
 & 12 \mu^2 \mu h \mu 1 \tau 1h \tau 11 \tau 2h + 8 \mu^2 \mu 1^2 \tau 1h \tau 11 \tau 2h + 12 \mu^3 \mu h \tau 11^2 \tau 2h + 8 \mu^2 \mu h^2 \tau 11^2 \tau 2h + \\
 & 4 \mu^2 \mu h \mu 1 \tau 11^2 \tau 2h + 12 \mu^3 \mu 1 \tau 11 \tau 2h^2 + 8 \mu^2 \mu h \mu 1 \tau 11 \tau 2h^2 + 4 \mu^2 \mu 1^2 \tau 11 \tau 2h^2 + \\
 & (6 \delta^2 \mu h^2 \mu 1^2 + 3 \text{ch } \mu \mu h (3 \delta \mu h \mu 1 + 4 \mu \mu h \tau 11 + 2 \mu \mu 1 (\tau 1h + \tau 2h)) + \\
 & \quad 2 \delta \mu \mu h \mu 1 (9 \mu (\tau 1h + \tau 2h) + 8 \mu 1 (\tau 1h + \tau 2h) + \mu h (4 \tau 1h + 3 \tau 11 + 5 \tau 2h)) + \\
 & \quad 4 \mu^2 (2 \mu 1^2 (\tau 1h + \tau 2h)^2 + 2 \mu h^2 \tau 11 (\tau 1h + 2 \tau 2h) + \mu h \mu 1 (\tau 1h + \tau 2h) \\
 & \quad (\tau 1h + 3 \tau 11 + 2 \tau 2h) + 3 \mu (\tau 1h + \tau 2h) (2 \mu h \tau 11 + \mu 1 (\tau 1h + \tau 2h))) \Big) \tau 21 + \\
 & 2 \mu \mu h (3 \text{ch } \mu \mu h + 3 \delta \mu h \mu 1 + 2 \mu ((\mu h + 2 \mu 1) \tau 1h + 2 (\mu h + \mu 1) \tau 2h + 3 \mu (\tau 1h + \tau 2h))) \\
 & \quad \tau 21^2 + 3 \text{cl } (\delta \mu h \mu 1 + 2 \mu \mu 1 (\tau 1h + \tau 2h) + \mu \mu h (\tau 11 + \tau 21)) \\
 & \quad (3 \delta \mu h \mu 1 + 2 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21))) \Big) / \\
 & (3 (\delta \mu h \mu 1 + 2 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21))) \\
 & \quad (3 \delta \mu h \mu 1 + 2 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21)))) \Big] ;
 \end{aligned}$$

$$\begin{aligned}
 \text{p2h} = \text{Simplify} \Big[& (9 \delta^2 \mu \mu h \mu 1^2 \tau 1h + 6 \delta^2 \mu h^2 \mu 1^2 \tau 1h + 6 \delta \mu^2 \mu 1^2 \tau 1h^2 + 6 \delta \mu \mu h \mu 1^2 \tau 1h^2 + \\
 & 24 \delta \mu^2 \mu h \mu 1 \tau 1h \tau 11 + 16 \delta \mu \mu h^2 \mu 1 \tau 1h \tau 11 + 10 \delta \mu \mu h \mu 1^2 \tau 1h \tau 11 + \\
 & 12 \mu^3 \mu 1 \tau 1h^2 \tau 11 + 8 \mu^2 \mu h \mu 1 \tau 1h^2 \tau 11 + 8 \mu^2 \mu 1^2 \tau 1h^2 \tau 11 + 12 \mu^3 \mu h \tau 1h \tau 11^2 + \\
 & 8 \mu^2 \mu h^2 \tau 1h \tau 11^2 + 8 \mu^2 \mu h \mu 1 \tau 1h \tau 11^2 + 9 \delta^2 \mu \mu h \mu 1^2 \tau 2h + 3 \delta^2 \mu h^2 \mu 1^2 \tau 2h + \\
 & 12 \delta \mu^2 \mu 1^2 \tau 1h \tau 2h + 6 \delta \mu \mu h \mu 1^2 \tau 1h \tau 2h + 24 \delta \mu^2 \mu h \mu 1 \tau 11 \tau 2h + 8 \delta \mu \mu h^2 \mu 1 \tau 11 \tau 2h + \\
 & 10 \delta \mu \mu h \mu 1^2 \tau 11 \tau 2h + 24 \mu^3 \mu 1 \tau 1h \tau 11 \tau 2h + 12 \mu^2 \mu h \mu 1 \tau 1h \tau 11 \tau 2h + \\
 & 16 \mu^2 \mu 1^2 \tau 1h \tau 11 \tau 2h + 12 \mu^3 \mu h \tau 11^2 \tau 2h + 4 \mu^2 \mu h^2 \tau 11^2 \tau 2h + 8 \mu^2 \mu h \mu 1 \tau 11^2 \tau 2h + \\
 & 6 \delta \mu^2 \mu 1^2 \tau 2h^2 + 12 \mu^3 \mu 1 \tau 11 \tau 2h^2 + 4 \mu^2 \mu h \mu 1 \tau 11 \tau 2h^2 + 8 \mu^2 \mu 1^2 \tau 11 \tau 2h^2 + \\
 & 4 \mu (\mu 1 (3 \mu + 2 \mu h + \mu 1) \tau 1h (2 \delta \mu h + \mu \tau 1h) + \mu \mu h (6 \mu + 4 \mu h + 3 \mu 1) \tau 1h \tau 11 + \\
 & \quad (2 \delta \mu h \mu 1 (3 \mu + \mu h + \mu 1) + \mu \mu 1 (6 \mu + 3 \mu h + 2 \mu 1) \tau 1h + \mu \mu h (6 \mu + 2 \mu h + 3 \mu 1) \tau 11) \\
 & \quad \tau 2h + \mu \mu 1 (3 \mu + \mu h + \mu 1) \tau 2h^2) \tau 21 + \\
 & 4 \mu^2 \mu h ((2 \mu h + \mu 1) \tau 1h + (\mu h + \mu 1) \tau 2h + 3 \mu (\tau 1h + \tau 2h)) \tau 21^2 + \\
 & 3 \text{cl } \mu \mu 1 (\tau 1h + \tau 2h) (3 \delta \mu h \mu 1 + 2 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21))) + \\
 & 3 \text{ch } (\delta \mu h \mu 1 + \mu \mu 1 (\tau 1h + \tau 2h) + 2 \mu \mu h (\tau 11 + \tau 21)) \\
 & \quad (3 \delta \mu h \mu 1 + 2 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21))) \Big) / \\
 & (3 (\delta \mu h \mu 1 + 2 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21))) \\
 & \quad (3 \delta \mu h \mu 1 + 2 \mu (\mu 1 (\tau 1h + \tau 2h) + \mu h (\tau 11 + \tau 21)))) \Big] ;
 \end{aligned}$$

$$\begin{aligned}
 \text{p21} = \text{Simplify} \Big[& (6 \delta^2 \mu h^2 \mu 1^2 \tau 11 + 18 \delta \mu^2 \mu h \mu 1 \tau 1h \tau 11 + 10 \delta \mu \mu h^2 \mu 1 \tau 1h \tau 11 + \\
 & 16 \delta \mu \mu h \mu 1^2 \tau 1h \tau 11 + 12 \mu^3 \mu 1 \tau 1h^2 \tau 11 + 8 \mu^2 \mu h \mu 1 \tau 1h^2 \tau 11 + 8 \mu^2 \mu 1^2 \tau 1h^2 \tau 11 +
 \end{aligned}$$

$$\begin{aligned}
& 6 \delta \mu \mu h^2 \mu l \tau_{11}^2 + 12 \mu^3 \mu h \tau_{1h} \tau_{11}^2 + 8 \mu^2 \mu h^2 \tau_{1h} \tau_{11}^2 + 8 \mu^2 \mu h \mu l \tau_{1h} \tau_{11}^2 + \\
& 18 \delta \mu^2 \mu h \mu l \tau_{11} \tau_{2h} + 8 \delta \mu \mu h^2 \mu l \tau_{11} \tau_{2h} + 16 \delta \mu \mu h \mu l^2 \tau_{11} \tau_{2h} + 24 \mu^3 \mu l \tau_{1h} \tau_{11} \tau_{2h} + \\
& 12 \mu^2 \mu h \mu l \tau_{1h} \tau_{11} \tau_{2h} + 16 \mu^2 \mu l^2 \tau_{1h} \tau_{11} \tau_{2h} + 12 \mu^3 \mu h \tau_{11}^2 \tau_{2h} + 4 \mu^2 \mu h^2 \tau_{11}^2 \tau_{2h} + \\
& 8 \mu^2 \mu h \mu l \tau_{11}^2 \tau_{2h} + 12 \mu^3 \mu l \tau_{11} \tau_{2h}^2 + 4 \mu^2 \mu h \mu l \tau_{11} \tau_{2h}^2 + 8 \mu^2 \mu l^2 \tau_{11} \tau_{2h}^2 + \\
& \left(3 \delta^2 \mu h^2 \mu l^2 + 2 \delta \mu \mu h \mu l \left(9 \mu \left(\tau_{1h} + \tau_{2h} \right) + 4 \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(5 \tau_{1h} + 3 \tau_{11} + 4 \tau_{2h} \right) \right) + \right. \\
& \quad \left. 4 \mu^2 \left(\mu l^2 \left(\tau_{1h} + \tau_{2h} \right)^2 + 2 \mu h^2 \tau_{11} \left(2 \tau_{1h} + \tau_{2h} \right) + \mu h \mu l \left(\tau_{1h} + \tau_{2h} \right) \right) \right) \tau_{21} + \\
& \quad \left(2 \tau_{1h} + 3 \tau_{11} + \tau_{2h} \right) + 3 \mu \left(\tau_{1h} + \tau_{2h} \right) \left(2 \mu h \tau_{11} + \mu l \left(\tau_{1h} + \tau_{2h} \right) \right) \tau_{21} + \\
& 4 \mu^2 \mu h \left(3 \mu \left(\tau_{1h} + \tau_{2h} \right) + \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(2 \tau_{1h} + \tau_{2h} \right) \right) \tau_{21}^2 + \\
& 3 c h \mu \mu h \left(\tau_{11} + \tau_{21} \right) \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) + \\
& 3 c l \left(\delta \mu h \mu l + 2 \mu \mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu \mu h \left(\tau_{11} + \tau_{21} \right) \right) \\
& \quad \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \Big/ \\
& \left(3 \left(\delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right. \\
& \quad \left. \left(3 \delta \mu h \mu l + 2 \mu \left(\mu l \left(\tau_{1h} + \tau_{2h} \right) + \mu h \left(\tau_{11} + \tau_{21} \right) \right) \right) \right) \Big] ;
\end{aligned}$$

```

In[268]:=  $\mu l = \mu h;$ 
 $\tau_{11} = \tau_{1h};$ 
 $\tau_{21} = \tau_{2h};$ 
 $c l = 0;$ 
Assuming[ $\tau_{1h} > \tau_{2h}$ , FullSimplify@Reduce[p1h < p2h && And @@ $Assumptions]]
Assuming[ $\tau_{1h} > \tau_{2h}$ , FullSimplify@Reduce[p1l < p2l && And @@ $Assumptions]]

```

```
Out[272]= True
```

```
Out[273]= True
```

```

In[274]:= cutofffxh = Simplify[xh[p1h, p1l, p2h, p2l]] ;
cutofffxl = Simplify[xl[p1h, p1l, p2h, p2l]] ;
cutofffy1 = Simplify[y1[p1h, p1l, p2h, p2l]] ;
cutofffy2 = Simplify[y2[p1h, p1l, p2h, p2l]] ;
Assuming[ $\tau_{1h} > \tau_{2h}$ , FullSimplify@Reduce[cutofffxh < 1/2 && And @@ $Assumptions]]
Assuming[ $\tau_{1h} > \tau_{2h}$ , FullSimplify@Reduce[cutofffxl < 1/2 && And @@ $Assumptions]]
Assuming[ $\tau_{1h} > \tau_{2h}$ , FullSimplify@Reduce[cutofffy1 == cutofffy2 && And @@ $Assumptions]]

```

```
Out[278]= True
```

```
Out[279]= True
```

```
Out[280]= True
```

```

In[281]:=  $\pi_1 = \text{Simplify}[\pi_{1h}[p1h, p1l, p2h, p2l] + \pi_{1l}[p1h, p1l, p2h, p2l]] ;$ 
 $\pi_2 = \text{Simplify}[\pi_{2h}[p1h, p1l, p2h, p2l] + \pi_{2l}[p1h, p1l, p2h, p2l]] ;$ 
Assuming[ $\tau_{1h} > \tau_{2h}$ , FullSimplify@Reduce[ $\pi_1 < \pi_2$  && And @@ $Assumptions]]

```

```
Out[283]= True
```

Asymmetric costs (ch)

```

In[284]:= Clear["Global`*"]
$Assumptions = { $\beta > 0$ ,  $\delta > 0$ ,  $\tau h > 0$ ,  $\tau l > 0$ ,  $\mu h > 0$ ,  $\mu l > 0$ ,  $\mu > 0$ ,  $c1h > 0$ ,
 $c1l > 0$ ,  $c2h > 0$ ,  $c2l > 0$ ,  $c1h > c1l$ ,  $c2h > c2l$ ,  $\delta > c1h - c1l$ ,  $\delta > c2h - c2l$ };

```

```
In[286]:= xh[p1h_, p1l_, p2h_, p2l_] = Solve[τh * xx + p1h == τh * (1 - xx) + p2h, xx][[1, 1, 2]];
x1[p1h_, p1l_, p2h_, p2l_] = Solve[τ1 * xx + p1l == τ1 * (1 - xx) + p2l, xx][[1, 1, 2]];
y1[p1h_, p1l_, p2h_, p2l_] = Solve[β + δ * yy - p1h == β - p1l, yy][[1, 1, 2]];
y2[p1h_, p1l_, p2h_, p2l_] = Solve[β + δ * yy - p2h == β - p2l, yy][[1, 1, 2]];
q1h[p1h_, p1l_, p2h_, p2l_] = xh[p1h, p1l, p2h, p2l] * μh + (1 - y1[p1h, p1l, p2h, p2l]) * μ;
q1l[p1h_, p1l_, p2h_, p2l_] = x1[p1h, p1l, p2h, p2l] * μl + y1[p1h, p1l, p2h, p2l] * μ;
q2h[p1h_, p1l_, p2h_, p2l_] =
  (1 - xh[p1h, p1l, p2h, p2l]) * μh + (1 - y2[p1h, p1l, p2h, p2l]) * μ;
q2l[p1h_, p1l_, p2h_, p2l_] = (1 - x1[p1h, p1l, p2h, p2l]) * μl + y2[p1h, p1l, p2h, p2l] * μ;
π1h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1h - c1h) * q1h[p1h, p1l, p2h, p2l]];
π1l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p1l - c1l) * q1l[p1h, p1l, p2h, p2l]];
π2h[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2h - c2h) * q2h[p1h, p1l, p2h, p2l]];
π2l[p1h_, p1l_, p2h_, p2l_] = Simplify[(p2l - c2l) * q2l[p1h, p1l, p2h, p2l]];
```

```
In[298]:= FullSimplify@Solve[{D[π1h[p1h, p1l, p2h, p2l] + π1l[p1h, p1l, p2h, p2l], p1h] == 0,
  D[π1h[p1h, p1l, p2h, p2l] + π1l[p1h, p1l, p2h, p2l], p1l] == 0,
  D[π2h[p1h, p1l, p2h, p2l] + π2l[p1h, p1l, p2h, p2l], p2h] == 0,
  D[π2h[p1h, p1l, p2h, p2l] + π2l[p1h, p1l, p2h, p2l], p2l] == 0}, {p1h, p1l, p2h, p2l}]
```

```
Out[298]= { {p1h -> 1/3 (2 c1h + c2h + 3 (2 μ + μh + μl) τh /
  (3 μl τh ((-c1h + c1l - c2h + c2l) μ μh - δ μh μl - 4 μ (2 μ + μh + μl) τh)) /
  (μh (δ μh μl + 4 μ (μl τh + μh τl))) + (c1h - c1l - c2h + c2l) μ μl τh) /
  (3 δ μh μl + 4 μ (μl τh + μh τl))),
  p1l -> ((2 c1l + c2l) μl^2 (δ μh + 4 μ τh) (3 δ μh + 4 μ τh) +
  μl (δ μh^2 (2 (4 c1h + 12 c1l + 5 c2h + 3 c2l) μ + 9 δ μl) + 4 μ μh (2 c1h μ + 14 c1l μ +
  4 c2h μ + 4 c2l μ + 18 δ μ + 9 δ μh + 12 δ μl) τh + 48 μ^2 (2 μ + μh + μl) τh^2) τl +
  4 μ μh (2 (c1h + 3 c1l + 2 c2h) μ μh + 3 δ μh μl + 12 μ (2 μ + μh + μl) τh) τl^2) /
  (3 (δ μh μl + 4 μ (μl τh + μh τl)) (3 δ μh μl + 4 μ (μl τh + μh τl))),
  p2h -> 1/3 (c1h + 2 c2h + 3 (2 μ + μh + μl) τh /
  (3 μl τh ((-c1h + c1l - c2h + c2l) μ μh - δ μh μl - 4 μ (2 μ + μh + μl) τh)) /
  (μh (δ μh μl + 4 μ (μl τh + μh τl))) + (-c1h + c1l + c2h - c2l) μ μl τh) /
  (3 δ μh μl + 4 μ (μl τh + μh τl))),
  p2l -> ((c1l + 2 c2l) μl^2 (δ μh + 4 μ τh) (3 δ μh + 4 μ τh) +
  μl (δ μh^2 (10 c1h μ + 6 c1l μ + 8 (c2h + 3 c2l) μ + 9 δ μl) + 4 μ μh (4 c1h μ + 4 c1l μ +
  2 c2h μ + 14 c2l μ + 18 δ μ + 9 δ μh + 12 δ μl) τh + 48 μ^2 (2 μ + μh + μl) τh^2) τl +
  4 μ μh (2 (2 c1h + c2h + 3 c2l) μ μh + 3 δ μh μl + 12 μ (2 μ + μh + μl) τh) τl^2) /
  (3 (δ μh μl + 4 μ (μl τh + μh τl)) (3 δ μh μl + 4 μ (μl τh + μh τl))) } }
```

```

In[299]:= p1h =  $\frac{1}{3} \left( 2 c1h + c2h + \frac{3 (2 \mu + \mu h + \mu l) \tau h}{\mu h} + \right.$ 

$$\left. (3 \mu l \tau h ((-c1h + c1l - c2h + c2l) \mu \mu h - \delta \mu h \mu l - 4 \mu (2 \mu + \mu h + \mu l) \tau h)) / \right.$$


$$\left. (\mu h (\delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l))) + \frac{(c1h - c1l - c2h + c2l) \mu \mu l \tau h}{3 \delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l)} \right);$$

p1l =  $((2 c1l + c2l) \mu l^2 (\delta \mu h + 4 \mu \tau h) (3 \delta \mu h + 4 \mu \tau h) +$ 

$$\mu l (\delta \mu h^2 (2 (4 c1h + 12 c1l + 5 c2h + 3 c2l) \mu + 9 \delta \mu l) + 4 \mu \mu h (2 c1h \mu + 14 c1l \mu +$$


$$4 c2h \mu + 4 c2l \mu + 18 \delta \mu + 9 \delta \mu h + 12 \delta \mu l) \tau h + 48 \mu^2 (2 \mu + \mu h + \mu l) \tau h^2) \tau l +$$


$$4 \mu \mu h (2 (c1h + 3 c1l + 2 c2h) \mu \mu h + 3 \delta \mu h \mu l + 12 \mu (2 \mu + \mu h + \mu l) \tau h) \tau l^2) /$$


$$(3 (\delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l)) (3 \delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l))) ;$$

p2h =  $\frac{1}{3} \left( c1h + 2 c2h + \frac{3 (2 \mu + \mu h + \mu l) \tau h}{\mu h} + \right.$ 

$$(3 \mu l \tau h ((-c1h + c1l - c2h + c2l) \mu \mu h - \delta \mu h \mu l - 4 \mu (2 \mu + \mu h + \mu l) \tau h)) /$$


$$(\mu h (\delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l))) + \frac{(-c1h + c1l + c2h - c2l) \mu \mu l \tau h}{3 \delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l)} \right);$$

p2l =  $((c1l + 2 c2l) \mu l^2 (\delta \mu h + 4 \mu \tau h) (3 \delta \mu h + 4 \mu \tau h) +$ 

$$\mu l (\delta \mu h^2 (10 c1h \mu + 6 c1l \mu + 8 (c2h + 3 c2l) \mu + 9 \delta \mu l) + 4 \mu \mu h (4 c1h \mu + 4 c1l \mu +$$


$$2 c2h \mu + 14 c2l \mu + 18 \delta \mu + 9 \delta \mu h + 12 \delta \mu l) \tau h + 48 \mu^2 (2 \mu + \mu h + \mu l) \tau h^2) \tau l +$$


$$4 \mu \mu h (2 (2 c1h + c2h + 3 c2l) \mu \mu h + 3 \delta \mu h \mu l + 12 \mu (2 \mu + \mu h + \mu l) \tau h) \tau l^2) /$$


$$(3 (\delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l)) (3 \delta \mu h \mu l + 4 \mu (\mu l \tau h + \mu h \tau l))) ;$$

In[302]:=  $\mu l = \mu h;$ 
 $\tau l = \tau h;$ 
 $c1l = 0;$ 
 $c2l = 0;$ 
Assuming[c1h > c2h, FullSimplify@Reduce[p1h > p2h && And @@ $Assumptions]]
Assuming[c1h > c2h, FullSimplify@Reduce[p1l < p2l && And @@ $Assumptions]]

Out[306]= True

Out[307]= True

In[308]:= cutofffxh = Simplify[xh[p1h, p1l, p2h, p2l]] ;
cutoffxl = Simplify[xl[p1h, p1l, p2h, p2l]] ;
cutoffy1 = Simplify[y1[p1h, p1l, p2h, p2l]] ;
cutoffy2 = Simplify[y2[p1h, p1l, p2h, p2l]] ;
Assuming[c1h > c2h, FullSimplify@Reduce[cutofffxh < 1/2 && And @@ $Assumptions]]
Assuming[c1h > c2h, FullSimplify@Reduce[cutoffxl > 1/2 && And @@ $Assumptions]]
Assuming[c1h > c2h, FullSimplify@Reduce[cutoffy1 > cutoffy2 && And @@ $Assumptions]]

Out[312]= True

Out[313]= True

Out[314]= True

In[315]:=  $\pi 1 = \text{Simplify}[\pi 1h[p1h, p1l, p2h, p2l] + \pi 1l[p1h, p1l, p2h, p2l]] ;$ 
 $\pi 2 = \text{Simplify}[\pi 2h[p1h, p1l, p2h, p2l] + \pi 2l[p1h, p1l, p2h, p2l]] ;$ 
Assuming[c1h > c2h, FullSimplify@Reduce[ $\pi 1 < \pi 2$  && And @@ $Assumptions]]

Out[317]= True

```