Supplementary Material for “Long memory estimation for complex-valued time series”

Marina I. Knight
Department of Mathematics, University of York
Matthew A. Nunes
Department of Mathematics and Statistics, Lancaster University

December 22, 2017

B Simulated performance of the Hurst exponent estimator

In this appendix we give additional details of the simulation study used to assess the performance of our Hurst estimation technique for both regularly and irregularly sampled real-valued and complex-valued long memory processes.

B.1 Real-valued processes

To investigate the performance of our $\mathbb{C}$LoMPE technique on real-valued long memory processes, we chose stationary long-range dependent processes of fractionally Gaussian noise and fractionally integrated processes; we also tested the technique for fractional Brownian motions, even though the results in the main article hold for stationary long memory processes only. We simulated the series for a range of Hurst exponents $H = 0.6, \ldots, 0.9$. The time series lengths used in this study were $n = 256$, 512, and 1024 points.

For each simulated process, $\underline{X}^k$, $k = 1, \ldots, K = 100$, we perform the estimation procedure described in Section 4 to obtain an estimate $\hat{H}^k$ and then compute the mean squared error defined by

$$\text{MSE} = K^{-1} \sum_{k=1}^{K} (H - \hat{H}^k)^2, \tag{B.1}$$

as well as the bias incurred by the average estimate for the Hurst exponent over the K simulated series, in other words,

$$\text{Bias}(\bar{\hat{H}}) = \bar{\hat{H}} - H = K^{-1} \sum_{k=1}^{K} \hat{H}^k - H, \tag{B.2}$$

where H is the true Hurst exponent value of the simulated process.

The estimation procedure described in Section 4 was implemented using an average of $P = 50$ estimated parameters (one for each lifting trajectory). In the tables below, we denote our proposed method by "ℂLoMPE"; the tables also report the performance of the real-valued lifting technique of Knight et al (2017) ("LoMPE") and for the residual variance method (Peng et al, 1994) (denoted "*Peng*"), reproduced from Knight et al (2017) with the permission of the authors.

The ℂLoMPE method was implemented in *R* using modifications to the code contained in the *CNLTreg* and *liftLRD* `R` packages (see Knight and Nunes (2012) and Knight and Nunes (2016) respectively), both on CRAN. The residual variance method is available within the *fArma* `R` package (Wuertz et al, 2013), whilst the LoMPE method was is available in the *liftLRD* package (Knight and Nunes, 2016).

B.1.1 Regularly sampled time series

Tables 1, 2 and 3 (resp. Tables 4, 5 and 6) show our simulation results for the range of Hurst parameters $H = 0.6, \ldots, 0.9$ across all three methods in terms of mean square error and bias. Overall, our proposed estimation method is competitive with other methods even for the particular case of regularly sampled real-valued processes; in particular, our technique shows improvement over the real-valued lifting method (LoMPE) on fractional Gaussian noise and fractionally integrated series, and attains better mean square error for fractional Brownian motion series, albeit at a cost of higher bias. Notably, these results extend the usability and usefulness of our proposed method beyond the realm of purely complex-valued persistent processes sampled with irregularities or missingness.

Table 1: Mean squared error ($\times 10^3$) for regularly spaced fractional Brownian motion series for a range of Hurst parameters for the estimation procedures described in the text. Numbers in brackets represent the standard deviation of estimation errors. Boxed numbers indicate best result.

	$n = 256$			$n = 512$			$n = 1024$		
H	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE
0.6	19 (30)	12 (21)	$\boxed{4}$ (4)	13 (22)	9 (15)	$\boxed{3}$ (3)	10 (14)	10 (11)	$\boxed{3}$ (2)
0.7	25 (35)	12 (12)	$\boxed{10}$ (8)	14 (16)	8 (11)	$\boxed{8}$ (5)	9 (12)	$\boxed{8}$ (9)	$\boxed{8}$ (3)
0.8	19 (23)	$\boxed{11}$ (11)	13 (9)	13 (18)	$\boxed{7}$ (10)	13 (7)	12 (16)	$\boxed{8}$ (10)	16 (9)
0.9	23 (39)	28 (39)	$\boxed{14}$ (12)	15 (23)	$\boxed{13}$ (20)	17 (12)	12 (16)	$\boxed{7}$ (9)	14 (8)

Table 2: Mean squared error ($\times 10^3$) for regularly spaced fractional Gaussian noise for a range of Hurst parameters for the estimation procedures described in the text. Numbers in brackets represent the standard deviation of estimation errors. Boxed numbers indicate best result.

	$n = 256$			$n = 512$			$n = 1024$		
H	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE
0.6	8 (11)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	4 (6)	$\boxed{1}$ (1)	2 (2)	2 (3)	$\boxed{1}$ (1)	$\boxed{1}$ (1)
0.7	7 (8)	$\boxed{2}$ (3)	$\boxed{2}$ (3)	3 (5)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	3 (3)	$\boxed{1}$ (1)	2 (2)
0.8	7 (11)	$\boxed{2}$ (3)	$\boxed{2}$ (3)	5 (6)	$\boxed{2}$ (3)	$\boxed{2}$ (3)	4 (6)	$\boxed{3}$ (2)	$\boxed{3}$ (3)
0.9	10 (13)	$\boxed{3}$ (4)	$\boxed{3}$ (6)	4 (5)	$\boxed{2}$ (3)	$\boxed{2}$ (2)	$\boxed{3}$ (5)	4 (2)	$\boxed{3}$ (3)

Table 3: Mean squared error ($\times 10^3$) for regularly spaced fractionally integrated series for a range of Hurst parameters, $H = d + 1/2$, for the estimation procedures described in the text. Numbers in brackets represent the standard deviation of estimation errors. Boxed numbers indicate best result.

	$n = 256$			$n = 512$			$n = 1024$		
H	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE
0.6	8 (9)	$\boxed{3}$ (4)	$\boxed{3}$ (4)	4 (6)	$\boxed{1}$ (2)	$\boxed{1}$ (2)	2 (2)	$\boxed{1}$ (1)	$\boxed{1}$ (1)
0.7	8 (11)	4 (5)	$\boxed{3}$ (4)	4 (5)	4 (4)	$\boxed{3}$ (3)	$\boxed{3}$ (3)	4 (3)	$\boxed{3}$ (3)
0.8	11 (16)	6 (8)	$\boxed{5}$ (7)	7 (8)	6 (5)	$\boxed{5}$ (6)	$\boxed{4}$ (5)	6 (4)	5 (4)
0.9	12 (15)	7 (8)	$\boxed{6}$ (7)	7 (10)	8 (7)	$\boxed{6}$ (7)	$\boxed{4}$ (6)	9 (5)	5 (5)

Table 4: Empirical bias ($\times 100$) of Hurst estimation on fractional Brownian motion realizations. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$			$n = 512$			$n = 1024$		
H	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE
0.6	−8 (10)	$\boxed{-4}$ (10)	$\boxed{-4}$ (4)	$\boxed{-5}$ (10)	−6 (7)	$\boxed{-5}$ (3)	$\boxed{-5}$ (8)	−8 (6)	$\boxed{-5}$ (2)
0.7	−10 (12)	$\boxed{-1}$ (11)	-9 (5)	−7 (9)	$\boxed{-5}$ (7)	-9 (3)	$\boxed{-5}$ (8)	−6 (6)	-9 (2)
0.8	−9 (11)	$\boxed{4}$ (10)	-10 (4)	−8 (9)	$\boxed{-1}$ (8)	-11 (3)	−7 (9)	$\boxed{-5}$ (8)	-12 (2)
0.9	$\boxed{-10}$ (11)	$\boxed{10}$ (13)	$\boxed{-10}$ (6)	−8 (9)	$\boxed{4}$ (11)	-12 (4)	−8 (8)	$\boxed{-1}$ (8)	-14 (3)

Table 5: Empirical bias ($\times 100$) of Hurst estimation on fractional Gaussian noise realizations. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$			$n = 512$			$n = 1024$		
H	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE
0.6	−4 (8)	$\boxed{-1}$ (4)	$\boxed{-1}$ (4)	−2 (6)	$\boxed{-1}$ (3)	$\boxed{-1}$ (3)	$\boxed{-2}$ (4)	$\boxed{-2}$ (2)	$\boxed{-2}$ (2)
0.7	−3 (8)	$\boxed{-2}$ (4)	$\boxed{-2}$ (4)	−3 (5)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	$\boxed{-2}$ (5)	−3 (2)	-4 (2)
0.8	−3 (8)	−2 (5)	$\boxed{-1}$ (5)	$\boxed{-3}$ (6)	−4 (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (6)	−5 (2)	-5 (2)
0.9	−4 (9)	$\boxed{-1}$ (5)	-3 (6)	−3 (6)	−4 (3)	$\boxed{-2}$ (4)	$\boxed{-3}$ (5)	−6 (2)	-5 (3)

Table 6: Empirical bias ($\times 100$) of Hurst estimation on fractionally integrated process realizations. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$			$n = 512$			$n = 1024$		
H	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE	Peng	LoMPE	ℂLoMPE
0.6	−4 (8)	$\boxed{-2}$ (5)	$\boxed{-2}$ (5)	−3 (6)	−3 (3)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	−3 (2)	-3 (2)
0.7	−5 (8)	−4 (5)	$\boxed{-3}$ (5)	$\boxed{-4}$ (5)	−6 (3)	$\boxed{-4}$ (3)	$\boxed{-3}$ (4)	−6 (2)	-5 (3)
0.8	−6 (9)	−6 (6)	$\boxed{-4}$ (6)	$\boxed{-5}$ (7)	−7 (4)	$\boxed{-5}$ (5)	$\boxed{-4}$ (5)	−8 (3)	-6 (3)
0.9	−6 (9)	−6 (6)	$\boxed{0}$ (8)	−5 (7)	−8 (4)	$\boxed{-3}$ (7)	$\boxed{-4}$ (5)	−9 (3)	-5 (5)

B.1.2 Irregularly sampled time series

To evaluate the $\mathbb{C}$LoMPE Hurst parameter estimation technique for irregular time series, we performed an additional study. After simulating time series with different long memory properties as above, we induced an irregular structure by randomly removing a portion of observations. More specifically, for each value of $H = 0.6, \ldots, 0.9$, we created irregular time series with increasing proportions of missingness, namely $p = 5\%, 10\%, 15\%, 20\%$. We then computed the average error and bias as before (see equations (B.1) and (B.2)), by performing our proposed $\mathbb{C}$LoMPE estimation method on the irregular time series. We compared these to the equivalent quantities for the only available competitor for irregularly sampled real-valued persistent processes, namely LoMPE.

Tables 7, 8 and 9 (resp. Tables 10, 11 and 12) report the mean square error and bias for the irregularly sampled time series for the different proportions of missingness for varying Hurst exponent. As in the real-valued lifting case, we observe a slight decrease in performance as the amount of missingness increases. Pleasingly, the tables also reflect similar observations to those in the regular sampling setting, that our $\mathbb{C}$LoMPE method improves over the LoMPE method for fractionally integrated series and fractional Gaussian noise, but there is a tradeoff between improvement in MSE and a decrease in bias performance.

Table 7: Mean squared error ($\times 10^3$) for irregularly spaced fractional Brownian motion series featuring different degrees of missing observations for a range of Hurst parameters for the LoMPE and ℂLoMPE estimation procedures. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$								$n = 512$								$n = 1024$							
	Missingness proportion, p								Missingness proportion, p								Missingness proportion, p							
	5%		10%		15%		20%		5%		10%		15%		20%		5%		10%		15%		20%	
H	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE
0.6	13 (22)	$\boxed{3}$ (4)	14 (23)	$\boxed{3}$ (3)	15 (25)	$\boxed{3}$ (4)	16 (25)	$\boxed{3}$ (4)	11 (16)	$\boxed{2}$ (3)	12 (17)	$\boxed{2}$ (2)	12 (18)	$\boxed{2}$ (2)	13 (19)	$\boxed{1}$ (2)	12 (12)	$\boxed{2}$ (2)	13 (13)	$\boxed{2}$ (1)	13 (13)	$\boxed{1}$ (1)	14 (13)	$\boxed{1}$ (1)
0.7	14 (17)	$\boxed{8}$ (8)	13 (17)	$\boxed{8}$ (8)	14 (18)	$\boxed{6}$ (7)	15 (20)	$\boxed{6}$ (7)	9 (12)	$\boxed{7}$ (5)	10 (13)	$\boxed{6}$ (4)	10 (13)	$\boxed{5}$ (4)	11 (14)	$\boxed{4}$ (4)	9 (11)	$\boxed{7}$ (3)	10 (11)	$\boxed{6}$ (3)	10 (12)	$\boxed{5}$ (3)	10 (12)	$\boxed{4}$ (3)
0.8	$\boxed{11}$ (13)	$\boxed{11}$ (8)	11 (12)	$\boxed{10}$ (8)	11 (13)	$\boxed{8}$ (8)	12 (14)	$\boxed{7}$ (7)	$\boxed{8}$ (11)	12 (7)	$\boxed{8}$ (12)	10 (7)	$\boxed{8}$ (12)	9 (6)	9 (13)	$\boxed{7}$ (6)	$\boxed{9}$ (12)	14 (5)	$\boxed{9}$ (12)	12 (5)	$\boxed{10}$ (12)	11 (5)	10 (13)	$\boxed{9}$ (4)
0.9	24 (35)	$\boxed{12}$ (11)	21 (34)	$\boxed{11}$ (11)	22 (33)	$\boxed{11}$ (11)	20 (30)	$\boxed{9}$ (9)	$\boxed{12}$ (19)	16 (10)	$\boxed{11}$ (16)	14 (9)	$\boxed{11}$ (16)	12 (9)	$\boxed{11}$ (17)	$\boxed{11}$ (8)	$\boxed{8}$ (10)	18 (8)	$\boxed{8}$ (11)	16 (7)	$\boxed{9}$ (11)	15 (7)	$\boxed{9}$ (12)	13 (6)

Table 8: Mean squared error ($\times 10^3$) for irregularly spaced fractional Gaussian noise featuring different degrees of missing observations for a range of Hurst parameters for the LoMPE and ℂLoMPE estimation procedures. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$								$n = 512$								$n = 1024$							
	Missingness proportion, p								Missingness proportion, p								Missingness proportion, p							
	5%		10%		15%		20%		5%		10%		15%		20%		5%		10%		15%		20%	
H	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE
0.6	$\boxed{2}$ (2)	$\boxed{2}$ (3)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (3)	3 (4)	$\boxed{3}$ (4)	$\boxed{3}$ (4)	$\boxed{1}$ (1)	$\boxed{1}$ (2)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	$\boxed{1}$ (2)	$\boxed{1}$ (2)	2 (2)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	2 (2)
0.7	$\boxed{3}$ (3)	$\boxed{3}$ (4)	$\boxed{3}$ (3)	4 (4)	$\boxed{3}$ (4)	4 (5)	$\boxed{3}$ (4)	4 (6)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	3 (2)	$\boxed{2}$ (2)	3 (2)	$\boxed{2}$ (2)	3 (3)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	3 (2)	$\boxed{3}$ (2)	$\boxed{3}$ (2)	$\boxed{3}$ (3)	4 (3)
0.8	$\boxed{3}$ (4)	$\boxed{3}$ (3)	$\boxed{3}$ (4)	$\boxed{3}$ (4)	$\boxed{3}$ (5)	$\boxed{3}$ (4)	$\boxed{4}$ (6)	5 (6)	$\boxed{3}$ (3)	$\boxed{3}$ (4)	$\boxed{3}$ (3)	4 (4)	$\boxed{4}$ (3)	$\boxed{4}$ (5)	$\boxed{4}$ (4)	5 (5)	$\boxed{3}$ (2)	4 (3)	$\boxed{4}$ (3)	$\boxed{4}$ (3)	$\boxed{4}$ (3)	5 (4)	$\boxed{5}$ (3)	6 (4)
0.9	$\boxed{3}$ (5)	4 (5)	$\boxed{4}$ (6)	$\boxed{4}$ (5)	$\boxed{4}$ (6)	$\boxed{4}$ (6)	$\boxed{4}$ (7)	$\boxed{4}$ (6)	$\boxed{3}$ (3)	$\boxed{3}$ (3)	4 (3)	$\boxed{3}$ (4)	$\boxed{4}$ (4)	$\boxed{4}$ (4)	$\boxed{4}$ (4)	$\boxed{4}$ (4)	$\boxed{4}$ (3)	$\boxed{4}$ (4)	$\boxed{5}$ (3)	$\boxed{5}$ (4)	$\boxed{6}$ (3)	$\boxed{6}$ (5)	$\boxed{6}$ (4)	$\boxed{6}$ (5)

Table 9: Mean squared error ($\times 10^3$) for irregularly spaced fractionally integrated process realizations featuring different degrees of missing observations for a range of Hurst parameters, $H = d + 1/2$, for the LoMPE and ℂLoMPE estimation procedures. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$								$n = 512$								$n = 1024$							
	Missingness proportion, p								Missingness proportion, p								Missingness proportion, p							
	5%		10%		15%		20%		5%		10%		15%		20%		5%		10%		15%		20%	
H	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE
0.6	$\boxed{2}$ (3)	3 (3)	$\boxed{3}$ (4)	$\boxed{3}$ (4)	$\boxed{3}$ (4)	$\boxed{3}$ (5)	$\boxed{3}$ (5)	4 (5)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	$\boxed{2}$ (3)	$\boxed{2}$ (1)	$\boxed{2}$ (2)	$\boxed{2}$ (1)	$\boxed{2}$ (2)	$\boxed{2}$ (1)	$\boxed{2}$ (2)	$\boxed{2}$ (1)	$\boxed{2}$ (2)
0.7	$\boxed{4}$ (5)	$\boxed{4}$ (5)	5 (6)	$\boxed{4}$ (6)	$\boxed{5}$ (6)	$\boxed{5}$ (6)	$\boxed{5}$ (5)	$\boxed{5}$ (6)	5 (4)	$\boxed{4}$ (4)	$\boxed{5}$ (4)	$\boxed{5}$ (4)	$\boxed{5}$ (4)	$\boxed{5}$ (5)	$\boxed{6}$ (4)	$\boxed{6}$ (5)	$\boxed{4}$ (3)	$\boxed{4}$ (4)	$\boxed{5}$ (3)	$\boxed{5}$ (4)	$\boxed{5}$ (4)	$\boxed{5}$ (4)	$\boxed{5}$ (4)	6 (5)
0.8	8 (9)	$\boxed{6}$ (8)	8 (9)	$\boxed{6}$ (8)	9 (10)	$\boxed{8}$ (9)	9 (9)	$\boxed{7}$ (9)	7 (6)	$\boxed{6}$ (6)	8 (6)	$\boxed{7}$ (7)	8 (7)	$\boxed{7}$ (8)	9 (7)	$\boxed{8}$ (8)	8 (5)	$\boxed{6}$ (5)	8 (5)	$\boxed{7}$ (5)	9 (6)	$\boxed{7}$ (6)	9 (6)	$\boxed{8}$ (6)
0.9	8 (8)	$\boxed{6}$ (7)	9 (10)	$\boxed{6}$ (8)	10 (10)	$\boxed{7}$ (9)	10 (10)	$\boxed{7}$ (9)	9 (7)	$\boxed{6}$ (8)	10 (8)	$\boxed{7}$ (8)	10 (8)	$\boxed{7}$ (9)	11 (10)	$\boxed{8}$ (10)	10 (6)	$\boxed{6}$ (6)	10 (6)	$\boxed{6}$ (6)	11 (7)	$\boxed{7}$ (6)	12 (7)	$\boxed{7}$ (7)

Table 10: Empirical bias ($\times 100$) of LoMPE estimation for fractional Brownian motion realizations with different degrees of missing observations for the LoMPE and ℂLoMPE estimation procedures. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$								$n = 512$								$n = 1024$							
	Missingness proportion, p								Missingness proportion, p								Missingness proportion, p							
	5%		10%		15%		20%		5%		10%		15%		20%		5%		10%		15%		20%	
H	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE
0.6	-5 (10)	$\boxed{-4}$ (4)	-6 (10)	$\boxed{-2}$ (4)	-6 (10)	$\boxed{-2}$ (5)	-7 (10)	$\boxed{-2}$ (5)	-8 (7)	$\boxed{-4}$ (3)	-8 (8)	$\boxed{-4}$ (3)	-8 (8)	$\boxed{-2}$ (3)	-8 (8)	$\boxed{-2}$ (3)	-9 (6)	$\boxed{-4}$ (2)	-9 (6)	$\boxed{-3}$ (2)	-10 (6)	$\boxed{-3}$ (2)	-10 (6)	$\boxed{-2}$ (2)
0.7	$\boxed{-2}$ (11)	-8 (5)	$\boxed{-3}$ (11)	-8 (5)	$\boxed{-4}$ (11)	-6 (5)	$\boxed{-4}$ (12)	-5 (6)	$\boxed{-6}$ (7)	-8 (3)	$\boxed{-7}$ (7)	$\boxed{-7}$ (3)	-7 (7)	$\boxed{-6}$ (3)	-7 (7)	$\boxed{-5}$ (3)	$\boxed{-8}$ (6)	$\boxed{-8}$ (2)	-8 (6)	$\boxed{-7}$ (2)	-8 (6)	$\boxed{-6}$ (2)	-8 (7)	$\boxed{-6}$ (2)
0.8	$\boxed{2}$ (10)	-9 (4)	$\boxed{1}$ (10)	-9 (5)	$\boxed{1}$ (11)	-8 (5)	$\boxed{0}$ (11)	-6 (5)	$\boxed{-3}$ (8)	-10 (3)	$\boxed{-3}$ (8)	-10 (3)	$\boxed{-4}$ (8)	-9 (4)	$\boxed{-4}$ (9)	-8 (4)	$\boxed{-6}$ (8)	-12 (2)	$\boxed{-6}$ (8)	-10 (2)	$\boxed{-6}$ (8)	-10 (2)	$\boxed{-7}$ (8)	-9 (2)
0.9	$\boxed{8}$ (13)	-9 (6)	$\boxed{7}$ (13)	-9 (6)	$\boxed{7}$ (13)	-8 (6)	$\boxed{6}$ (13)	-7 (6)	$\boxed{2}$ (11)	-12 (4)	$\boxed{1}$ (10)	-11 (4)	$\boxed{1}$ (11)	-10 (4)	$\boxed{1}$ (11)	-10 (4)	$\boxed{-2}$ (9)	-13 (3)	$\boxed{-3}$ (9)	-12 (3)	$\boxed{-3}$ (9)	-12 (3)	$\boxed{-3}$ (9)	-11 (3)

Table 11: Empirical bias ($\times 100$) of LoMPE estimation for fractional Gaussian noise realizations with different degrees of missing observations for the LoMPE and ℂLoMPE estimation procedures. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$								$n = 512$								$n = 1024$							
	Missingness proportion, p								Missingness proportion, p								Missingness proportion, p							
	5%		10%		15%		20%		5%		10%		15%		20%		5%		10%		15%		20%	
H	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE
0.6	$\boxed{-2}$ (4)	$\boxed{-2}$ (4)	$\boxed{-2}$ (4)	$\boxed{-2}$ (4)	$\boxed{-2}$ (4)	-3 (4)	$\boxed{-2}$ (5)	$\boxed{-2}$ (5)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	$\boxed{-2}$ (3)	-3 (3)	$\boxed{-2}$ (2)	-3 (2)	$\boxed{-2}$ (2)	-3 (2)	$\boxed{-2}$ (2)	-3 (2)	$\boxed{-3}$ (2)	-4 (2)
0.7	$\boxed{-2}$ (5)	-3 (5)	$\boxed{-3}$ (5)	-4 (5)	$\boxed{-3}$ (4)	-4 (5)	$\boxed{-3}$ (4)	-4 (5)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	-4 (3)	$\boxed{-3}$ (3)	-4 (3)	$\boxed{-4}$ (3)	-5 (3)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	-5 (2)	$\boxed{-5}$ (2)	$\boxed{-5}$ (2)	$\boxed{-5}$ (2)	-6 (2)
0.8	$\boxed{-2}$ (5)	$\boxed{-2}$ (5)	$\boxed{-3}$ (5)	$\boxed{-3}$ (5)	$\boxed{-3}$ (5)	$\boxed{-3}$ (5)	$\boxed{-4}$ (5)	$\boxed{-4}$ (6)	$\boxed{-5}$ (3)	$\boxed{-5}$ (3)	$\boxed{-5}$ (3)	$\boxed{-5}$ (3)	$\boxed{-6}$ (3)	$\boxed{-6}$ (4)	$\boxed{-6}$ (3)	$\boxed{-6}$ (4)	$\boxed{-5}$ (2)	-6 (2)	$\boxed{-6}$ (2)	$\boxed{-6}$ (3)	$\boxed{-6}$ (2)	-7 (3)	$\boxed{-6}$ (2)	-7 (3)
0.9	-2 (5)	$\boxed{1}$ (6)	-3 (5)	$\boxed{0}$ (6)	-3 (6)	$\boxed{0}$ (7)	-4 (5)	$\boxed{-1}$ (7)	-5 (3)	$\boxed{-3}$ (4)	-5 (3)	$\boxed{-4}$ (4)	-6 (3)	$\boxed{-4}$ (4)	-6 (3)	$\boxed{-4}$ (5)	$\boxed{-6}$ (2)	$\boxed{-6}$ (3)	-7 (2)	$\boxed{-6}$ (3)	$\boxed{-7}$ (2)	$\boxed{-7}$ (3)	$\boxed{-7}$ (2)	$\boxed{-7}$ (3)

Table 12: Empirical bias ($\times 100$) of LoMPE estimation for fractionally integrated process realizations with different degrees of missing observations for the LoMPE and ℂLoMPE estimation procedures. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result.

	$n = 256$								$n = 512$								$n = 1024$							
	Missingness proportion, p								Missingness proportion, p								Missingness proportion, p							
	5%		10%		15%		20%		5%		10%		15%		20%		5%		10%		15%		20%	
H	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE	LoMPE	ℂLoMPE
0.6	$\boxed{-2}$ (4)	-3 (5)	$\boxed{-2}$ (5)	-3 (5)	$\boxed{-2}$ (4)	-3 (5)	$\boxed{-3}$ (5)	-4 (5)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (3)	$\boxed{-3}$ (4)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)	$\boxed{-4}$ (2)
0.7	$\boxed{-4}$ (4)	$\boxed{-4}$ (5)	-5 (5)	$\boxed{-4}$ (5)	$\boxed{-5}$ (5)	$\boxed{-5}$ (5)	$\boxed{-5}$ (4)	$\boxed{-5}$ (5)	$\boxed{-6}$ (3)	$\boxed{-6}$ (3)	$\boxed{-6}$ (3)	$\boxed{-6}$ (4)	-7 (3)	$\boxed{-6}$ (3)	$\boxed{-7}$ (3)	$\boxed{-7}$ (4)	$\boxed{-6}$ (2)	$\boxed{-6}$ (3)	$\boxed{-6}$ (2)	$\boxed{-6}$ (3)	$\boxed{-7}$ (3)	$\boxed{-7}$ (3)	$\boxed{-7}$ (2)	$\boxed{-7}$ (3)
0.8	-7 (6)	$\boxed{-5}$ (6)	-7 (6)	$\boxed{-5}$ (6)	-7 (6)	$\boxed{-6}$ (7)	-8 (5)	$\boxed{-6}$ (6)	-8 (4)	$\boxed{-6}$ (5)	-8 (4)	$\boxed{-7}$ (5)	-8 (4)	$\boxed{-7}$ (5)	-8 (4)	$\boxed{-7}$ (5)	-8 (3)	$\boxed{-7}$ (3)	$\boxed{-8}$ (3)	$\boxed{-8}$ (3)	-9 (3)	$\boxed{-8}$ (4)	-9 (3)	$\boxed{-8}$ (4)
0.9	-7 (6)	$\boxed{-2}$ (7)	-8 (6)	$\boxed{-2}$ (8)	-8 (6)	$\boxed{-3}$ (8)	-8 (6)	$\boxed{-3}$ (8)	-9 (4)	$\boxed{-4}$ (6)	-9 (4)	$\boxed{-5}$ (7)	-9 (4)	$\boxed{-5}$ (7)	-10 (5)	$\boxed{-5}$ (7)	-9 (3)	$\boxed{-6}$ (5)	-10 (3)	$\boxed{-6}$ (5)	-10 (4)	$\boxed{-6}$ (5)	-10 (4)	$\boxed{-7}$ (5)

B.2 Complex-valued processes

To investigate the performance of our proposed $\mathbb{C}$LoMPE estimation method on complex-valued processes, we performed a simulation study on circularly symmetric complex-valued fractional Brownian motion, as well as improper complex-valued fractional Gaussian noise, as described in the articles Coeurjolly and Porcu (2017a) and Sykulski and Percival (2016) respectively. We assessed the performance using the same measures as above, for regular and irregularly spaced series (for which we induce different levels of missingness). For the regularly spaced fractional Brownian motion series, we compare our method against the technique in Coeurjolly and Porcu (2017b), denoted “CP”.

Table 13: Mean squared error ($\times 10^3$) for irregularly spaced complex-valued fractional Brownian motion series featuring different degrees of missing observations for a range of Hurst parameters for the LoMPE estimation procedure. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result for the regular setting.

	$n = 256$						$n = 512$						$n = 1024$					
	Missingness proportion, p						Missingness proportion, p						Missingness proportion, p					
	CP	$\mathbb{C}$LoMPE					CP	$\mathbb{C}$LoMPE					CP	$\mathbb{C}$LoMPE				
H	0%	0%	5%	10%	15%	20%	0%	0%	5%	10%	15%	20%	0%	0%	5%	10%	15%	20%
0.6	2 (3)	$\boxed{1}$ (2)	1 (2)	1 (1)	2 (2)	2 (3)	$\boxed{1}$ (2)	$\boxed{1}$ (1)	0 (0)	0 (1)	1 (1)	1 (1)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	0 (0)	0 (0)	0 (0)	0 (0)
0.7	2 (3)	$\boxed{1}$ (2)	1 (1)	1 (2)	1 (2)	2 (3)	$\boxed{1}$ (1)	$\boxed{1}$ (1)	1 (1)	1 (1)	1 (1)	1 (1)	$\boxed{0}$ (1)	2 (1)	1 (1)	1 (1)	1 (1)	0 (0)
0.8	3 (3)	$\boxed{2}$ (2)	2 (2)	1 (2)	2 (2)	2 (2)	$\boxed{1}$ (2)	2 (2)	1 (2)	1 (2)	1 (1)	1 (2)	$\boxed{1}$ (1)	3 (2)	2 (2)	2 (1)	2 (1)	1 (1)
0.9	$\boxed{2}$ (3)	3 (4)	2 (3)	2 (3)	2 (3)	2 (2)	$\boxed{1}$ (2)	2 (2)	2 (3)	2 (2)	2 (2)	2 (2)	$\boxed{2}$ (2)	$\boxed{2}$ (2)	3 (2)	3 (2)	3 (2)	2 (2)

Table 14: Empirical bias ($\times 100$) of $\mathbb{C}$LoMPE estimation for complex-valued fractional Brownian motion realizations with different degrees of missing observations. Numbers in brackets are the estimation errors' standard deviation. Boxed numbers indicate best result for the regular setting.

	$n = 256$						$n = 512$						$n = 1024$					
	Missingness proportion, p						Missingness proportion, p						Missingness proportion, p					
	CP	$\mathbb{C}$LoMPE					CP	$\mathbb{C}$LoMPE					CP	$\mathbb{C}$LoMPE				
H	0%	0%	5%	10%	15%	20%	0%	0%	5%	10%	15%	20%	0%	0%	5%	10%	15%	20%
0.6	$\boxed{0}$ (5)	$\boxed{0}$ (3)	1 (3)	1 (3)	2 (4)	2 (4)	$\boxed{0}$ (4)	-2 (2)	-1 (2)	0 (2)	1 (2)	1 (2)	$\boxed{0}$ (2)	-2 (1)	-1 (1)	-1 (1)	0 (1)	1 (2)
0.7	$\boxed{-1}$ (4)	-2 (3)	0 (3)	0 (3)	0 (4)	1 (4)	$\boxed{0}$ (3)	-3 (2)	-2 (2)	-2 (2)	-1 (2)	0 (3)	$\boxed{0}$ (2)	-4 (1)	-3 (1)	-2 (1)	-2 (2)	-1 (2)
0.8	$\boxed{0}$ (5)	-2 (4)	-1 (4)	-1 (4)	0 (4)	0 (4)	$\boxed{0}$ (4)	-3 (2)	-3 (2)	-2 (2)	-2 (3)	-1 (3)	$\boxed{0}$ (2)	-5 (2)	-4 (2)	-4 (2)	-4 (2)	-3 (2)
0.9	$\boxed{0}$ (5)	1 (5)	2 (5)	1 (4)	0 (4)	0 (4)	$\boxed{0}$ (3)	-2 (4)	-3 (3)	-3 (3)	-3 (3)	-3 (3)	$\boxed{0}$ (2)	-4 (2)	-5 (2)	-5 (2)	-5 (2)	-4 (2)

Table 15: Mean squared error ($\times 10^3$) for irregularly spaced complex-valued fractional Gaussian noise featuring different degrees of missing observations for a range of Hurst parameters for the $\mathbb{C}$LoMPE estimation procedure. Numbers in brackets are the estimation errors' standard deviation.

	$n = 256$					$n = 512$					$n = 1024$				
	Proportion of missingness, p					Proportion of missingness, p					Proportion of missingness, p				
H	0%	5%	10%	15%	20%	0%	5%	10%	15%	20%	0%	5%	10%	15%	20%
0.6	1 (2)	1 (2)	1 (2)	1 (2)	2 (2)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
0.7	2 (2)	2 (2)	2 (2)	2 (3)	2 (3)	1 (1)	2 (2)	2 (2)	3 (2)	3 (2)	2 (1)	2 (1)	2 (1)	3 (2)	3 (2)
0.8	2 (2)	2 (3)	2 (3)	2 (3)	3 (5)	2 (2)	3 (3)	3 (3)	4 (3)	4 (4)	2 (2)	3 (2)	3 (2)	4 (2)	5 (3)
0.9	3 (4)	3 (3)	3 (3)	3 (4)	3 (5)	2 (2)	2 (3)	3 (3)	3 (3)	3 (3)	2 (2)	3 (2)	3 (2)	4 (3)	4 (3)

Table 16: Empirical bias ($\times 100$) of $\mathbb{C}$LoMPE estimation for complex-valued fractional Gaussian noise realizations with different degrees of missing observations. Numbers in brackets are the estimation errors' standard deviation.

	$n = 256$					$n = 512$					$n = 1024$				
	Proportion of missingness, p					Proportion of missingness, p					Proportion of missingness, p				
H	0%	5%	10%	15%	20%	0%	5%	10%	15%	20%	0%	5%	10%	15%	20%
0.6	-1 (3)	-2 (3)	-2 (3)	-2 (3)	-2 (3)	-2 (2)	-2 (2)	-2 (2)	-2 (2)	-2 (2)	-2 (1)	-2 (2)	-3 (2)	-3 (1)	-3 (2)
0.7	-1 (3)	-2 (3)	-2 (4)	-3 (4)	-3 (4)	-3 (2)	-4 (2)	-4 (2)	-4 (2)	-5 (2)	-4 (1)	-4 (1)	-5 (1)	-5 (2)	-6 (1)
0.8	0 (4)	-1 (4)	-2 (4)	-2 (4)	-3 (4)	-4 (3)	-4 (3)	-5 (3)	-6 (3)	-6 (3)	-5 (2)	-5 (2)	-6 (2)	-6 (2)	-7 (2)
0.9	2 (5)	0 (5)	0 (5)	0 (5)	-2 (5)	-2 (3)	-3 (4)	-4 (4)	-4 (3)	-4 (4)	-4 (2)	-5 (2)	-5 (2)	-6 (2)	-6 (2)

Tables 13 – 16 report the MSE and bias performance for the $\mathbb{C}$LoMPE approach for the complex-valued processes considered. Whilst designed for irregularly spaced series, our method performs competitively with the CP method on regularly sampled series. Note that departing from sampling regularity, there is only a mild decrease in the performance of $\mathbb{C}$LoMPE in terms of MSE and bias when the irregularity increases. Our proposed algorithm also performs well for regularly and irregularly spaced complex fractional Gaussian noise, with quite consistent behaviour with higher degrees of missingness.

References

Coeurjolly JF, Porcu E (2017a) Fast and exact simulation of complex-valued stationary Gaussian processes through embedding circulant matrix. J Comput Graph Stat (to appear)

Coeurjolly JF, Porcu E (2017b) Properties and Hurst exponent estimation of the circularly-symmetric fractional Brownian motion. Stat Prob Lett 128:21 – 27

Knight MI, Nunes MA (2012) nlt: a nondecimated lifting scheme algorithm. URL http://CRAN.R-project.org/package=nlt, R package version 2.1-3

Knight MI, Nunes MA (2016) liftLRD: Wavelet lifting estimators for the Hurst exponent for regular and irregular time series. R package version 1.0-5

Knight MI, Nason GP, Nunes MA (2017) A wavelet lifting approach to long-memory estimation. Stat Comput 27(6):1453–1471

Peng CK, Buldyrev SV, Havlin S, Simons M, Stanley HE, Goldberger AL (1994) Mosaic organization of DNA nucleotides. Phys Rev E 49(2):1685

Sykulski AM, Percival DB (2016) Exact simulation of noncircular or improper complex-valued stationary Gaussian processes using circulant embedding. In: IEEE 26th Intern. Work. Mach. Learn. Sig. Proc. (MLSP) 2016, pp 1–6

Wuertz D, et al (2013) fARMA: ARMA Time Series Modelling. URL http://CRAN.R-project.org/package=fArma, R package version 3010.79