Supplementary Material: A New Look at the Flexible Generalized Skew-Normal Family: A Trimodal Extension and Numerical Insights

Michele Bufalo[1] and Andrea Nigri[2,*]

[1]Department of Economics, Management and Business Law, University of Bari Aldo Moro, Bari, Italy. michele.bufalo@uniba.it
[2]Department of Social Sciences, University of Foggia, Foggia, Italy. andrea.nigri@unifg.it
[*]Corresponding author: Andrea Nigri, andrea.nigri@unifg.it

0.1 Bivariate FGSN density

The bivariate extension of the density given in Definition 2.1 is introduced, as outlined in [Azzalini, 2013, Section 7.2.1] and discussed by Mahdavi et al. [2021].

Definition 0.1. A random vector (X_0, X_1) is said to be (standard) bivariate flexible generalized skew-normally distributed if its probability density function is given by

$$f_{(X_0,X_1)}(x_0, x_1) = 2\varphi(x_0, x_1; \rho)\Phi(W(x_0, x_1)) \qquad ((x_0, x_1) \in \mathbb{R}^2) \tag{1}$$

where

$$\varphi(x_0, x_1; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x_0^2 - 2\rho x_0 x_1 + x_1^2\right]\right\}$$

is the (standard) bivariate normal density with correlation coefficient $\rho \in (-1, 1)$ and

$$\begin{aligned} W(x_0, x_1) = {} & \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_0^3 + \alpha_3 x_1^3 + \alpha_4 x_0^5 + \alpha_5 x_1^5 + \alpha_6 x_0 x_1^2 + \\ & \alpha_7 x_0^2 x_1 + \alpha_8 x_0 x_1^4 + \alpha_9 x_0^4 x_1 + \alpha_{10} x_0^2 x_1^3 + \alpha_{11} x_0^3 x_1^2 \end{aligned} \tag{2}$$

is a bivariate (odd) polynomial of order $K = 5$. According to the definition in the univariate case, we assume $\alpha_0 = \alpha_1 = 0$.

Remark 0.2. Observe that the joint density (1) is a well-defined bivariate density as explained in Mahdavi et al. [2021]. Indeed, more generally, Azzalini and Capitanio [2003], showed that it is well-defined the following multivariate probability density function

$$f(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = 2|\boldsymbol{\Sigma}|^{-1/2} f_0(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{x} - \boldsymbol{\mu}))G_0(w(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{x} - \boldsymbol{\mu}))) \qquad (\boldsymbol{x} \in \mathbb{R}^n),$$

where $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ are the location vector and the scale covariance matrix, respectively; f_0 is a n-variate centrally symmetric pdf with respect to the origin (i.e., $f_0(\boldsymbol{x}) = f_0(-\boldsymbol{x})$); G_0 is a univariate cdf satisfying $G_0(-x) = 1 - G_0(x)$; and w odd real-valued function.
However, up now, it remains unknown the copula function that links the joint density (1) to its marginals except for the particular case of the multivariate skew-normal density (see [Yoshiba et al., 2023, Lemma 1]). The reason of this lack may depend from the absence of a stochastic representation of a flexible generalized skew-normal variable (see, e.g., Remark 0.3)

It should be noted that each coefficient $\alpha_i; (i = 0, 1, \ldots, 11)$ depends on the (rescaled) shape parameters of X_0, X_1, and the correlation parameter ρ, as demonstrated by Azzalini and Dalla Valle [1996] in the simplified case where $K = 1$. However, for $K > 1$, no explicit expressions for the coefficients α_i are currently available, as the stochastic representation of a random variable with density given by Definition 2.1 remains unknown. This gap in the literature will be addressed in future research. By application of Bayes' theorem, the conditional density of $(X_1 \mid X_0)$ is expressed as follows:

$$f_{(X_1|X_0)}(x) = \frac{\varphi(x; x_0, \rho)\Phi(W(x))}{\varphi(x_0)\Phi(w(x_0))} \qquad (x \in \mathbb{R}), \tag{3}$$

where $W(x)$ denotes the polynomial (2) with respect to the only variable x, which can be rewritten as

$$W(x) = \sum_{i=0}^{5} \tilde{\alpha}_i x^i, \tag{4}$$

with

$$\tilde{\alpha}_0 = \alpha_0 x_0 + \alpha_2 x_0^3 + \alpha_4 x_0^5, \quad \tilde{\alpha}_1 = \alpha_1 + \alpha_7 x_0^2 + \alpha_9 x_0^4, \quad \tilde{\alpha}_2 = \alpha_6 x_0 + \alpha_{11} x_0^3,$$
$$\tilde{\alpha}_3 = \alpha_3 + \alpha_{10} x_0^2, \quad \tilde{\alpha}_4 = a_8 x_0, \quad \tilde{\alpha}_5 = \alpha_5. \tag{5}$$

The above Definition 0.1 is still valid for the benchmarks densities of Section 3. More precisely:

- the bivariate FGSN of Ma and Genton [2004] is given by

  $$f_{(X_0,X_1)}(x_0, x_1) = 2\varphi(x_0, x_1; \rho)\Phi(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_0^3 + \alpha_3 x_1^3 + \alpha_4 x_0 x_1^2 + \alpha_5 x_0^2 x_1) \qquad ((x_0, x_1) \in \mathbb{R}^2) \tag{6}$$

  (see Mahdavi et al. [2021]);

- the bivariate Azzalini SN density is given by

  $$f_{(X_0,X_1)}(x_0, x_1) = 2\varphi(x_0, x_1; \rho)\Phi(\alpha_1 x_0 + \alpha_2 x_1) \qquad ((x_0, x_1) \in \mathbb{R}^2) \tag{7}$$

  where the coefficients α_0, α_1 are expressed in closed formula through the rescaled shape parameters $\delta_i = \frac{\alpha_i}{\sqrt{1+\alpha_i^2}}$ of X_i $(i = 0, 1)$, i.e.,

  $$\begin{cases} \alpha_0 = \frac{\delta_0 - \delta_1 \rho}{\sqrt{(1-\rho^2)(1-\rho^2-\delta_0^2-\delta_1^2+2\delta_0\delta_1\rho)}} \\ \alpha_1 = \frac{\delta_1 - \delta_0 \rho}{\sqrt{(1-\rho^2)(1-\rho^2-\delta_0^2-\delta_1^2+2\delta_0\delta_1\rho)}} \end{cases} \tag{8}$$

  (see [Azzalini and Dalla Valle, 1996, Section 3]).

Remark 0.3. As a further research, we strongly believe that relation (8) may be extended for the FGSN densities (1) and (6). As a consequence, any α_i's could be expressed just as function of the basic parameters δ_0, δ_1 and ρ, with a considerable gain from a computational point of view. This would be possible under a stochastic representation of a FGSN random variable, that is not available, up now.

References

A. Azzalini. *The skew-normal and related families*, volume 3. Cambridge University Press, 2013.

A. Azzalini and A. Capitanio. Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t distribution. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65:367–389, 2003.

A. Azzalini and A. Dalla Valle. The multivariate skew-normal distribution. *Biometrika*, 83(4):715–726, 1996.

Y. Ma and M. G. Genton. Flexible class of skew-symmetric distributions. *Scandinavian Journal of Statistics*, 31:459–468, 2004.

A. Mahdavi, V. Amirzadeh, A. Jamalizadeh, and T. Lin. A multivariate flexible skew-symmetric-normal distribution: Scale-shape mixtures and parameter estimation via selection representation. *Symmetry*, 13:1343, 2021.

T. Yoshiba, T. Koike, and S. Kato. On a measure of tail asymmetry for the bivariate skew-normal copula. *Symmetry*, 15(17):1410, 2023.