

Supplementary Material for “Mathematical Formulations and an Enhanced Column Generation for the Hybrid Flow Shop Scheduling Problem”

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S1 Additional Experiments

Here we used up to five methods or algorithms in three complementary computational experiments. These algorithms are described in Table 1. The name and a description are in Column 1 and 2, respectively. Column 3 shows the section that the algorithms/models appears.

All additional experiments were executed on the same hardware used in the main study: an Intel Core i7-4790K 4.00 GHz processor, 16 GB RAM, and Ubuntu 18.04 LTS. The tests were conducted on a subset of 15 instances from the original benchmark set, using Python 3.10.16 and CPLEX 22.1.2. In Section S5 the tests conducted in CPLEX 12.10 used the same machines, but Python 3.6.9.

The experiments were designed to further analyze the fairness of algorithmic comparisons, the acceleration of the pricing problem, and the impact of the solver version.

Table 1: Description of the algorithms and models used in the experiments.

Algorithm or Model	Description	Sections
FB_1H	FB model, as described in the article, with a runtime limit of 1 hour.	Secs. 2
FB_2H	FB model, as described in the article, with a runtime limit of 2 hours.	Sec. 2, 3, 4, and 5
ECG	Column generation algorithm, as described in the article. The column generation phase has unlimited time, while the master problem is solved with integrality activated and limited to 1 hour.	Sec. 5
HCG	Modified version of the column generation algorithm in which the first phase (column generation) behaves heuristically. Thus, it does not generate all columns and does not guarantee a valid lower bound.	Sec. 3
ECG_HPS	The column generation algorithm is modified so that the pricing (still solved by CPLEX) behaves heuristically (e.g., stops after the first integer solution or emphasizes feasibility heuristics), producing columns faster.	Sec. 4

Unless stated otherwise, all tables in this Supplementary Material follow the same format. The first columns describe the problem instances, including the number of jobs, machines, processing and setup time, and a replication index, which serves as a counter to identify instances generated under the same parameter settings but differing in random seeds. Subsequent columns report performance indicators such as upper and lower bounds, runtime, and optimality gaps, upper bound and lower bound deviation as defined in the main text.

S2 Comparison of FB running with 1 or 2 hours

We conduct an experiment to evaluate the effect of extending the execution time of the FB algorithm from one hour (FB_1H) to two hours (FB_2H). The goal was to verify whether longer computational times would significantly improve the obtained upper bounds, since both HCG and ECG_HPS converge within approximately 1.5 hour for the largest instances.

Tables 2 and 3 summarize the comparative results between the FB_1H and FB_2H formulations across all benchmark instances. Tables formatting and column descriptions are consistent with those presented in the introductory section (Section S1) of this Supplementary Material.

Table 2: Bounds (UB/LB) and runtime for FB_1H and FB_2H.

Jobs	Mach.	Time	Rep.	UB		LB		Time (s)	
				FB_1H	FB_2H	FB_1H	FB_2H	FB_1H	FB_2H
6	2	0005-0030	1	295	295	245	245	3	3
6	2	0005-0030	2	279	279	222	222	10	10
6	5	0005-0030	1	207	207	199	199	1	1
6	5	0005-0030	2	190	190	181	181	2	1
6	5	0005-0030	3	196	196	185	185	2	2
16	2	0005-0030	1	1,602	1,605	1,121	1,121	3,601	7,201
16	2	0005-0030	2	1,985	1,985	1,414	1,414	3,600	7,200
16	5	0005-0030	1	908	916	774	774	3,601	7,202
16	5	0005-0030	2	893	893	787	787	3,601	7,202
16	5	0005-0030	3	843	844	727	727	3,601	7,201
30	2	0005-0030	1	8,445	8,445	3,851	3,851	3,602	7,202
30	2	0005-0030	2	8,259	8,185	3,032	3,032	3,602	7,202
30	5	0005-0030	1	3,439	3,439	1,693	1,693	3,602	7,203
30	5	0005-0030	2	3,234	3,234	1,831	1,831	3,600	7,202
30	5	0005-0030	3	2,818	2,746	1,790	1,790	3,602	7,202

Table 3: Relative indicators for FB_1H and FB_2H: GAP and deviations.

Jobs	Mach.	Time	Rep.	GAP (%)		UB Deviation (%)		LB Deviation (%)	
				FB_1H	FB_2H	FB_1H	FB_2H	FB_1H	FB_2H
6	2	0005-0030	1	17.12	17.12	0.00	0.00	16.55	16.55
6	2	0005-0030	2	20.31	20.31	0.00	0.00	20.02	20.02
6	5	0005-0030	1	3.86	3.86	0.00	0.00	3.86	3.86
6	5	0005-0030	2	5.00	5.00	0.00	0.00	5.00	5.00
6	5	0005-0030	3	5.61	5.61	0.00	0.00	6.09	6.09
16	2	0005-0030	1	30.00	30.13	4.98	5.18	26.32	26.32
16	2	0005-0030	2	28.77	28.77	8.12	8.12	22.99	22.99
16	5	0005-0030	1	14.74	15.48	1.57	2.46	13.31	13.31
16	5	0005-0030	2	11.83	11.83	0.45	0.45	11.03	11.03
16	5	0005-0030	3	13.80	13.93	1.69	1.81	12.13	12.13
30	2	0005-0030	1	54.40	54.40	56.53	56.53	28.38	28.38
30	2	0005-0030	2	63.29	62.95	70.92	69.39	36.82	36.82
30	5	0005-0030	1	50.77	50.77	50.64	50.64	24.19	24.19
30	5	0005-0030	2	43.39	43.39	36.63	36.63	21.08	21.08
30	5	0005-0030	3	36.47	34.81	23.22	20.07	20.26	20.26

The results, presented in Tables 2 and 3, clearly indicate that extending the FB runtime from one hour to two hours yields only marginal benefit. Across almost all instances, the relative deviations between FB_1H and FB_2H remain below 5%,

demonstrating that the FB formulation effectively stabilizes within the first hour of computation. Moreover, although slight improvements can be observed in the upper bounds when allowing two hours, these gains do not translate into any substantial reduction in optimality gaps, which remain consistently around 50–52% for the 30-job instances. Taken together, these findings reinforce that additional runtime does not materially change the performance of the FB model, and that one hour is sufficient for it to reach its practical limit of progress.

S3 Fair-Time Comparison Between FB and Column Generation Variants

In this experiment, the Enhanced Column Generation (ECG) algorithm was modified to operate in a heuristic manner by interrupting the iterative column generation cycle after the construction of the initial pool of columns. Once this pool was generated, the Restricted Master Problem (RMP) was solved only once, without further column generation iterations. This algorithm, denominated Heuristic Column Generation (HCG), enables a fair comparison with the Flow-Based (FB) formulation under comparable time limits.

Table 4, presents the results of this experiment. Table formatting and column descriptions are consistent with those presented in the introductory section (Section S1) of this Supplementary Material.

The results show that HCG consistently delivers superior upper bounds compared with the FB formulation, including on the largest 30-job instances, where convergence was reached after approximately 1.5 hour. This behavior aligns with the expected characteristics of heuristic column generation, which typically yields high-quality feasible solutions early in the search. Furthermore, HCG exhibited considerably shorter runtimes, reinforcing that terminating the generation process immediately after constructing the initial pool is sufficient to obtain strong feasible solutions efficiently.

S4 Enhanced Column Generation with Heuristic Pricing Subproblem (ECG_HPS)

Another experiment was conducted to evaluate the impact of heuristic settings on the pricing subproblems of Enhanced Column Generation (ECG). In this configuration, the CPLEX solver was adjusted to operate in a heuristic mode, prioritizing the identification of feasible integer solutions at early stages of the optimization.

Table 4: Comparison between FB_2H and HCG across instances.

Jobs	Mach.	Time	Rep.	UB		LB		GAP (%)		UB Dev. (%)		LB Dev. (%)		Time (s)	
				FB_2H	HCG	FB_2H	HCG	FB_2H	HCG	FB_2H	HCG	FB_2H	HCG	FB_2H	HCG
6	2	0005-0030	1	295	295	245	293	17.12	0.68	0.00	0.00	16.55	0.00	3	6
6	2	0005-0030	2	279	281	222	278	20.31	1.07	0.00	0.72	20.02	0.00	10	2
6	5	0005-0030	1	207	207	199	207	3.86	0.00	0.00	0.00	3.86	0.00	1	2
6	5	0005-0030	2	190	190	181	190	5.00	0.00	0.00	0.00	5.00	0.00	2	16
6	5	0005-0030	3	196	197	185	197	5.61	0.00	0.00	0.51	6.09	0.00	2	1
16	2	0005-0030	1	1,605	1,526	1,121	1,522	30.13	0.26	5.18	0.00	26.32	0.00	7,201	256
16	2	0005-0030	2	1,985	1,836	1,414	1,836	28.77	0.00	8.12	0.00	22.99	0.00	7,200	260
16	5	0005-0030	1	916	896	774	890	15.48	0.67	2.23	0.00	13.02	0.00	7,202	469
16	5	0005-0030	2	893	890	787	882	11.83	0.90	0.34	0.00	10.73	0.00	7,202	474
16	5	0005-0030	3	844	830	727	827	13.90	0.36	1.69	0.00	12.13	0.00	7,201	431
30	2	0005-0030	1	8,445	5,395	3,851	5,320	54.40	1.39	56.53	0.00	27.61	0.00	7,202	2,037
30	2	0005-0030	2	8,185	4,908	3,032	4,751	62.95	3.20	66.77	0.00	36.18	0.00	7,202	2,036
30	5	0005-0030	1	3,439	2,296	1,693	2,190	50.77	4.62	49.78	0.00	22.70	0.00	7,203	4,167
30	5	0005-0030	2	3,234	2,376	1,831	2,318	43.39	2.44	36.11	0.00	21.01	0.00	7,202	4,408
30	5	0005-0030	3	2,746	2,287	1,790	2,223	34.81	2.80	20.07	0.00	19.47	0.00	7,202	4,118

When no improving column was found through this heuristic process, the algorithm automatically reverted to the exact formulation to ensure convergence.

The main goal of this experiment was to investigate whether this modification could accelerate ECG convergence by enabling faster generation of promising columns in the initial iterations.

Tables 5 and 6 formatting and column descriptions are consistent with those presented in the introductory section (Section S1) of this Supplementary Material.

As reported in Tables 5 and 6, the Enhanced Column Generation with Heuristic Pricing Subproblem (ECG_HPS) notably reduces computational time while preserving the quality of the obtained solutions. Across all tested instances, the method consistently achieved optimality gaps below 2.2%, and was able to complete even the largest scenarios, 30 jobs on 5 machines, in under 1.5 hour. These results confirm that incorporating heuristic pricing retains the performance guarantees of the exact framework while enabling substantial efficiency gains.

These results indicate that introducing heuristic elements to solve the pricing subproblem can substantially improve the efficiency of the column generation process. The heuristic pricing configuration therefore represents a promising balance between computational speed and solution quality, offering a practical enhancement to the ECG framework.

S5 Solver Version Update

Another experiment was conducted to examine the effect of the solver version on the performance of both the Flow-Based (FB) and Enhanced Column Generation (ECG)

Table 5: Upper and lower bounds (UB/LB) and computational time for FB_2H and ECG_HPS formulations.

Jobs	Mach.	Time	Rep.	UB		LB		Time (s)	
				FB_2H	ECG_HPS	FB_2H	ECG_HPS	FB_2H	ECG_HPS
6	2	0005-0030	1	295	295	245	293	3	6
6	2	0005-0030	2	279	281	222	278	10	2
6	5	0005-0030	1	207	207	199	207	1	3
6	5	0005-0030	2	190	190	181	190	2	17
6	5	0005-0030	3	196	197	185	197	2	2
16	2	0005-0030	1	1,605	1,526	1,121	1,517	7,201	245
16	2	0005-0030	2	1,985	1,839	1,414	1,836	7,200	244
16	5	0005-0030	1	916	894	774	893	7,202	448
16	5	0005-0030	2	893	889	787	885	7,202	463
16	5	0005-0030	3	844	829	727	825	7,201	444
30	2	0005-0030	1	8,445	5,414	3,851	5,377	7,202	2,111
30	2	0005-0030	2	8,185	4,832	3,032	4,799	7,202	2,231
30	5	0005-0030	1	3,439	2,283	1,693	2,233	7,203	4,171
30	5	0005-0030	2	3,234	2,367	1,831	2,320	7,202	4,179
30	5	0005-0030	3	2,746	2,293	1,790	2,245	7,202	4,148

Table 6: GAP and deviation values for FB_2H and ECG_HPS formulations.

Jobs	Mach.	Time	Rep.	GAP (%)		UB Deviation (%)		LB Deviation (%)	
				FB_2H	ECG_HPS	FB_2H	ECG_HPS	FB_2H	ECG_HPS
6	2	0005-0030	1	17.12	0.68	0.00	0.00	16.55	0.00
6	2	0005-0030	2	20.31	1.07	0.00	0.72	20.02	0.00
6	5	0005-0030	1	3.86	0.00	0.00	0.00	3.86	0.00
6	5	0005-0030	2	5.00	0.00	0.00	0.00	5.00	0.00
6	5	0005-0030	3	5.61	0.00	0.00	0.51	6.09	0.00
16	2	0005-0030	1	30.13	0.59	5.18	0.00	26.08	0.00
16	2	0005-0030	2	28.77	0.19	7.94	0.00	22.97	0.00
16	5	0005-0030	1	15.48	0.11	2.46	0.00	13.31	0.00
16	5	0005-0030	2	11.83	0.45	0.45	0.00	11.03	0.00
16	5	0005-0030	3	13.90	0.48	1.81	0.00	11.92	0.00
30	2	0005-0030	1	54.40	0.68	55.98	0.00	28.38	0.00
30	2	0005-0030	2	62.95	0.68	69.39	0.00	36.82	0.00
30	5	0005-0030	1	50.77	2.19	50.64	0.00	24.19	0.00
30	5	0005-0030	2	43.39	1.99	36.63	0.00	21.08	0.00
30	5	0005-0030	3	34.81	2.09	19.76	0.00	20.26	0.00

formulations. The results compared the runs obtained with CPLEX 22.1.2 with those of CPLEX 12.10, in the article, under identical computational settings.

Tables 7 and 8 report the effect of the CPLEX version on the performance of the FB_1H formulation and the ECG algorithm, respectively. In both tables, the first four columns identify each instance consistent with those presented in the introductory section (Section S1) of this Supplementary Material. The following two columns present the upper bounds (UB) obtained with CPLEX 22 and CPLEX 12. The last two columns show the total runtime (in seconds) under the identical solver versions. For FB_1H (Table 7), the UB values are largely stable across versions for small and medium instances, while differences appear for the largest cases (e.g., 30 jobs, 2 machines). For ECG (Table 8), the upper bounds exhibit more variability across versions, and the computational times of the ECG formulation varied between instances, sometimes decreasing and sometimes increasing with the newer version.

Table 7: FB_1H: upper bounds (UB) and runtime with CPLEX 22 vs CPLEX 12.

Jobs	Mach.	Time	Rep.	UB		Time (s)	
				CPLEX 22	CPLEX 12	CPLEX 22	CPLEX 12
6	2	0005-0030	1	295	295	3	4
6	2	0005-0030	2	279	279	10	10
6	5	0005-0030	1	207	207	1	1
6	5	0005-0030	2	190	190	2	2
6	5	0005-0030	3	196	196	2	2
16	2	0005-0030	1	1,602	1,606	3,601	3,600
16	2	0005-0030	2	1,985	1,956	3,600	3,601
16	5	0005-0030	1	908	896	3,601	3,601
16	5	0005-0030	2	893	915	3,601	3,601
16	5	0005-0030	3	843	850	3,601	3,601
30	2	0005-0030	1	8,445	9,381	3,602	3,600
30	2	0005-0030	2	8,259	8,775	3,602	3,600
30	5	0005-0030	1	3,439	3,439	3,600	3,600
30	5	0005-0030	2	3,234	3,234	3,600	3,600
30	5	0005-0030	3	2,818	2,767	3,602	3,600

These mixed results suggest that the solver update did not systematically enhance the quality of the solution or the runtime. In the case of the ECG approach, part of this variation may also come from the stochastic nature of the heuristic that generates initial columns, which introduces natural fluctuations in computational effort and convergence time.

The solver version had no significant impact on the relative performance of the formulations: the ECG model continues to outperform the FB formulation in bound quality, but with higher computational effort.

Table 8: ECG: upper bounds (UB) and runtime with CPLEX 22 vs CPLEX 12.

Jobs	Mach.	Time	Rep.	UB		Time (s)	
				CPLEX 22	CPLEX 12	CPLEX 22	CPLEX 12
6	2	0005-0030	1	295	295	10	8
6	2	0005-0030	2	281	280	17	16
6	5	0005-0030	1	207	207	12	6
6	5	0005-0030	2	190	190	30	24
6	5	0005-0030	3	196	197	22	11
16	2	0005-0030	1	1,527	1,524	897	686
16	2	0005-0030	2	1,839	1,836	772	683
16	5	0005-0030	1	976	970	4,993	4,580
16	5	0005-0030	2	889	889	2,808	3,438
16	5	0005-0030	3	854	856	5,385	5,036
30	2	0005-0030	1	5,432	5,375	11,927	10,816
30	2	0005-0030	2	4,865	4,874	21,723	15,071
30	5	0005-0030	1	3,809	3,681	32,860	26,565
30	5	0005-0030	2	12,999	13,288	26,868	32,195
30	5	0005-0030	3	4,593	3,722	37,122	29,453