On Some Properties of Cronbach's α Coefficient for Interval-Valued Data in Questionnaires (Supplementary Information)

José García-García[1], María Ángeles Gil[1,2], María Asunción Lubiano[1*]

[1]Department of Statistics, OR, and Didactics of Mathematics, University of Oviedo, Faculty of Sciences, Oviedo, 33007, Spain.
[2]Spanish Royal Academy of Sciences, C/ Valverde, 22, Madrid, 28004, Spain.

*Corresponding author(s). E-mail(s): lubiano@uniovi.es;
Contributing authors: garciagarjose@uniovi.es; magil@uniovi.es;

Submitted to

Advances in Data Analysis and Classification

Supplementary Information, Part 1: Proofs of Main Results in Sections 2 and 3

Proof of Proposition 1.*iii)*

Let $\mathbb{H}_2 = \{f : \{-1,1\} \to \mathbb{R}, f \in L^2(\{-1,1\}, \lambda_1)\}$ where $\lambda_1 \equiv \mathrm{U}\{-1,1\}$.

On $\mathbb{H}_2$ one can define the θ-*inner product* given by

$$\langle f, g\rangle_\theta = \sum_{u\in\{-1,1\}} \frac{(f(u)-f(-u))\cdot(g(u)-g(-u))}{8} + \theta \sum_{u\in\{-1,1\}} \frac{(f(u)+f(-u))\cdot(g(u)+g(-u))}{8}$$

and the associated θ-*norm*, that is, $\|f\|_\theta = \sqrt{\langle f, f\rangle_\theta}$.

On the one hand, we will have that $\mathbb{H}_2$ with the usual functional arithmetic and the inner product and norm above is a Hilbert space of functions. Moreover, the Cauchy-Schwarz inequality for inner products and norms states that $|\langle f, g\rangle_\theta| \le \|f\|_\theta \cdot \|g\|_\theta$ with equality iff f and g are linearly dependent.

On the other hand, the *support function* of a nonempty compact interval $A \in \mathcal{K}_c(\mathbb{R})$, suggested by Minkowski (1903) in a more general setting, characterizes a kind of 'outer cortex' of A and it is given by the mapping $\eta_A : \{-1,1\} \to \mathbb{R}$ such that $\eta_A(u) = \sup\{a.u : a \in A\}$, so that $\eta_A(-1) = -\inf A$ and $\eta_A(1) = \sup A$.

In particular, if $A, B \in \mathcal{K}_c(\mathbb{R})$, then

$$\langle \eta_A, \eta_B\rangle_\theta = \mathrm{mid}\, A \cdot \mathrm{mid}\, B + \theta \cdot \mathrm{spr}\, A \cdot \mathrm{spr}\, B,$$

and whatever random intervals $\mathcal{X}$ and $\mathcal{Y}$ may be

$$\eta_{\mathrm{E}(\mathcal{X})} = E(\eta_{\mathcal{X}}), \quad \mathrm{Var}(\mathcal{X}) = E\left(\|\eta_{\mathcal{X}} - \eta_{\mathrm{E}(\mathcal{X})}\|_\theta^2\right),$$

$$\mathrm{Cov}(\mathcal{X}, \mathcal{Y}) = E\left(\langle \eta_{\mathcal{X}} - \eta_{\mathrm{E}(\mathcal{X})}, \eta_{\mathcal{Y}} - \eta_{\mathrm{E}(\mathcal{Y})}\rangle_\theta\right).$$

By using the Cauchy-Schwarz inequality for inner product-norm ($\star$) along with the Cauchy-Schwarz inequality for expected values of real-valued random variables ($\star\star$), we have that

$$
\begin{aligned}
|\mathrm{Cov}(\mathcal{X},\mathcal{Y})| &= \left|E\left(\langle \eta_{\mathcal{X}} - \eta_{\mathrm{E}(\mathcal{X})}, \eta_{\mathcal{Y}} - \eta_{\mathrm{E}(\mathcal{Y})}\rangle_\theta\right)\right| \leq E\left(\left|\langle \eta_{\mathcal{X}} - \eta_{\mathrm{E}(\mathcal{X})}, \eta_{\mathcal{Y}} - \eta_{\mathrm{E}(\mathcal{Y})}\rangle_\theta\right|\right) \\
&\leq^{[\star]} E\left(\|\eta_{\mathcal{X}} - \eta_{\mathrm{E}(\mathcal{X})}\|_\theta \cdot \|\eta_{\mathcal{Y}} - \eta_{\mathrm{E}(\mathcal{Y})}\|_\theta\right) = \left|E\left(\|\eta_{\mathcal{X}} - \eta_{\mathrm{E}(\mathcal{X})}\|_\theta \cdot \|\eta_{\mathcal{Y}} - \eta_{\mathrm{E}(\mathcal{Y})}\|_\theta\right)\right| \\
&\leq^{[\star\star]} \sqrt{E\left(\|\eta_{\mathcal{X}} - \eta_{\mathrm{E}(\mathcal{X})}\|_\theta^2\right) \cdot E\left(\|\eta_{\mathcal{Y}} - \eta_{\mathrm{E}(\mathcal{Y})}\|_\theta^2\right)} = \sqrt{\mathrm{Var}(\mathcal{X}) \cdot \mathrm{Var}(\mathcal{Y})}. \qquad \square
\end{aligned}
$$

Proof of Proposition 3

On the one hand, it holds that

$$
\begin{aligned}
\sum_{i=1}^{k}\sum_{j=1}^{k}\left(\mathrm{Var}\left(\mathcal{X}_i\right) + \mathrm{Var}\left(\mathcal{X}_j\right)\right) &= \sum_{i=1}^{k}\left(k\mathrm{Var}\left(\mathcal{X}_i\right) + \sum_{j=1}^{k}\mathrm{Var}\left(\mathcal{X}_j\right)\right) \\
&= k\sum_{i=1}^{k}\mathrm{Var}\left(\mathcal{X}_i\right) + k\sum_{j=1}^{k}\mathrm{Var}\left(\mathcal{X}_j\right) = 2k\sum_{i=1}^{k}\mathrm{Var}\left(\mathcal{X}_i\right).
\end{aligned} \tag{1}
$$

On the other hand, this expression can be expanded in a similar way as

$$
\begin{aligned}
\sum_{i=1}^{k}\sum_{j=1}^{k}\left(\mathrm{Var}\left(\mathcal{X}_i\right) + \mathrm{Var}\left(\mathcal{X}_j\right)\right) &= \sum_{i=1}^{k}\sum_{j=i}\left(\mathrm{Var}\left(\mathcal{X}_i\right) + \mathrm{Var}\left(\mathcal{X}_j\right)\right) \\
+\sum_{i=1}^{k}\sum_{j\neq i}\left(\mathrm{Var}\left(\mathcal{X}_i\right) + \mathrm{Var}\left(\mathcal{X}_j\right)\right) &= 2\sum_{i=1}^{k}\mathrm{Var}\left(\mathcal{X}_i\right) + \sum_{i=1}^{k}\sum_{j\neq i}\left(\mathrm{Var}\left(\mathcal{X}_i\right) + \mathrm{Var}\left(\mathcal{X}_j\right)\right).
\end{aligned} \tag{2}
$$

Then, by matching (1) and (2) it can be concluded that

$$
2(k-1)\sum_{i=1}^{k}\mathrm{Var}\left(\mathcal{X}_i\right) = \sum_{i=1}^{k}\sum_{j\neq i}\left(\mathrm{Var}\left(\mathcal{X}_i\right) + \mathrm{Var}\left(\mathcal{X}_j\right)\right)
$$

and, from Proposition 3.*iii)*, it is straightforward to see that

$$
\mathrm{Var}\left(\mathcal{X}_i\right) + \mathrm{Var}\left(\mathcal{X}_j\right) \geq 2 \cdot \mathrm{Cov}\left(\mathcal{X}_i, \mathcal{X}_j\right) \qquad \forall\, i, j = 1, \ldots, k; i \neq j,
$$

whence

$$
\sum_{i=1}^{k}\mathrm{Var}\left(\mathcal{X}_i\right) \geq \frac{1}{k-1}\sum_{i=1}^{k}\sum_{j\neq i}\mathrm{Cov}\left(\mathcal{X}_i, \mathcal{X}_j\right).
$$

Finally, by taking into account that, due to Proposition 1.*i)*, the θ-Fréchet-type variance of $\mathcal{X}_{total}$ can be alternatively computed as

$$
\mathrm{Var}(\mathcal{X}_{total}) = \sum_{i=1}^{k}\mathrm{Var}(\mathcal{X}_i) + \sum_{i=1}^{k}\sum_{j\neq i}\mathrm{Cov}(\mathcal{X}_i, \mathcal{X}_j),
$$

it holds that

$$\sum_{i=1}^{k} \sum_{j \neq i} \operatorname{Cov}\left(\mathcal{X}_i, \mathcal{X}_j\right) \leq \frac{k-1}{k} \operatorname{Var}\left(\mathcal{X}_{total}\right),$$

so

$$\alpha = \frac{k}{k-1} \frac{\sum_{i=1}^{k} \sum_{j \neq i} \operatorname{Cov}(\mathcal{X}_i, \mathcal{X}_j)}{\operatorname{Var}(\mathcal{X}_{total})} \leq 1.$$

Moreover, if the responses to all the items coincide, it holds that $\sum_{j=1}^{k} \operatorname{Var}(\mathcal{X}_j) = k \cdot \operatorname{Var}(\mathcal{X}_1)$ and, since θ-Fréchet-type variance of random intervals is 'squared equivariant' under the product by a scalar, it also holds that

$$\operatorname{Var}(\mathcal{X}_{total}) = \operatorname{Var}(k \cdot \mathcal{X}_1) = k^2 \cdot \operatorname{Var}(\mathcal{X}_1).$$

Consequently, it is straightforward to see that $\alpha = 1$. On the other hand, if the responses to all the items are (stochastically) independent, according to Proposition 1.*ii)*, we have that $\operatorname{Var}\left(\mathcal{X}_{total}\right) = \sum_{j=1}^{k} \operatorname{Var}\left(\mathcal{X}_j\right)$, whence it can be immediately deduced that $\alpha = 0$. □

Proof of Proposition 4

Let $\mathcal{Y}_j = \gamma \cdot \mathcal{X}_j + A$ where $\gamma \in \mathbb{R}$ and $A \in \mathcal{K}_c(\mathbb{R})$ denote the linear transformation of the interval-valued responses to the j-th item of the given questionnaire ($j = 1, \ldots, k$). Thus, it is straightforward to see that the total score of the 'translated construct' can be written as $\mathcal{Y}_{total} = \gamma \cdot \mathcal{X}_{total} + k \cdot A$. Therefore, since θ-Fréchet-type variance of random intervals is invariant under translation and 'squared equivariant' under the product by a scalar, it follows that

$$\sum_{j=1}^{k} \operatorname{Var}\left(\mathcal{Y}_j\right) = \gamma^2 \cdot \sum_{j=1}^{k} \operatorname{Var}\left(\mathcal{X}_j\right) \quad \text{and} \quad \operatorname{Var}\left(\mathcal{Y}_{total}\right) = \gamma^2 \cdot \operatorname{Var}\left(\mathcal{X}_{total}\right),$$

whence the result statement can be immediately deduced. □

Proof of Proposition 5

First, since the sample θ-Fréchet-type variance of the responses to the j-th item in the questionnaire, $S^2_{\mathcal{X}_j}$, is a real-valued random variable for each $j = 1, \ldots, k$ and $f\left(x_1, \ldots, x_k\right) = \sum_{j=1}^{k} x_j$ is a continuous function, it follows that $\sum_{j=1}^{k} S^2_{\mathcal{X}_j}$ is another real-valued random variable (see, for instance, Rohatgi and Ehsanes Saleh, 2000,

p. 128). Then, since the sample θ-Fréchet-type variance associated to the construct's total score, $S^2_{\mathcal{X}_{total}}$, never vanishes –in order to let the sample Cronbach's α coefficient for interval-valued data, $\widehat{\alpha}$, be always well-defined– by considering the function

$$g(x,y) = \frac{k}{k-1}\left(1 - \frac{x}{y}\right),$$

which is continuous whenever $y \neq 0$, and the real-valued random variables $\sum_{j=1}^{k} S^2_{\mathcal{X}_j}$ and $S^2_{\mathcal{X}_{total}}$, it is straightforward to show that $\widehat{\alpha}$ is a real-valued random variable too (see, for instance, Rohatgi and Ehsanes Saleh, 2000, p. 128, again). $\square$

Proof of Theorem 1

For simplicity in the notation, it is assumed that the involved questionnaire contains only two different items, whose responses are modeled by means of two random intervals, $\mathcal{X}\colon \Omega \to \mathcal{K}_c(\mathbb{R})$ and $\mathcal{Y}\colon \Omega \to \mathcal{K}_c(\mathbb{R})$, associated with the probability space $(\Omega, \mathcal{A}, P)$. Assume that the population Aumann-type means of both random intervals exist, so that $\mathrm{E}(\mathcal{X}) = [m_{\mathcal{X}} \mp d_{\mathcal{X}}]$ and $\mathrm{E}(\mathcal{Y}) = [m_{\mathcal{Y}} \mp d_{\mathcal{Y}}]$, where $m_{\mathcal{X}} = E(\operatorname{mid}\mathcal{X})$, $d_{\mathcal{X}} = E(\operatorname{spr}\mathcal{X})$, $m_{\mathcal{Y}} = E(\operatorname{mid}\mathcal{Y})$, and $d_{\mathcal{Y}} = E(\operatorname{spr}\mathcal{Y})$. So, the four following two-dimensional column vectors can be considered,

$$\boldsymbol{M} = \begin{pmatrix} \operatorname{mid}\mathcal{X} \\ \operatorname{mid}\mathcal{Y} \end{pmatrix}; \quad \boldsymbol{m} = \begin{pmatrix} m_{\mathcal{X}} \\ m_{\mathcal{Y}} \end{pmatrix}; \quad \boldsymbol{D} = \begin{pmatrix} \operatorname{spr}\mathcal{X} \\ \operatorname{spr}\mathcal{Y} \end{pmatrix}; \quad \boldsymbol{d} = \begin{pmatrix} d_{\mathcal{X}} \\ d_{\mathcal{Y}} \end{pmatrix}.$$

It should be remarked that, on the one hand, it holds that

$$\begin{aligned} S^2_{\mathcal{X}} = \frac{1}{n}\sum_{i=1}^{n} d^2_{\theta}\left(\mathcal{X}_i, \overline{\mathcal{X}}\right) &= \frac{1}{n}\sum_{i=1}^{n}\left(\left(\operatorname{mid}\mathcal{X}_i - \operatorname{mid}\overline{\mathcal{X}}\right)^2 + \theta\left(\operatorname{spr}\mathcal{X}_i - \operatorname{spr}\overline{\mathcal{X}}\right)^2\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\left(\operatorname{mid}\mathcal{X}_i - \overline{\operatorname{mid}\mathcal{X}}\right)^2 + \theta\left(\operatorname{spr}\mathcal{X}_i - \overline{\operatorname{spr}\mathcal{X}}\right)^2\right) \\ = \left(\frac{1}{n}\sum_{i=1}^{n}(\operatorname{mid}\mathcal{X}_i)^2 - \left(\overline{\operatorname{mid}\mathcal{X}}\right)^2\right) &+ \theta\left(\frac{1}{n}\sum_{i=1}^{n}(\operatorname{spr}\mathcal{X}_i)^2 - \left(\overline{\operatorname{spr}\mathcal{X}}\right)^2\right) \end{aligned}$$

and, analogously, on the other hand, it is straightforward to see that

$$S_{\mathcal{X}\mathcal{Y}} = \left(\frac{1}{n}\sum_{i=1}^{n}\operatorname{mid}\mathcal{X}_i \operatorname{mid}\mathcal{Y}_i - \overline{\operatorname{mid}\mathcal{X}}\,\overline{\operatorname{mid}\mathcal{Y}}\right) + \theta\left(\frac{1}{n}\sum_{i=1}^{n}\operatorname{spr}\mathcal{X}_i \operatorname{spr}\mathcal{Y}_i - \overline{\operatorname{spr}\mathcal{X}}\,\overline{\operatorname{spr}\mathcal{Y}}\right).$$

Likewise, it should be pointed out that the sample moments of the real-valued random variables involved in the reasonings below will be denoted as follows,

$$m_{\mathrm{mid}\,\mathcal{X}} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{mid}\,\mathcal{X}_i, \qquad m_{\mathrm{spr}\,\mathcal{X}} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{spr}\,\mathcal{X}_i,$$
$$m_{(\mathrm{mid}\,\mathcal{X})^2} = \frac{1}{n}\sum_{i=1}^{n} (\mathrm{mid}\,\mathcal{X}_i)^2, \qquad m_{(\mathrm{spr}\,\mathcal{X})^2} = \frac{1}{n}\sum_{i=1}^{n} (\mathrm{spr}\,\mathcal{X}_i)^2,$$
$$m_{\mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{mid}\,\mathcal{X}_i\,\mathrm{mid}\,\mathcal{Y}_i, \qquad m_{\mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{spr}\,\mathcal{X}_i\,\mathrm{spr}\,\mathcal{Y}_i.$$

It can be assumed without loss of generality that $m_{\mathcal{X}} = m_{\mathcal{Y}} = 0$. Thus, in case either $m_{\mathcal{X}} \neq 0$ or $m_{\mathcal{Y}} \neq 0$, the preceding assumption would be equivalent to work with auxiliary real-valued random variables, defined as $\mathrm{mid}\,\mathcal{X} - m_{\mathcal{X}}$ or $\mathrm{mid}\,\mathcal{Y} - m_{\mathcal{Y}}$, respectively, for which population expectations equal 0. Then, by taking into account that for a real-valued random variable X it holds that $E\left(X^2\right) = \mathrm{Var}\,(X) + E\,(X)^2$, these sample moments along with their related population values can be used to define the two following 10-dimensional column vectors,

$$\boldsymbol{M}_n = \begin{pmatrix} m_{\mathrm{mid}\,\mathcal{X}} \\ m_{\mathrm{mid}\,\mathcal{Y}} \\ m_{\mathrm{spr}\,\mathcal{X}} \\ m_{\mathrm{spr}\,\mathcal{Y}} \\ m_{(\mathrm{mid}\,\mathcal{X})^2} \\ m_{\mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}} \\ m_{(\mathrm{mid}\,\mathcal{Y})^2} \\ m_{(\mathrm{spr}\,\mathcal{X})^2} \\ m_{\mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}} \\ m_{(\mathrm{spr}\,\mathcal{Y})^2} \end{pmatrix}; \qquad \boldsymbol{m}_n = \begin{pmatrix} m_{\mathcal{X}} \\ m_{\mathcal{Y}} \\ d_{\mathcal{X}} \\ d_{\mathcal{Y}} \\ \mathrm{Var}(\mathrm{mid}\,\mathcal{X}) \\ \mathrm{Cov}(\mathrm{mid}\,\mathcal{X}, \mathrm{mid}\,\mathcal{Y}) \\ \mathrm{Var}(\mathrm{mid}\,\mathcal{Y}) \\ \mathrm{Var}(\mathrm{spr}\,\mathcal{X}) + d_{\mathcal{X}}^2 \\ \mathrm{Cov}(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{Y}) + d_{\mathcal{X}} d_{\mathcal{Y}} \\ \mathrm{Var}(\mathrm{spr}\,\mathcal{Y}) + d_{\mathcal{Y}}^2 \end{pmatrix},$$

From the Central Limit Theorem (Ferguson, 1996), it follows that

$$\sqrt{n}\,(\boldsymbol{M}_n - \boldsymbol{m}_n) \xrightarrow{\mathcal{L}} \mathcal{N}_{10}(\mathbf{0}, \boldsymbol{\Sigma}) \quad \text{as } n \longrightarrow \infty,$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of the random vector given by

$$\begin{pmatrix} \operatorname{mid}\mathcal{X} \\ \operatorname{mid}\mathcal{Y} \\ \operatorname{spr}\mathcal{X} \\ \operatorname{spr}\mathcal{Y} \\ (\operatorname{mid}\mathcal{X})^2 \\ \operatorname{mid}\mathcal{X}\operatorname{mid}\mathcal{Y} \\ (\operatorname{mid}\mathcal{Y})^2 \\ (\operatorname{spr}\mathcal{X})^2 \\ \operatorname{spr}\mathcal{X}\operatorname{spr}\mathcal{Y} \\ (\operatorname{spr}\mathcal{Y})^2 \end{pmatrix}.$$

On the other hand, in order to find the desired asymptotic distribution, let $\boldsymbol{g} : \mathbb{R}^{10} \to \mathbb{R}^3$ be a function such that when it is evaluated at $\boldsymbol{M}_n$ yields

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{M}_n) &= \begin{pmatrix} \left(m_{(\operatorname{mid}\mathcal{X})^2} - (m_{\operatorname{mid}\mathcal{X}})^2\right) + \theta\left(m_{(\operatorname{spr}\mathcal{X})^2} - (m_{\operatorname{spr}\mathcal{X}})^2\right) \\ (m_{\operatorname{mid}\mathcal{X}\operatorname{mid}\mathcal{Y}} - m_{\operatorname{mid}\mathcal{X}}\, m_{\operatorname{mid}\mathcal{Y}}) + \theta(m_{\operatorname{spr}\mathcal{X}\operatorname{spr}\mathcal{Y}} - m_{\operatorname{spr}\mathcal{X}}\, m_{\operatorname{spr}\mathcal{Y}}) \\ \left(m_{(\operatorname{mid}\mathcal{Y})^2} - (m_{\operatorname{mid}\mathcal{Y}})^2\right) + \theta\left(m_{(\operatorname{spr}\mathcal{Y})^2} - (m_{\operatorname{spr}\mathcal{Y}})^2\right) \end{pmatrix} \\ &= \begin{pmatrix} S^2_{\operatorname{mid}\mathcal{X}} + \theta \cdot S^2_{\operatorname{spr}\mathcal{X}} \\ S_{\operatorname{mid}\mathcal{X}\operatorname{mid}\mathcal{Y}} + \theta \cdot S_{\operatorname{spr}\mathcal{X}\operatorname{spr}\mathcal{Y}} \\ S^2_{\operatorname{mid}\mathcal{Y}} + \theta \cdot S^2_{\operatorname{spr}\mathcal{Y}} \end{pmatrix} = \begin{pmatrix} S^2_{\mathcal{X}} \\ S_{\mathcal{X}\mathcal{Y}} \\ S^2_{\mathcal{Y}} \end{pmatrix} \end{aligned} \tag{3}$$

and, consequently, when it is evaluated at $\boldsymbol{m}_n$, holds that

$$\boldsymbol{g}(\boldsymbol{m}_n) = \begin{pmatrix} \operatorname{Var}(\operatorname{mid}\mathcal{X}) + \theta \cdot \operatorname{Var}(\operatorname{spr}\mathcal{X}) \\ \operatorname{Cov}(\operatorname{mid}\mathcal{X}, \operatorname{mid}\mathcal{Y}) + \theta \cdot \operatorname{Cov}(\operatorname{spr}\mathcal{X}, \operatorname{spr}\mathcal{Y}) \\ \operatorname{Var}(\operatorname{mid}\mathcal{Y}) + \theta \cdot \operatorname{Var}(\operatorname{spr}\mathcal{Y}) \end{pmatrix} = \begin{pmatrix} \operatorname{Var}(\mathcal{X}) \\ \operatorname{Cov}(\mathcal{X}, \mathcal{Y}) \\ \operatorname{Var}(\mathcal{Y}) \end{pmatrix}. \tag{4}$$

Furthermore, the Jacobian matrix with the first-order derivatives of $\boldsymbol{g}$, when it is evaluated at $\boldsymbol{M}_n$, is given by

$$\boldsymbol{J_g}(\boldsymbol{M}_n) = \left(\frac{\partial \boldsymbol{g}}{\partial m_{\mathrm{mid}\,\mathcal{X}}} \; \frac{\partial \boldsymbol{g}}{\partial m_{\mathrm{mid}\,\mathcal{Y}}} \; \cdots \; \frac{\partial \boldsymbol{g}}{\partial m_{(\mathrm{spr}\,\mathcal{Y})^2}} \right) (\boldsymbol{M}_n)$$

$$= \begin{pmatrix} -2m_{\mathrm{mid}\,\mathcal{X}} & 0 & -2\theta m_{\mathrm{spr}\,\mathcal{X}} & 0 & 1 & 0 & 0 & \theta & 0 & 0 \\ -m_{\mathrm{mid}\,\mathcal{Y}} & -m_{\mathrm{mid}\,\mathcal{X}} & -\theta m_{\mathrm{spr}\,\mathcal{Y}} & -\theta m_{\mathrm{spr}\,\mathcal{X}} & 0 & 1 & 0 & 0 & \theta & 0 \\ 0 & -2m_{\mathrm{mid}\,\mathcal{Y}} & 0 & -2\theta m_{\mathrm{spr}\,\mathcal{Y}} & 0 & 0 & 1 & 0 & 0 & \theta \end{pmatrix}$$

and, similarly, when it is evaluated at $\boldsymbol{m}_n$, yields

$$\boldsymbol{J_g}(\boldsymbol{m}_n) = \begin{pmatrix} 0 & 0 & -2\theta d_{\mathcal{X}} & 0 & 1 & 0 & 0 & \theta & 0 & 0 \\ 0 & 0 & -\theta d_{\mathcal{Y}} & -\theta d_{\mathcal{X}} & 0 & 1 & 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & -2\theta d_{\mathcal{Y}} & 0 & 0 & 1 & 0 & 0 & \theta \end{pmatrix}.$$

Hence, from Delta Method (Ferguson, 1996), it can be concluded that

$$\sqrt{n}\left(\boldsymbol{g}\left(\boldsymbol{M}_n\right) - \boldsymbol{g}\left(\boldsymbol{m}_n\right)\right) \xrightarrow{\mathcal{L}} \mathcal{N}_3\left(\boldsymbol{0}, \boldsymbol{J_g}\left(\boldsymbol{m}_n\right) \boldsymbol{\Sigma} \, \boldsymbol{J_g}\left(\boldsymbol{m}_n\right)^T\right), \quad \text{as } n \longrightarrow \infty,$$

which, due to (3) and (4), is equivalent to say that

$$\sqrt{n}\left(\begin{pmatrix} S^2_{\mathcal{X}} \\ S_{\mathcal{X}\mathcal{Y}} \\ S^2_{\mathcal{Y}} \end{pmatrix} - \begin{pmatrix} \mathrm{Var}(\mathcal{X}) \\ \mathrm{Cov}(\mathcal{X},\mathcal{Y}) \\ \mathrm{Var}(\mathcal{Y}) \end{pmatrix} \right) \xrightarrow{\mathcal{L}} \mathcal{N}_3\left(\boldsymbol{0}, \boldsymbol{J_g}\left(\boldsymbol{m}_n\right) \boldsymbol{\Sigma} \, \boldsymbol{J_g}\left(\boldsymbol{m}_n\right)^T\right), \quad \text{as } n \longrightarrow \infty.$$

Finally, it should be checked that $\boldsymbol{J_g}\left(\boldsymbol{m}_n\right) \boldsymbol{\Sigma} \, \boldsymbol{J_g}\left(\boldsymbol{m}_n\right)^T$ coincides with $\boldsymbol{\Gamma}$, which is defined as the covariance matrix of the random vector given by

$$\mathrm{vech}\left(\left(\boldsymbol{M} - \boldsymbol{m}\right)\left(\boldsymbol{M} - \boldsymbol{m}\right)^T\right) + \theta \cdot \mathrm{vech}\left(\left(\boldsymbol{D} - \boldsymbol{d}\right)\left(\boldsymbol{D} - \boldsymbol{d}\right)^T\right)$$
$$= \begin{pmatrix} (\mathrm{mid}\,\mathcal{X} - m_{\mathcal{X}})^2 \\ (\mathrm{mid}\,\mathcal{X} - m_{\mathcal{X}})(\mathrm{mid}\,\mathcal{Y} - m_{\mathcal{Y}}) \\ (\mathrm{mid}\,\mathcal{Y} - m_{\mathcal{Y}})^2 \end{pmatrix} + \theta \begin{pmatrix} (\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}})^2 \\ (\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}})(\mathrm{spr}\,\mathcal{Y} - d_{\mathcal{Y}}) \\ (\mathrm{spr}\,\mathcal{Y} - d_{\mathcal{Y}})^2 \end{pmatrix},$$

where $m_{\mathcal{X}} = m_{\mathcal{Y}} = 0$. However, it should be pointed out that extremely tedious calculations are needed to prove the identity between both 9×9 matrices. For this reason, only the elements on the first row and the first two columns of these matrices will be compared. The remaining checks would be carried out in a similar way. First, the elements on the first row and first column of $\boldsymbol{J_g}(\boldsymbol{m}_n)\,\boldsymbol{\Sigma}\,\boldsymbol{J_g}(\boldsymbol{m}_n)^T$ and $\boldsymbol{\Gamma}$ will be computed and compared. Thus, on the one hand, the element on the first row and the first column of $\boldsymbol{J_g}(\boldsymbol{m}_n)\,\boldsymbol{\Sigma}\,\boldsymbol{J_g}(\boldsymbol{m}_n)^T$ is given by

$$
\begin{aligned}
&-2\theta d_{\mathcal{X}}\left(-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\mathrm{spr}\,\mathcal{X}\right)+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{X}\right)+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{X}\right)\right)\\
&-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\left(\mathrm{mid}\,\mathcal{X}\right)^2\right)+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{mid}\,\mathcal{X}\right)^2\right)+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\left(\mathrm{mid}\,\mathcal{X}\right)^2\right)\\
&+\theta\left(-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)\right)\\
&=4\theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\mathrm{spr}\,\mathcal{X}\right)-4d_{\mathcal{X}}\theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{X}\right)-4\theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)\\
&+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{mid}\,\mathcal{X}\right)^2\right)+2\theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)+\theta^2\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right).
\end{aligned}
$$

On the other hand, by taking into account the properties of the variance of real-valued random variables, it is straightforward to see that the element on the first row and first column of $\boldsymbol{\Gamma}$ equals

$$
\begin{aligned}
&\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2+\theta\left(\mathrm{spr}\,\mathcal{X}-d_{\mathcal{X}}\right)^2,\left(\mathrm{mid}\,\mathcal{X}\right)^2+\theta\left(\mathrm{spr}\,\mathcal{X}-d_{\mathcal{X}}\right)^2\right)\\
&=\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{mid}\,\mathcal{X}\right)^2\right)+2\theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}-d_{\mathcal{X}}\right)^2\right)\\
&+\theta^2\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}-d_{\mathcal{X}}\right)^2,\left(\mathrm{spr}\,\mathcal{X}-d_{\mathcal{X}}\right)^2\right)\\
&=\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{mid}\,\mathcal{X}\right)^2\right)+2\theta\left(\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)-2d_{\mathcal{X}}\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{X}\right)\right)\\
&+\theta^2\left(\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)-4d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)\right)+4d_{\mathcal{X}}^2\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)\\
&=4\theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\mathrm{spr}\,\mathcal{X}\right)-4d_{\mathcal{X}}\theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{X}\right)-4\theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)\\
&+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{mid}\,\mathcal{X}\right)^2\right)+2\theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right)+\theta^2\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\left(\mathrm{spr}\,\mathcal{X}\right)^2\right).
\end{aligned}
$$

Consequently, the elements on the first row and first column of both matrices coincide. In a similar way, the elements on the first row and second column of $\boldsymbol{J_g}(\boldsymbol{m}_n)\,\boldsymbol{\Sigma}\,\boldsymbol{J_g}(\boldsymbol{m}_n)^T$ and $\boldsymbol{\Gamma}$ are also equal. Indeed, on the one hand, the element on this position in $\boldsymbol{J_g}\left(\boldsymbol{m}_n\right)\,\boldsymbol{\Sigma}\,\boldsymbol{J_g}\left(\boldsymbol{M}_n\right)^T$ is given by

$$
\begin{aligned}
&-\theta d_{\mathcal{Y}}\left(-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\mathrm{spr}\,\mathcal{X}\right)+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{X}\right)+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{X}\right)\right)\\
&-\theta d_{\mathcal{X}}\left(-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X},\mathrm{spr}\,\mathcal{Y}\right)+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{Y}\right)+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2,\mathrm{spr}\,\mathcal{Y}\right)\right)
\end{aligned}
$$

(Continued on next page)

(Continued from previous page)

$$
\begin{aligned}
&-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) + \mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) \\
&+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) + \theta\left(-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right)\right. \\
&\left.+\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right) + \theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right)\right) \\
&= 2\theta^2 d_{\mathcal{X}} d_{\mathcal{Y}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{X}\right) - \theta d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\right) - \theta^2 d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\right) \\
&+2\theta^2 d_{\mathcal{X}}^2\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{Y}\right) - \theta d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{Y}\right) - \theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{Y}\right) \\
&-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) + \mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) \\
&+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) - 2\theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right) \\
&+\theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right) + \theta^2\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right).
\end{aligned}
$$

On the other hand, the element on the first row and second column of $\mathbf{\Gamma}$, due to the properties of the variance of real-valued random variables again, is

$$
\begin{aligned}
&\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2 + \theta\left(\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y} + \theta\left(\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}}\right)\left(\mathrm{spr}\,\mathcal{Y} - d_{\mathcal{Y}}\right)\right) \\
&= \mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) + \theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \left(\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}}\right)\left(\mathrm{spr}\,\mathcal{Y} - d_{\mathcal{Y}}\right)\right) \\
&+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) + \theta^2\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}}\right)^2, \left(\mathrm{spr}\,\mathcal{X} - d_{\mathcal{X}}\right)\left(\mathrm{spr}\,\mathcal{Y} - d_{\mathcal{Y}}\right)\right) \\
&= \mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) + \theta\left(\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right)\right. \\
&\left.-d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\right) - d_{\mathcal{X}}\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{Y}\right)\right) \\
&+\theta\left(\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) - 2d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right)\right) \\
&+\theta^2\left(\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right) - d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\right) - d_{\mathcal{X}}\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{Y}\right)\right. \\
&\left.-2d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right) + 2d_{\mathcal{X}}d_{\mathcal{Y}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{X}\right) + 2d_{\mathcal{X}}^2\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{Y}\right)\right) \\
&= 2\theta^2 d_{\mathcal{X}} d_{\mathcal{Y}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{X}\right) - \theta d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\right) - \theta^2 d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\right) \\
&+2\theta^2 d_{\mathcal{X}}^2\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{Y}\right) - \theta d_{\mathcal{Y}}\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{Y}\right) - \theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{Y}\right) \\
&-2\theta d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) + \mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) \\
&+\theta\,\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{mid}\,\mathcal{X}\,\mathrm{mid}\,\mathcal{Y}\right) - 2\theta^2 d_{\mathcal{X}}\mathrm{Cov}\left(\mathrm{spr}\,\mathcal{X}, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right) \\
&+\theta\,\mathrm{Cov}\left(\left(\mathrm{mid}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right) + \theta^2\mathrm{Cov}\left(\left(\mathrm{spr}\,\mathcal{X}\right)^2, \mathrm{spr}\,\mathcal{X}\,\mathrm{spr}\,\mathcal{Y}\right).
\end{aligned}
$$

Consequently, the elements on the fist row and first column of both matrices also coincide. Then, as it has been explained above, the remaining checks can be done in an analogous way, yielding the identity between $\boldsymbol{J_g}(\boldsymbol{m}_n)\,\boldsymbol{\Sigma}\,\boldsymbol{J_g}(\boldsymbol{m}_n)^T$ and $\boldsymbol{\Gamma}$. Therefore,

$$\sqrt{n}\left(\begin{pmatrix} S^2_{\mathcal{X}} \\ S_{\mathcal{X}\mathcal{Y}} \\ S^2_{\mathcal{Y}} \end{pmatrix} - \begin{pmatrix} \mathrm{Var}(\mathcal{X}) \\ \mathrm{Cov}(\mathcal{X},\mathcal{Y}) \\ \mathrm{Var}(\mathcal{Y}) \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}_3(\mathbf{0},\boldsymbol{\Gamma}) \quad \text{as } n \longrightarrow \infty.$$

This reasoning can be straightforwardly extended to the k-dimensional case in order to reach theorem's statement. □

Proof of Theorem 3

From Proposition 2.ii), it is known that $S^2_{\mathcal{X}_j} \xrightarrow{\text{a.s. }[P]} \mathrm{Var}(\mathcal{X}_j)$ for $j = 1,\ldots,k$ and $S^2_{\mathcal{X}_{total}} \xrightarrow{\text{a.s. }[P]} \mathrm{Var}(\mathcal{X}_{total})$ as $n \to \infty$. Then, by applying the Continuous Mapping Theorem for almost sure convergence (van der Vaart, 1998) and taking into account that $f\,(x_1,\ldots,x_k) = \sum_{j=1}^k x_j$ is a continuous function in $\mathbb{R}^k$, it is straightforward to see that

$$\sum_{j=1}^{k} S^2_{\mathcal{X}_j} \xrightarrow{\text{a.s. }[P]} \sum_{j=1}^{k} \mathrm{Var}(\mathcal{X}_j) \qquad \text{as } n \longrightarrow \infty.$$

Finally, since the population Fréchet variance associated to the construct's total score and its sample estimator, $\mathrm{Var}(\mathcal{X}_{total})$ and $S^2_{\mathcal{X}_{total}}$, are supposed to never vanish –in order to let both population and sample Cronbach's α coefficients for interval-valued data, α and $\widehat{\alpha}$, be well-defined– by using the Continuous Mapping Theorem for almost sure convergence (van der Vaart, 1998) again, but considering, in this case, the function

$$g\,(x,y) = \frac{k}{k-1}\left(1 - \frac{x}{y}\right),$$

which is continuous whenever $y \neq 0$, it can be immediately concluded that

$$\widehat{\alpha} \xrightarrow{\text{a.s. }[P]} \alpha, \qquad \text{as } n \longrightarrow \infty.$$

Therefore, $\widehat{\alpha}$ is a strongly consistent estimator of α. □

References

Ferguson TS (1996) A Course in Large Sample Theory. Chapman and Hall, London

Minkowski H (1903) Volumen und oberfäche. Math Ann 57:447–495

Rohatgi VK, Ehsanes Saleh AKM (2000) An Introduction to Probability and Statistics. Wiley Series in Probability and Statistics, John Wiley & Sons, New York, https://doi.org/10.1002/9781118165676

van der Vaart A (1998) Asymptotic Statistics. Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, https://doi.org/10.1017/CBO9780511802256

Supplementary Information, Part 2: Additional Simulation Studies to those in Section 4

Section 4.1. Empirical analysis of the asymptotic distribution of the sample estimator of Cronbach's α coefficient for interval-valued data

SCENARIO 1: High internal consistency ($\varepsilon_{ij} \sim \mathcal{N}_1(.1, .0025)$)

▶ **Results for $\beta(1, 1)$ distribution**

Table 1 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(1, 1)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.9009	$6.5874 \cdot 10^{-6}$	.0035	.0000	.0000	.8992	.9009	.9027
Empirical	.9008	$6.5595 \cdot 10^{-6}$	.0034	-.1641	-.0108	.8991	.9009	.9026

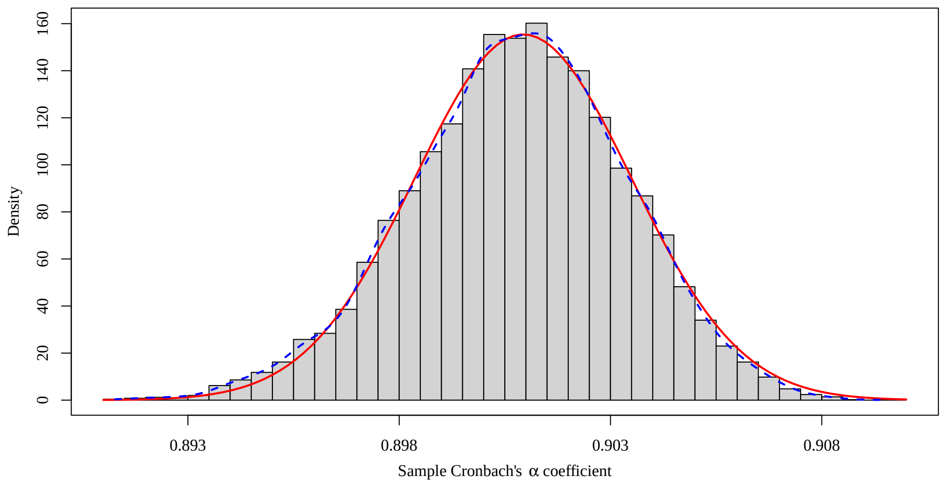

Fig. 1 Theoretical (—) and empirical (- - -) asymptotic distributions of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(1, 1)$ distribution

▶ Results for $\beta(.75, .75)$ distribution

Table 2 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(.75, .75)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.9093	$5.3582 \cdot 10^{-6}$	.0031	.0000	.0000	.9077	.9093	.9108
Empirical	.9092	$5.3736 \cdot 10^{-6}$	.0031	-.1199	.0146	.9077	.9093	.9108

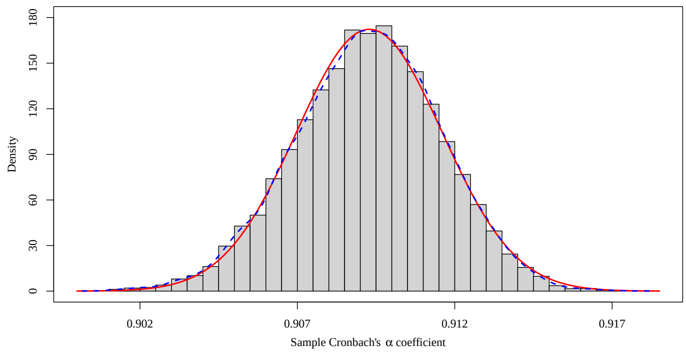

Fig. 2 Theoretical (——) and empirical (- - -) asymptotic distributions of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(.75, .75)$ distribution

▶ Results for $\beta(2, 2)$ distribution

Table 3 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(2, 2)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.9178	$5.3342 \cdot 10^{-6}$	.0031	.0000	.0000	.9163	.9178	.9194
Empirical	.9177	$5.3579 \cdot 10^{-6}$	.0031	-.1078	.0332	.9162	.9178	.9193

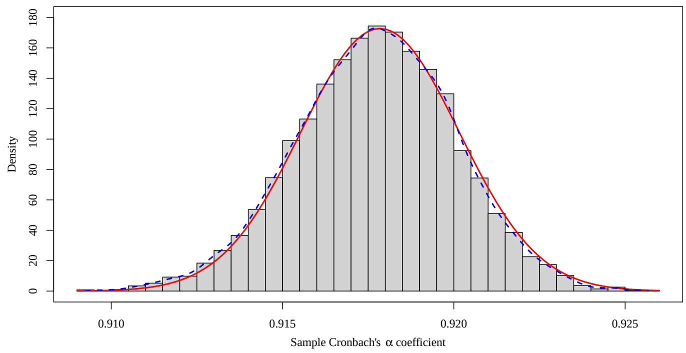

Fig. 3 Theoretical (——) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(2, 2)$ distribution

▶ Results for $\beta(4,2)$ distribution

Table 4 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(4,2)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.8828	$1.8915 \cdot 10^{-5}$	.0059	.0000	.0000	.8799	.8828	.8858
Empirical	.8826	$1.8900 \cdot 10^{-6}$	.0058	-.2168	.0818	.8798	.8827	.8856

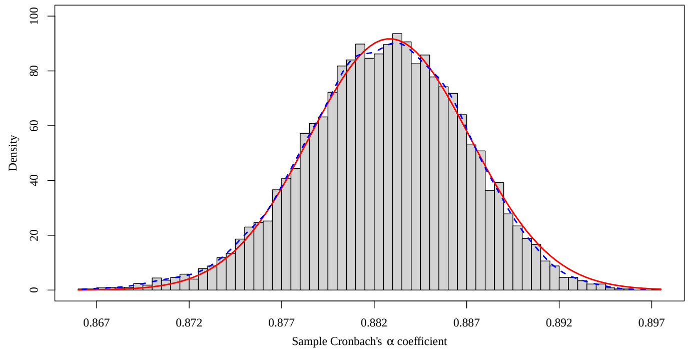

Fig. 4 Theoretical (—) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(4,2)$ distribution

▶ Results for $\beta(6,1)$ distribution

Table 5 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6,1)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.8987	$4.3714 \cdot 10^{-5}$	.0089	.0000	.0000	.8942	.8987	.9031
Empirical	.8982	$4.4539 \cdot 10^{-5}$	.0089	-.2771	.1444	.8939	.8984	.9028

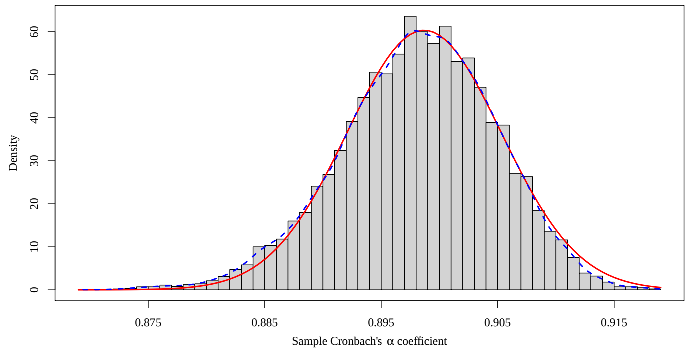

Fig. 5 Theoretical (—) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6,1)$ distribution

▶ Results for $\beta(6, 10)$ distribution

Table 6 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6, 10)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.9147	$1.5649 \cdot 10^{-6}$	.0053	.0000	.0000	.9120	.9147	.9174
Empirical	.9145	$1.6080 \cdot 10^{-6}$	.0053	-.1805	.0542	.9119	.9146	.9172

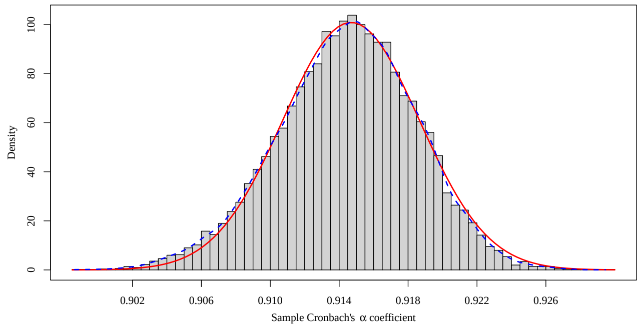

Fig. 6 Theoretical (——) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6, 10)$ distribution

SCENARIO 2: Low internal consistency ($\varepsilon_{ij} \sim \mathcal{N}_1\,(.1, .0625)$)

▶ Results for $\beta(1, 1)$ distribution

Table 7 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(1, 1)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.5541	$1.0601 \cdot 10^{-3}$	.0439	.0000	.0000	.5321	.5541	.5760
Empirical	.5517	$1.0701 \cdot 10^{-3}$	.0440	-.2922	.1610	.5306	.5531	.5746

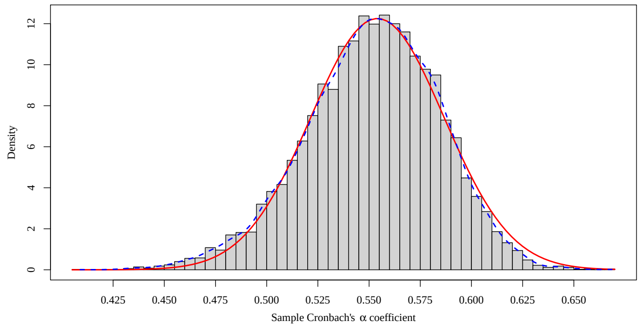

Fig. 7 Theoretical (——) and empirical (- - -) asymptotic distributions of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(1, 1)$ distribution

▶ Results for $\beta(.75, .75)$ distribution

Table 8 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(.75, .75)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.5559	$1.0668 \cdot 10^{-3}$	.0441	.0000	.0000	.5339	.5559	.5780
Empirical	.5544	$1.0946 \cdot 10^{-3}$	.0443	-.2767	.1292	.5332	.5560	.5774

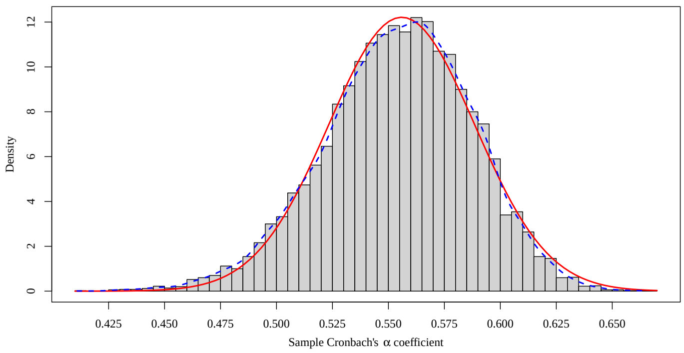

Fig. 8 Theoretical (——) and empirical (- - -) asymptotic distributions of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(.75, .75)$ distribution

▶ Results for $\beta(2, 2)$ distribution

Table 9 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(2, 2)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.5262	$1.2453 \cdot 10^{-3}$	.0476	.0000	.0000	.5024	.5262	.5500
Empirical	.5242	$1.2461 \cdot 10^{-6}$	.0478	-.2248	.0321	.5010	.5255	.5488

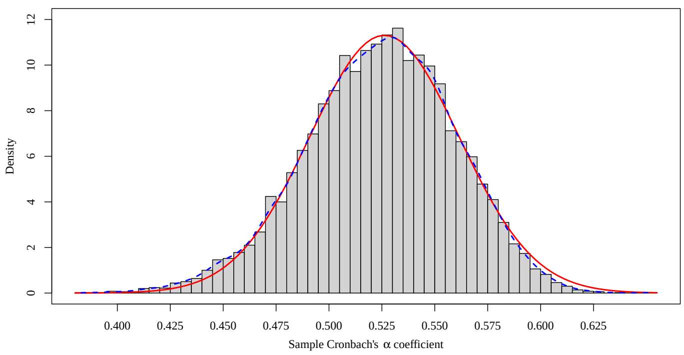

Fig. 9 Theoretical (——) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(2, 2)$ distribution

▶ Results for $\beta(4,2)$ distribution

Table 10 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(4,1)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.4481	$1.8729 \cdot 10^{-3}$	.0584	.0000	.0000	.4189	.4481	.4773
Empirical	.4450	$1.8856 \cdot 10^{-3}$	.0582	-.2464	.0214	.4169	.4469	.4751

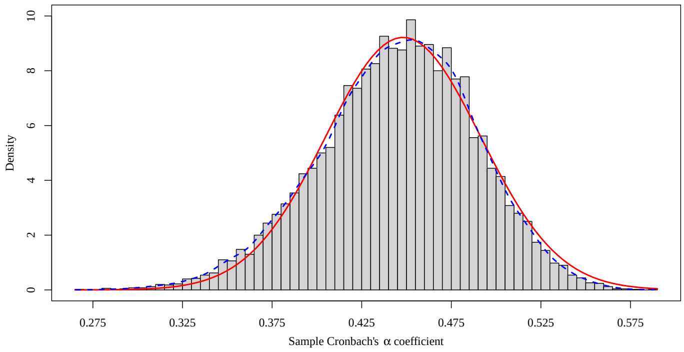

Fig. 10 Theoretical (——) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(4,2)$ distribution

▶ Results for $\beta(6,1)$ distribution

Table 11 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6,1)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.4961	$2.2416 \cdot 10^{-3}$	.0639	.0000	.0000	.4641	.4961	.5280
Empirical	.4916	$2.2314 \cdot 10^{-3}$	.0634	-.2670	.1406	.4612	.4935	.5246

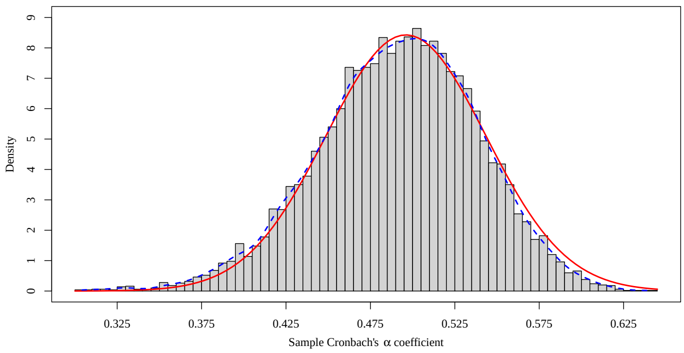

Fig. 11 Theoretical (——) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6,1)$ distribution

▶ Results for $\beta(6, 10)$ distribution

Table 12 Summary measures of theoretical and empirical asymptotic distribution of the sample Cronbach α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6, 10)$ distribution

	Mean	Variance	IQR	Skewness	Kurtosis	P25	P50	P75
Theoretical	.5584	$9.9980 \cdot 10^{-4}$	.0427	.0000	.0000	.5370	.5584	.5797
Empirical	.5567	$1.0156 \cdot 10^{-3}$	.0429	-.2818	.1778	.5361	.5582	.5790

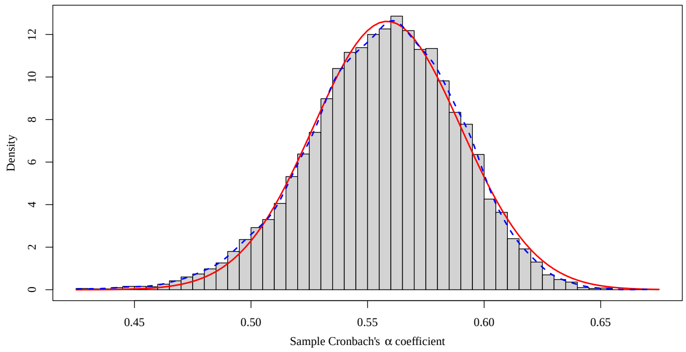

Fig. 12 Theoretical (——) and empirical (- - -) asymptotic distributions of Cronbach's α coefficient for $n = 300$ respondents, $k = 10$ items, and $\beta(6, 10)$ distribution

Section 4.2. Empirical analysis of the influence of the numbers of respondents to a questionnarie and items in a construct on sample Cronbach's α coefficient for interval-valued data

SCENARIO 1: High internal consistency ($\varepsilon_{ij} \sim \mathcal{N}_1\,(.1, .0025)$)

▶ **Results for $\beta(1,1)$ distribution**

Table 13 Sample Cronbach's α coefficient for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(1,1)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.7571	.7734	.7572	.7635	.7636	.7581	.7603	.7631	.7639	.7647
7	.8399	.8477	.8391	.8395	.8422	.8412	.8414	.8430	.8431	.8445
10	.9130	.9135	.9102	.9100	.9105	.9099	.9109	.9122	.9119	.9122
15	.9417	.9407	.9385	.9387	.9390	.9393	.9397	.9398	.9394	.9398
20	.9598	.9588	.9574	.9577	.9584	.9588	.9590	.9591	.9588	.9590
30	.9766	.9762	.9753	.9756	.9758	.9760	.9761	.9761	.9761	.9761
40	.9838	.9835	.9829	.9830	.9832	.9833	.9834	.9834	.9833	.9833
50	.9880	.9880	.9875	.9877	.9878	.9878	.9879	.9879	.9878	.9878
100	.9952	.9952	.9950	.9951	.9951	.9951	.9951	.9951	.9951	.9951

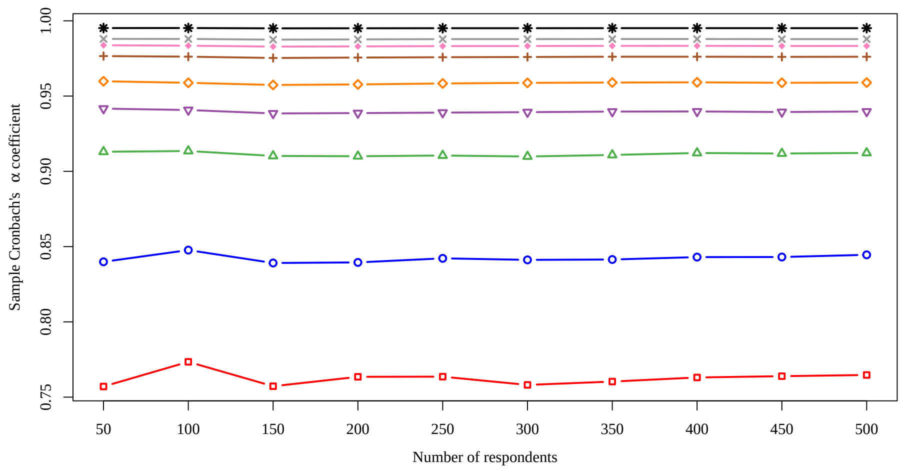

Fig. 13 Sample Cronbach's α coefficient for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (♦), 50 (×), and 100 (✱) items in a construct and $\beta\,(1,1)$ distribution

▶ Results for $\beta(.75, .75)$ distribution

Table 14 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(.75,.75)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.7883	.8209	.8247	.8209	.8235	.8203	.8191	.8170	.8193	.8196
7	.8233	.8508	.8499	.8445	.8465	.8458	.8445	.8425	.8438	.8442
10	.9168	.9261	.9256	.9244	.9246	.9246	.9236	.9229	.9234	.9235
15	.9353	.9394	.9387	.9377	.9378	.9374	.9368	.9366	.9365	.9364
20	.9560	.9593	.9586	.9579	.9579	.9577	.9572	.9570	.9571	.9571
30	.9755	.9770	.9768	.9765	.9765	.9764	.9761	.9759	.9760	.9759
40	.9831	.9843	.9841	.9841	.9841	.9840	.9837	.9836	.9837	.9836
50	.9874	.9883	.9882	.9882	.9881	.9881	.9879	.9878	.9878	.9878
100	.9944	.9947	.9947	.9947	.9947	.9946	.9946	.9946	.9946	.9946

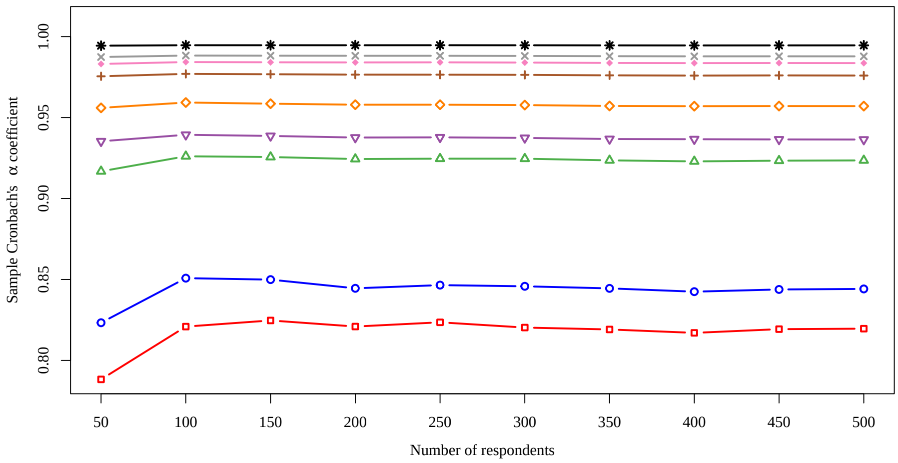

Fig. 14 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(.75,.75)$ distribution

▶ **Results for $\beta(2,2)$ distribution**

Table 15 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta(2,2)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.7524	.7576	.7736	.7824	.7834	.7808	.7817	.7855	.7863	.7873
7	.7872	.8069	.8180	.8230	.8212	.8205	.8207	.8227	.8247	.8272
10	.9065	.9165	.9211	.9232	.9222	.9221	.9220	.9232	.9244	.9251
15	.9336	.9424	.9439	.9449	.9439	.9433	.9429	.9440	.9443	.9449
20	.9466	.9558	.9562	.9564	.9555	.9547	.9546	.9554	.9558	.9563
30	.9696	.9744	.9748	.9746	.9742	.9738	.9737	.9741	.9743	.9747
40	.9757	.9801	.9805	.9802	.9800	.9797	.9796	.9799	.9801	.9804
50	.9830	.9861	.9863	.9862	.9860	.9858	.9858	.9860	.9861	.9863
100	.9930	.9942	.9943	.9942	.9941	.9941	.9941	.9942	.9942	.9943

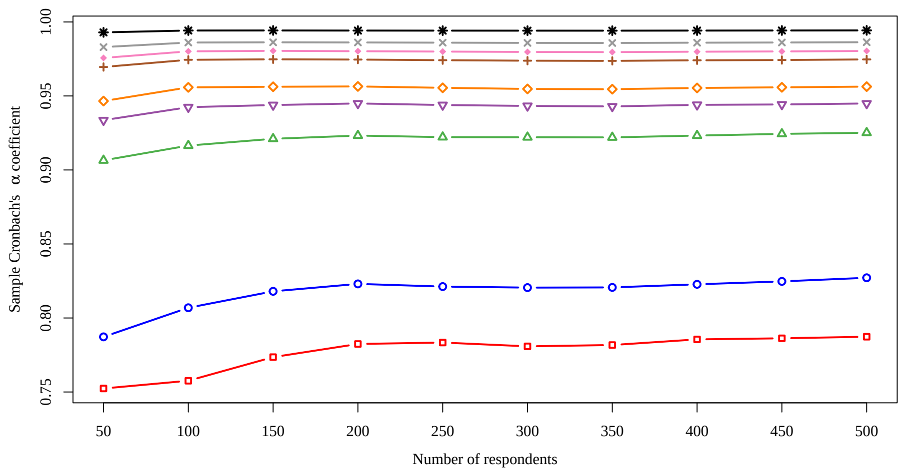

Fig. 15 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta(2,2)$ distribution

▶ Results for $\beta(4, 2)$ distribution

Table 16 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta(4, 2)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.7871	.8178	.8097	.8155	.8217	.8213	.8236	.8248	.8261	.8246
7	.8447	.8648	.8605	.8650	.8685	.8689	.8699	.8702	.8702	.8678
10	.9101	.9198	.9187	.9211	.9230	.9244	.9247	.9245	.9244	.9232
15	.9453	.9503	.9491	.9498	.9509	.9519	.9521	.9520	.9519	.9514
20	.9620	.9651	.9648	.9652	.9660	.9664	.9662	.9660	.9660	.9656
30	.9782	.9796	.9795	.9799	.9801	.9803	.9803	.9803	.9803	.9802
40	.9857	.9866	.9865	.9867	.9869	.9870	.9871	.9871	.9871	.9869
50	.9887	.9895	.9893	.9896	.9897	.9898	.9899	.9898	.9898	.9897
100	.9944	.9948	.9947	.9948	.9948	.9949	.9949	.9949	.9948	.9948

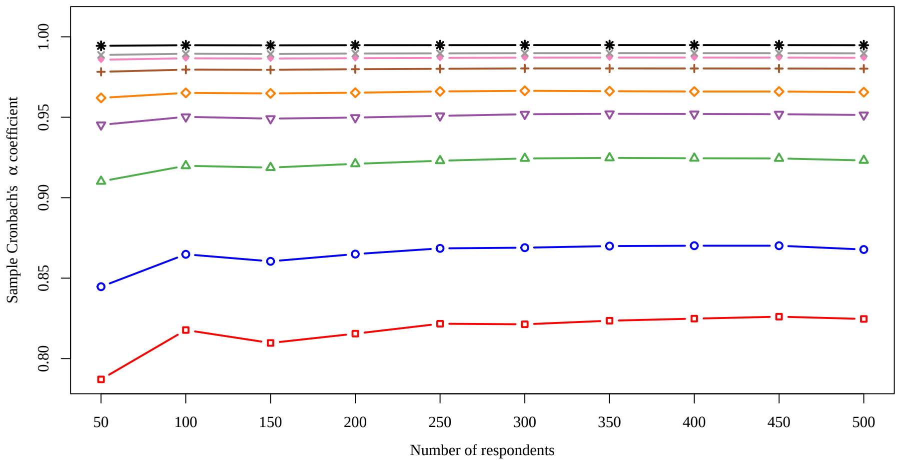

Fig. 16 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta(4, 2)$ distribution

▶ **Results for $\beta(6, 1)$ distribution**

Table 17 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta(6, 1)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.7520	.7764	.7924	.7918	.7929	.7802	.7869	.7920	.7945	.7991
7	.8392	.8493	.8565	.8543	.8540	.8420	.8473	.8516	.8555	.8591
10	.8831	.8982	.9028	.9000	.9003	.8938	.8987	.9002	.9022	.9050
15	.9391	.9469	.9481	.9465	.9467	.9433	.9460	.9460	.9474	.9485
20	.9546	.9610	.9618	.9603	.9601	.9568	.9590	.9587	.9597	.9608
30	.9694	.9747	.9750	.9741	.9738	.9716	.9732	.9729	.9735	.9743
40	.9784	.9820	.9820	.9812	.9809	.9794	.9805	.9803	.9807	.9812
50	.9798	.9829	.9829	.9821	.9819	.9807	.9818	.9815	.9819	.9824
100	.9910	.9924	.9923	.9919	.9919	.9914	.9919	.9917	.9919	.9921

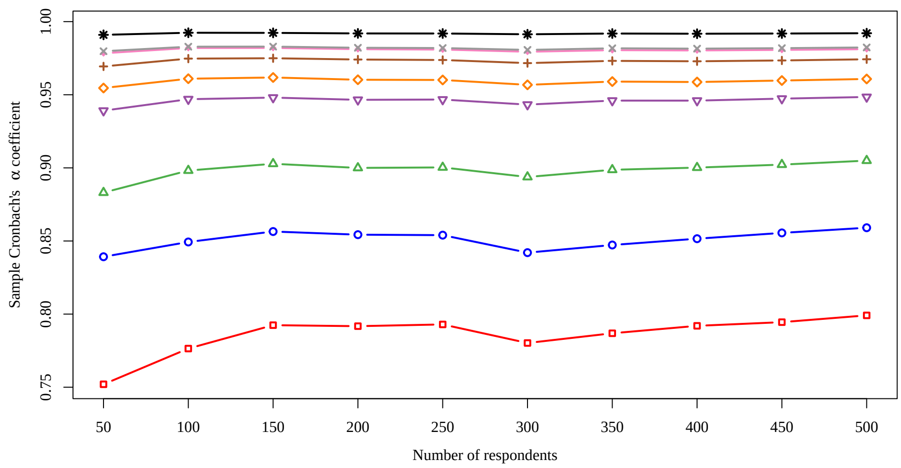

Fig. 17 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta(6, 1)$ distribution

▶ **Results for $\beta(6, 10)$ distribution**

Table 18 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(6, 10)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.7961	.8143	.8155	.8227	.8181	.8237	.8277	.8266	.8282	.8310
7	.8579	.8693	.8718	.8773	.8757	.8776	.8782	.8776	.8782	.8795
10	.9170	.9215	.9245	.9272	.9241	.9246	.9251	.9254	.9264	.9276
15	.9322	.9349	.9380	.9382	.9360	.9355	.9364	.9368	.9378	.9387
20	.9559	.9565	.9585	.9587	.9576	.9571	.9580	.9583	.9589	.9593
30	.9769	.9774	.9782	.9786	.9778	.9775	.9781	.9782	.9785	.9786
40	.9822	.9826	.9831	.9834	.9829	.9827	.9830	.9831	.9832	.9834
50	.9860	.9862	.9865	.9867	.9863	.9861	.9864	.9865	.9865	.9867
100	.9932	.9933	.9935	.9936	.9933	.9932	.9934	.9935	.9935	.9936

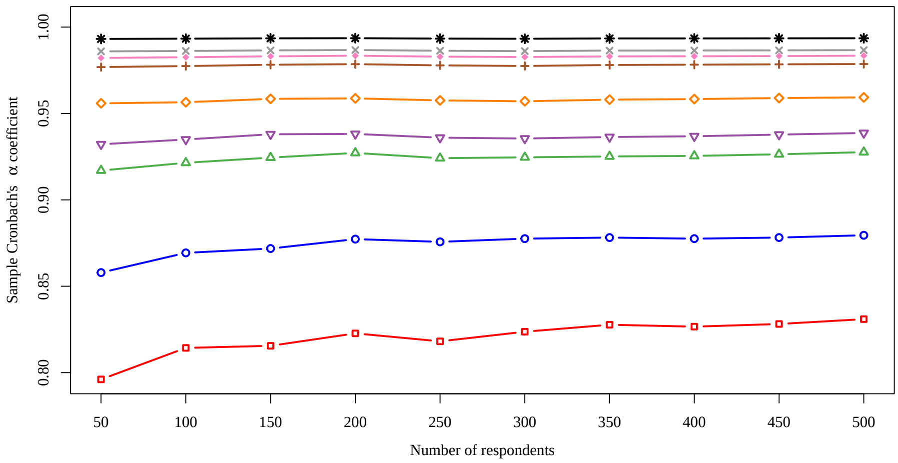

Fig. 18 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(6, 10)$ distribution

SCENARIO 2: Low internal consistency ($\varepsilon_{ij} \sim \mathcal{N}_1\,(.1, .0625)$)

▸ Results for $\beta(1,1)$ distribution

Table 19 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(1,1)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.2660	.4222	.4088	.3929	.4269	.4156	.4537	.4563	.4346	.4310
7	.3759	.5040	.4939	.4869	.5093	.4898	.5146	.5219	.5106	.5126
10	.4531	.5425	.5418	.5223	.5511	.5357	.5495	.5536	.5490	.5474
15	.6927	.7386	.7378	.7234	.7394	.7383	.7418	.7426	.7461	.7434
20	.8273	.8452	.8454	.8322	.8353	.8322	.8333	.8362	.8357	.8362
30	.8887	.8959	.8979	.8908	.8924	.8894	.8885	.8905	.8896	.8906
40	.9231	.9289	.9289	.9214	.9229	.9214	.9214	.9230	.9227	.9236
50	.9306	.9354	.9348	.9277	.9302	.9298	.9306	.9319	.9319	.9325
100	.9656	.9699	.9698	.9662	.9677	.9676	.9675	.9681	.9683	.9682

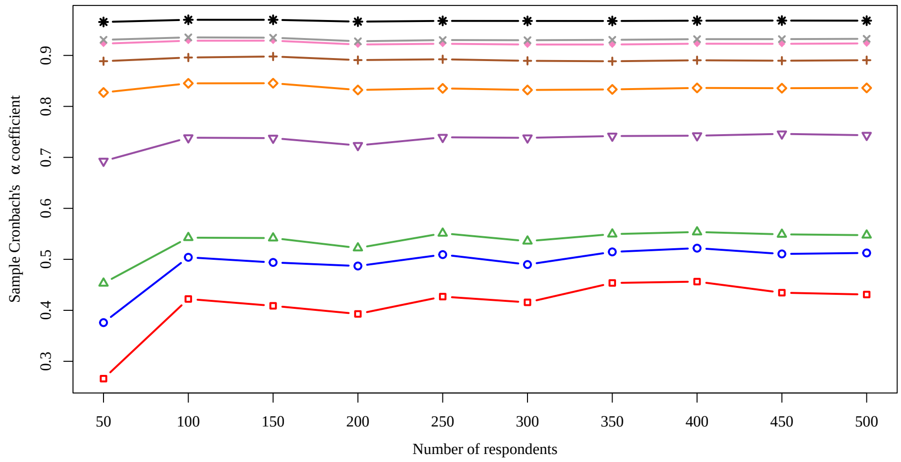

Fig. 19 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(1,1)$ distribution

▶ Results for $\beta(.75, .75)$ distribution

Table 20 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(.75, .75)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.4086	.3942	.4297	.4733	.5055	.5030	.4880	.5017	.4943	.5164
7	.4602	.4868	.5025	.5483	.5575	.5440	.5374	.5493	.5404	.5493
10	.6822	.6799	.6612	.6815	.6869	.6814	.6772	.6761	.6681	.6705
15	.8181	.8060	.8046	.8145	.8146	.8113	.8070	.8023	.7981	.7979
20	.8698	.8682	.8697	.8724	.8742	.8731	.8717	.8703	.8682	.8692
30	.9123	.9084	.9066	.9070	.9093	.9100	.9089	.9090	.9079	.9089
40	.9333	.9326	.9324	.9320	.9334	.9339	.9330	.9331	.9329	.9336
50	.9380	.9393	.9390	.9386	.9399	.9398	.9386	.9383	.9380	.9388
100	.9736	.9743	.9734	.9729	.9742	.9739	.9734	.9732	.9728	.9730

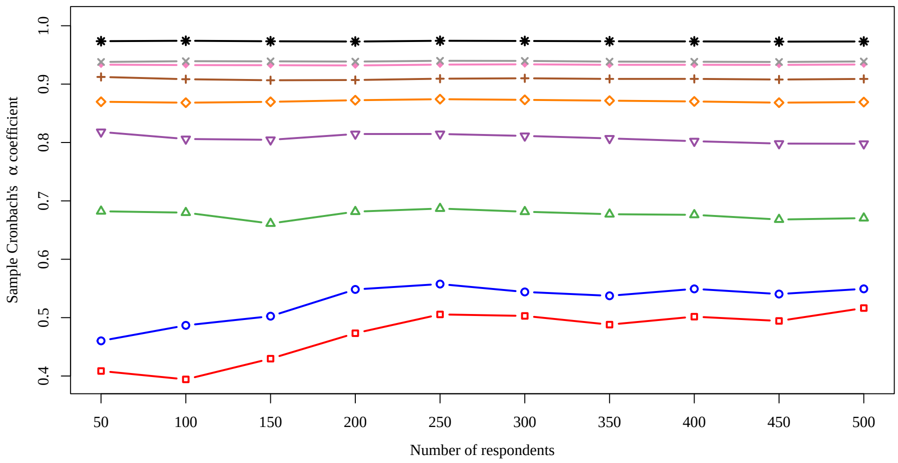

Fig. 20 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(.75, .75)$ distribution

▶ Results for $\beta(2,2)$ distribution

Table 21 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(2,2)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.3871	.3428	.2761	.3299	.3520	.3497	.3432	.3591	.3887	.4098
7	.6027	.6184	.5899	.6016	.6095	.6120	.5921	.5971	.6118	.6245
10	.6484	.6892	.6467	.6558	.6775	.6828	.6688	.6691	.6787	.6914
15	.7121	.7270	.6879	.7049	.7250	.7329	.7214	.7178	.7244	.7293
20	.8018	.8057	.7789	.7905	.8001	.8036	.7962	.7909	.7942	.7979
30	.8626	.8508	.8421	.8500	.8525	.8548	.8503	.8474	.8479	.8490
40	.9003	.8914	.8838	.8877	.8892	.8900	.8872	.8852	.8861	.8857
50	.9224	.9156	.9088	.9110	.9116	.9119	.9090	.9083	.9096	.9092
100	.9578	.9551	.9516	.9520	.9520	.9524	.9514	.9503	.9505	.9503

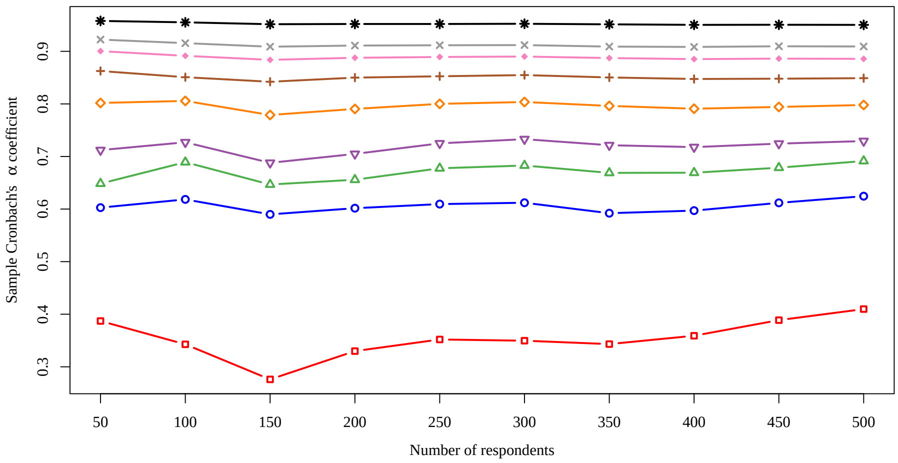

Fig. 21 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(2,2)$ distribution

▶ **Results for $\beta(4, 2)$ distribution**

Table 22 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(4, 2)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.2466	.2789	.2481	.2731	.2744	.3835	.3822	.3766	.3712	.3617
7	.4369	.4239	.4062	.4427	.4527	.4956	.4846	.4848	.4853	.4779
10	.5282	.5114	.5154	.5413	.5342	.5581	.5598	.5679	.5704	.5732
15	.5922	.5557	.5465	.5553	.5555	.5809	.5917	.6095	.6190	.6112
20	.6918	.6768	.6635	.6684	.6620	.6834	.6970	.7096	.7158	.7124
30	.8147	.8079	.8053	.8092	.8037	.8116	.8186	.8244	.8257	.8233
40	.8557	.8571	.8524	.8533	.8482	.8545	.8565	.8610	.8610	.8590
50	.8645	.8725	.8725	.8721	.8655	.8735	.8777	.8825	.8825	.8807
100	.9319	.9370	.9342	.9333	.9306	.9335	.9349	.9375	.9376	.9366

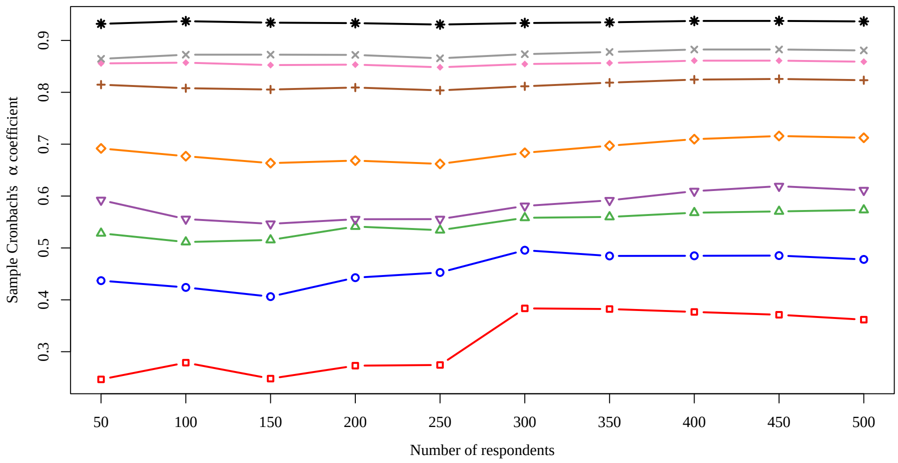

Fig. 22 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(4, 2)$ distribution

▶ **Results for $\beta(6, 1)$ distribution**

Table 23 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(6, 1)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	-.0119	.1867	.1681	.1320	.1165	.1161	.1695	.1800	.1954	.2004
7	.1350	.3192	.2996	.2700	.2740	.2989	.3294	.3473	.3481	.3508
10	.3966	.3979	.4063	.3812	.3908	.4117	.3959	.4051	.3987	.4074
15	.4473	.4829	.4914	.5221	.5390	.5461	.5376	.5357	.5324	.5334
20	.5775	.6441	.6126	.6215	.6320	.6406	.6325	.6333	.6290	.6289
30	.6301	.6984	.6710	.6812	.7042	.7217	.7205	.7207	.7138	.7172
40	.7026	.7411	.7034	.7135	.7272	.7457	.7512	.7545	.7516	.7507
50	.7330	.7749	.7327	.7326	.7462	.7610	.7698	.7703	.7678	.7653
100	.8681	.8767	.8590	.8565	.8599	.8725	.8778	.8809	.8791	.8764

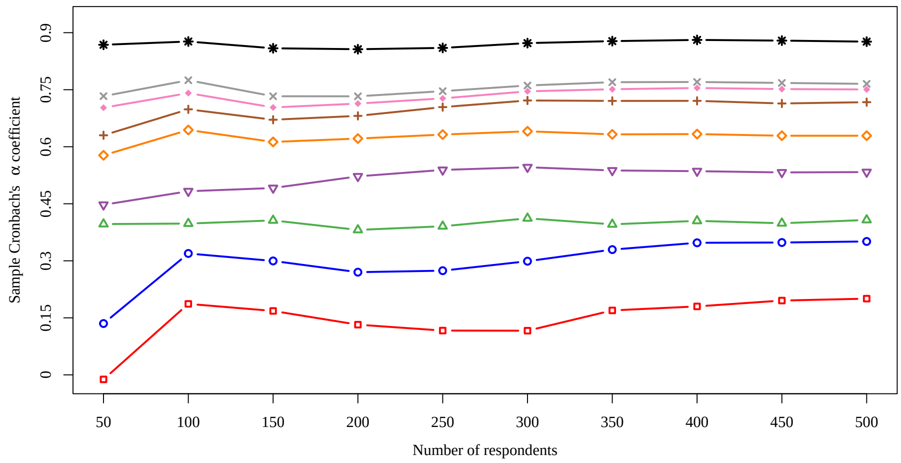

Fig. 23 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(6, 1)$ distribution

▶ Results for $\beta(6,10)$ distribution

Table 24 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire (n) and items in a construct (k) and $\beta\,(6,10)$ distribution

k/n	50	100	150	200	250	300	350	400	450	500
5	.1688	.2244	.1850	.2091	.1901	.1810	.1867	.2065	.1613	.1404
7	.3356	.3582	.3364	.3864	.3658	.3502	.3684	.3697	.3393	.3212
10	.3794	.4269	.4050	.4315	.4288	.4164	.4266	.4218	.4117	.4011
15	.4313	.4733	.4809	.4865	.5009	.5072	.5086	.5004	.4906	.4750
20	.5321	.5316	.5596	.5664	.5686	.5910	.5949	.5873	.5788	.5698
30	.6098	.6181	.6644	.6615	.6670	.6927	.6934	.6974	.6928	.6869
40	.6979	.6862	.7264	.7246	.7307	.7480	.7453	.7472	.7430	.7402
50	.7475	.7512	.7724	.7794	.7807	.7923	.7895	.7893	.7849	.7830
100	.8806	.8809	.8882	.8914	.8929	.8959	.8948	.8944	.8913	.8903

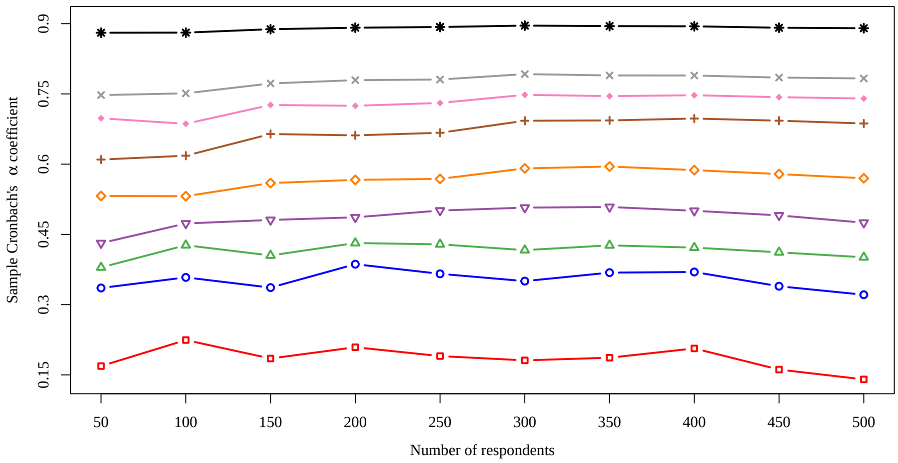

Fig. 24 Sample Cronbach's α coefficients for different numbers of respondents to a questionnaire with 5 (□), 7 (○), 10 (△), 15 (▽), 20 (◇), 30 (+), 40 (◆), 50 (×), and 100 (✱) items in a construct and $\beta\,(6,10)$ distribution