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Supplementary Material for "Robust mixture modeling using the truncated *t* distribution: recent advances and new results"

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Abstract This supplementary document offers a more detailed exposition of both the theoretical and numerical aspects presented in the main paper. Appendix A provides the proof of Proposition 1, and Appendix B contains the proofs of Equations (2) and (3). Appendix C presents the proof of Proposition 2, and Appendix D outlines the proofs of Equations (6) and (7). Appendix E gives the proofs of Equations (8), (9) and (10), and Appendix F outlines the proof of Proportion 3. Appendix G summarizes detailed derivations of the ECME algorithm, and Appendix H details the derivation of Equation (20). Additional figures and tables corresponding to the numerical results are included in Appendix I and Appendix J, respectively.

Appendix A Proof of Proposition 1

(a) Let $\phi(u)$ denote the pdf of $U \sim N(0,1)$ evaluated at $U = u$. A direct application of differentiation yields the following identity:

$$\begin{aligned}\frac{d(z^{k+1}\phi(z))}{dz} &= (k+1)z^k\phi(z) + z^{k+1}\frac{d}{dz}\phi(z)\\ &= (k+1)z^k\phi(z) + z^{k+1}\phi(z)(-z)\\ &= (k+1)z^k\phi(z) - z^{k+2}\phi(z).\end{aligned} \tag{A.1}$$

Integrating (A.1) on both sides over the interval (α,β) and dividing the constant $\Phi(\beta)-\Phi(\alpha)$, we obtain

$$\begin{aligned}&\int_\alpha^\beta \frac{d}{dz}\left(z^{k+1}\phi(z)\right)\frac{1}{\Phi(\beta)-\Phi(\alpha)}\,dz\\ &= \int_\alpha^\beta (k+1)z^k\phi(z)\frac{1}{\Phi(\beta)-\Phi(\alpha)}\,dz - \int_\alpha^\beta z^{k+2}\phi(z)\frac{1}{\Phi(\beta)-\Phi(\alpha)}\,dz.\end{aligned}$$

Applying Leibniz's integral rule to the left-hand side yields the following recursive relation:

$$\frac{1}{\Phi(\beta)-\Phi(\alpha)}\left[\beta^{k+1}\phi(\beta)-\alpha^{k+1}\phi(\alpha)\right] = (k+1)E(Z^k) - E(Z^{k+2}),$$

where $Z \sim TN(0,1;(\alpha,\beta))$. Notice that this identity also holds for $k=-1$, in which case $(k+1)E(Z^k)=0$. This completes the derivation of the desired expression.

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(b) Let $X = \mu + \sigma Z$, where $Z \sim TN(0,1;(\alpha,\beta))$, so that $X \sim TN(\mu,\sigma^2;(a,b))$ with $a = \mu + \sigma\alpha$ and $b = \mu + \sigma\beta$. The n-th moment of X can be computed by a change of variables and an application of the binomial theorem, allowing the moment to be expressed in terms of the moments of Z. Specifically,

$$E(X^n) = E\left[(\mu + \sigma Z)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k E(Z^k).$$

Appendix B Proofs of Equations (2) and (3)

(a) Taking the partial derivative of $\log \Phi\left((a,b);\mu,\sigma^2\right)$ with respect to μ, we obtain

$$\begin{aligned}
\frac{\partial \log \Phi\left((a,b);\mu,\sigma^2\right)}{\partial \mu} &= \frac{1}{\Phi\left((a,b);\mu,\sigma^2\right)} \int_a^b \frac{\partial}{\partial \mu} \phi\left(x;\mu,\sigma^2\right) dx \\
&= \frac{1}{\Phi\left((a,b);\mu,\sigma^2\right)} \int_a^b \frac{\partial}{\partial \mu} \left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right] dx \\
&= \frac{1}{\Phi\left((a,b);\mu,\sigma^2\right)} \frac{1}{\sqrt{2\pi}\sigma} \int_a^b \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \left(\frac{x-\mu}{\sigma^2}\right) dx. \qquad \text{(B.1)}
\end{aligned}$$

Letting $w = x - \mu$, we can rewrite (B.1) as

$$\frac{1}{\sigma^2} \int_{a-\mu}^{b-\mu} \frac{1}{\Phi\left((a,b),\mu,\sigma^2\right)} \frac{w}{\sqrt{2\pi}\sigma} \exp\left(-\frac{w^2}{2\sigma^2}\right) dw = \frac{1}{\sigma^2} M_{TN}^{(1)},$$

where $M_{TN}^{(1)} = E(W)$ is the first moment of $W \sim TN(0,\sigma^2;(a-\mu,b-\mu))$.

(b) Taking the partial derivative of $\log \Phi\left((a,b);\mu,\sigma^2\right)$ with respect to σ^2 yields

$$\begin{aligned}
\frac{\partial \log \Phi\left((a,b);\mu,\sigma^2\right)}{\partial \sigma^2} &= \frac{1}{\Phi((a,b);\mu,\sigma^2)} \int_a^b \frac{\partial}{\partial \sigma^2} \phi(x;\mu,\sigma^2)\, dx \\
&= \frac{1}{\Phi((a,b);\mu,\sigma^2)} \left[\int_a^b \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{2}\right) (\sigma^2)^{-\frac{3}{2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \right. \\
&\quad \left. + \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \left(\frac{(x-\mu)^2}{2\sigma^4}\right) dx \right] \\
&= \frac{1}{\Phi((a,b);\mu,\sigma^2)} \left[-\frac{1}{2\sigma^2} \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \right. \\
&\quad \left. + \frac{1}{2\sigma^4} \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) (x-\mu)^2\, dx \right] \\
&= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} \int_a^b \frac{1}{\Phi((a,b);\mu,\sigma^2)} \frac{(x-\mu)^2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \qquad \text{(B.2)}
\end{aligned}$$

By the substitution $w = x - \mu$ again and using the identity $\Phi((a-\mu,b-\mu);0,\sigma^2) = \Phi((a,b);\mu,\sigma^2)$, Eq. (B.2) can be written as

$$-\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} \frac{\Phi((a-\mu,b-\mu);0,\sigma^2)}{\Phi((a,b);\mu,\sigma^2)} \int_{a-\mu}^{b-\mu} \frac{w^2 \phi\left(w;0,\sigma^2\right)}{\Phi((a-\mu,b-\mu);0,\sigma^2)} dw = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} M_{TN}^{(2)},$$

where $M_{TN}^{(2)} = E(W^2)$ is the second moment of $W \sim TN(0,\sigma^2;(a-\mu,b-\mu))$.

Appendix C Proof of Proposition 2

Consider $X_1,\dots,X_n$ as a random sample from the $TN(\mu,\sigma^2;(a,b))$ distribution.

(a) The likelihood function of $\boldsymbol{\theta}$ is given by

$$\begin{aligned}\mathscr{L}(\boldsymbol{\theta};\boldsymbol{x}) &= \prod_{j=1}^{n}\left[\frac{1}{\Phi((a,b);\mu,\sigma^2)}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x_j-\mu)^2}{2\sigma^2}\right)\right]\\ &= \frac{1}{[\Phi((a,b);\mu,\sigma^2)]^n}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n\exp\left(\sum_{j=1}^{n}-\frac{1}{2\sigma^2}(x_j-\mu)^2\right).\end{aligned}$$

Obviously, the log-likelihood function of $\boldsymbol{\theta}$ is

$$\ell(\boldsymbol{\theta};\boldsymbol{x}) = -n\log\Phi((a,b);\mu,\sigma^2)-\frac{n}{2}\log\sigma^2-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j-\mu)^2. \tag{C.1}$$

(b) Taking the partial derivative of (C.1) with respect to μ yields

$$s_\mu = \frac{\partial}{\partial\mu}\ell(\boldsymbol{\theta};\boldsymbol{x}) = -n\frac{\partial}{\partial\mu}\log\Phi((a,b);\mu,\sigma^2)+\frac{1}{\sigma^2}\sum_{j=1}^{n}(x_j-\mu). \tag{C.2}$$

Applying Eq. (2), we can express the score function (C.2) as

$$s_\mu = \frac{n}{\sigma^2}(\bar{x}-M_{TN}^{(1)}-\mu),$$

where $\bar{x}=n^{-1}\sum_{j=1}^{n}x_j$ denotes the sample mean. Similarly, taking the partial derivative of (C.1) with respect to σ^2 gives

$$s_{\sigma^2} = \frac{\partial}{\partial\sigma^2}\ell(\boldsymbol{\theta};\boldsymbol{x}) = -n\frac{\partial}{\partial\sigma^2}\log\Phi((a,b);\mu,\sigma^2)-\frac{n}{2\sigma^2}+\sum_{j=1}^{n}(x_j-\mu)^2\frac{1}{2\sigma^4}.$$

Applying Eq. (3), the score function becomes

$$s_{\sigma^2} = \frac{1}{2\sigma^4}\left[\sum_{j=1}^{n}(x_j-\mu)^2-nM_{TN}^{(2)}\right].$$

(c) To compute the Hessian matrix of the TN distribution, we begin by calculating the partial derivatives of $\Phi((a,b);\mu,\sigma^2)$ with respect to the parameters μ and σ^2. Applying Leibnizs integral rule, the partial derivative of $\Phi((a,b);\mu,\sigma^2)$ with respect to μ is

$$\begin{aligned}\frac{\partial}{\partial\mu}\Phi((a,b);\mu,\sigma^2) &= \frac{\partial}{\partial\mu}\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx\\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(a-\mu)^2}{2\sigma^2}\right\}-\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(b-\mu)^2}{2\sigma^2}\right\}\\ &= \phi(a;\mu,\sigma^2)-\phi(b;\mu,\sigma^2).\end{aligned} \tag{C.3}$$

Analogously, the partial derivative with respect to σ^2 is given by

$$\begin{aligned}\frac{\partial}{\partial\sigma^2}\Phi((a,b);\mu,\sigma^2) &= \frac{\partial}{\partial\sigma^2}\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx\\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(\frac{b-\mu}{\sigma}\right)^2}{2}\right\}\left(-\frac{b-\mu}{2\sigma^3}\right)-\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(\frac{a-\mu}{\sigma}\right)^2}{2}\right\}\left(-\frac{a-\mu}{2\sigma^3}\right)\\ &= \phi(a;\mu,\sigma^2)\left(\frac{a-\mu}{2\sigma^2}\right)-\phi(b;\mu,\sigma^2)\left(\frac{b-\mu}{2\sigma^2}\right).\end{aligned} \tag{C.4}$$

Both (C.3) and (C.4) are essential for deriving the partial derivatives of $M_{TN}^{(1)}$ and $M_{TN}^{(2)}$ with respect to μ and σ^2. Recall that $M_{TN}^{(1)}$ and $M_{TN}^{(2)}$ are given explicitly by

$$M_{TN}^{(1)} = -\sigma \frac{\phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \tag{C.5}$$

and

$$M_{TN}^{(2)} = \sigma^2 - \sigma^2 \frac{\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right) - \left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi((a,b);\mu,\sigma^2)}. \tag{C.6}$$

To facilitate the computation of the second-order derivatives of the log-likelihood function, we derive the partial derivatives of (C.5) and (C.6) with respect to μ. Applying the quotient rule and chain rule, we obtain

$$\begin{aligned}
\frac{\partial}{\partial \mu} M_{TN}^{(1)} &= -\frac{\sigma\left[\phi(b;\mu,\sigma^2)\left(\frac{b-\mu}{\sigma}\right) - \phi(a;\mu,\sigma^2)\left(\frac{a-\mu}{\sigma}\right)\right]\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^2} \\
&\quad + \frac{\sigma\left[\phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)\right]\left(-\frac{1}{\sigma}\right)\left[\phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)\right]}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^2} \\
&= \frac{-\left[\phi(b;\mu,\sigma^2)(b-\mu) - \phi(a;\mu,\sigma^2)(a-\mu)\right]\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^2} \\
&\quad - \frac{\left[\phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)\right]^2}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^2} \\
&= \frac{-\phi(b;\mu,\sigma^2)(b-\mu) + \phi(a;\mu,\sigma^2)(a-\mu)}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^2} - \frac{\left[\phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)\right]^2}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^2} \\
&= \frac{\phi(a;\mu,\sigma^2)(a-\mu) - \phi(b;\mu,\sigma^2)(b-\mu)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} - \left(\frac{M_{TN}^{(1)}}{\sigma}\right)^2.
\end{aligned} \tag{C.7}$$

Before deriving the partial derivative of (C.6) with respect to μ, we need an intermediate identity involving the derivative of a key component:

$$\begin{aligned}
&\frac{\partial}{\partial \mu}\left\{\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right) - \left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)\right\} \\
&= -\frac{1}{\sigma}\phi\left(\frac{b-\mu}{\sigma}\right) + \left(\frac{b-\mu}{\sigma}\right)\phi(\beta)\frac{\beta}{\sigma} + \frac{1}{\sigma}\phi(\alpha) - \phi(\alpha)\frac{\alpha^2}{\sigma} \\
&= -\frac{1}{\sigma}\phi(\beta) + \frac{\beta^2}{\sigma}\phi(\beta) + \frac{1}{\sigma}\phi(\alpha) - \frac{\alpha^2}{\sigma}\phi(\alpha) \\
&= \phi(\beta)\left(\frac{\beta^2}{\sigma} - \frac{1}{\sigma}\right) + \phi(\alpha)\left(\frac{1}{\sigma} - \frac{\alpha^2}{\sigma}\right).
\end{aligned} \tag{C.8}$$

Using (C.8), we can obtain

$$
\begin{aligned}
\frac{\partial}{\partial\mu}M_{TN}^{(2)} &= -\frac{\sigma}{\Phi((a,b);\mu,\sigma^2)}\left[\phi(\beta)(\beta^2-1)+\phi(\alpha)(1-\alpha^2)\right] \\
&\quad + \frac{\sigma}{\left[\Phi((a,b);\mu,\sigma^2)\right]^2}\left[\phi(\beta)\beta-\phi(\alpha)\alpha\right]\left[\phi(\alpha)-\phi(\beta)\right] \\
&= -\sigma\left\{\frac{\phi(\beta)\beta^2-\phi(\alpha)\alpha^2+\phi(\alpha)-\phi(\beta)}{\Phi((a,b);\mu,\sigma^2)} - \frac{\phi(\alpha)-\phi(\beta)}{\Phi((a,b);\mu,\sigma^2)}\cdot\frac{\phi(\beta)\beta-\phi(\alpha)\alpha}{\Phi((a,b);\mu,\sigma^2)}\right\} \\
&= -\sigma\left\{\frac{\phi(\beta)\beta^2-\phi(\alpha)\alpha^2}{\Phi((a,b);\mu,\sigma^2)} + \frac{M_{TN}^{(1)}}{\sigma} - \frac{M_{TN}^{(1)}}{\sigma}\left[\frac{\phi(\beta)\beta-\phi(\alpha)\alpha}{\Phi((a,b);\mu,\sigma^2)}\right]\right\} \\
&= -\sigma\left[\frac{\phi(\beta)\beta^2-\phi(\alpha)\alpha^2}{\Phi((a,b);\mu,\sigma^2)}\right] - M_{TN}^{(1)} + M_{TN}^{(1)}\left[\frac{\phi(\beta)\beta-\phi(\alpha)\alpha}{\Phi((a,b);\mu,\sigma^2)}\right].
\end{aligned}
\tag{C.9}
$$

To compute $H_{\sigma^2\sigma^2}$, we start by deriving the partial derivative of $M_{TN}^{(2)}$ with respect to σ^2, as shown below:

$$
\begin{aligned}
\frac{\partial}{\partial\sigma^2}M_{TN}^{(2)} &= 1 - \frac{\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right) - \left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi((a,b);\mu,\sigma^2)} \\
&\quad -\sigma^2\frac{(b-\mu)\left[-\frac{1}{2\sigma^2}+\frac{(b-\mu)^2}{2\sigma^4}\right]\phi\left(b;\mu,\sigma^2\right) - (a-\mu)\left[-\frac{1}{2\sigma^2}+\frac{(a-\mu)^2}{2\sigma^4}\right]\phi\left(a;\mu,\sigma^2\right)}{\Phi((a,b);\mu,\sigma^2)} \\
&\quad +\sigma^2\frac{\left[\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right) - \left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)\right]\left[\phi\left(a;\mu,\sigma^2\right)\left(\frac{a-\mu}{2\sigma^2}\right) - \phi\left(b;\mu,\sigma^2\right)\left(\frac{b-\mu}{2\sigma^2}\right)\right]}{\left[\Phi((a,b);\mu,\sigma^2)\right]^2} \\
&= 1 - \frac{\phi(\beta)\beta-\phi(\alpha)\alpha}{\Phi((a,b);\mu,\sigma^2)} - \frac{(\beta^3-\beta)\phi(\beta)-(\alpha^3-\alpha)\phi(\alpha)}{2\Phi((a,b);\mu,\sigma^2)} - \frac{\left[\beta\phi(\beta)-\alpha\phi(\alpha)\right]^2}{2\left[\Phi((a,b);\mu,\sigma^2)\right]^2}.
\end{aligned}
\tag{C.10}
$$

We now proceed to compute the second-order partial derivative of the log-likelihood function with respect to μ. Applying Eq. (C.7), we obtain

$$
\begin{aligned}
H_{\mu\mu} &= \frac{\partial s_\mu}{\partial\mu} = \frac{\partial}{\partial\mu}\left[\frac{n}{\sigma^2}(\bar{X}-M_{TN}^{(1)}-\mu)\right] = \frac{n}{\sigma^2}\left[-\frac{\partial}{\partial\mu}M_{TN}^{(1)}-1\right] \\
&= \frac{n\left(M_{TN}^{(1)}\right)^2}{\sigma^4} - \frac{n\left[\phi(a;\mu,\sigma^2)(a-\mu)-\phi(b;\mu,\sigma^2)(b-\mu)\right]}{\Phi((a,b);\mu,\sigma^2)\sigma^2} - \frac{n}{\sigma^2} \\
&= \frac{n\left(M_{TN}^{(1)}\right)^2}{\sigma^4} - \frac{n\left[\phi(\alpha)\alpha-\phi(\beta)\beta\right]}{\Phi((a,b);\mu,\sigma^2)\sigma^2} - \frac{n}{\sigma^2}.
\end{aligned}
$$

The mixed second-order derivative of the log-likelihood with respect to μ and σ^2 is computed as follows:

$$
\begin{aligned}
H_{\mu\sigma^2} &= \frac{\partial^2}{\partial\mu\partial\sigma^2}\ell(\boldsymbol{\theta};\boldsymbol{x}) = \frac{\partial}{\partial\mu}\left\{\frac{1}{2\sigma^4}\left[\sum_{j=1}^{n}(x_j-\mu)^2 - nM_{TN}^{(2)}\right]\right\} \\
&= -\frac{\sum_{j=1}^{n}(x_j-\mu)}{\sigma^4} + \frac{n}{2\sigma^3}\frac{\phi(\beta)\beta^2-\phi(\alpha)\alpha^2}{\Phi((a,b);\mu,\sigma^2)} + \frac{n}{2\sigma^4}M_{TN}^{(1)} - \frac{n}{2\sigma^4}M_{TN}^{(1)}\frac{\phi(\beta)\beta-\phi(\alpha)\alpha}{\Phi((a,b);\mu,\sigma^2)}.
\end{aligned}
$$

Finally, the second-order derivative of the log-likelihood with respect to σ^2 is given by

$$
\begin{aligned}
H_{\sigma^2\sigma^2} &= \frac{\partial s_{\sigma^2}}{\partial\sigma^2\partial\sigma^2} = \frac{\partial}{\partial\sigma^2}\left\{\frac{1}{2\sigma^4}\left[\sum_{j=1}^{n}(x_j-\mu)^2 - nM_{TN}^{(2)}\right]\right\} \\
&= -\frac{1}{\sigma^6}\sum_{j=1}^{n}(x_j-\mu)^2 + \frac{n}{\sigma^6}M_{TN}^{(2)} - \frac{n}{2\sigma^4}\frac{\partial}{\partial\sigma^2}M_{TN}^{(2)}.
\end{aligned}
\tag{C.11}
$$

Applying (C.10), Eq. (C.11) can be rewritten as

$$
\begin{aligned}
H_{\sigma^2\sigma^2} &= -\frac{1}{\sigma^6}\sum_{j=1}^{n}(x_j-\mu)^2+\frac{n}{\sigma^6}M_{TN}^{(2)} \\
&\quad -\frac{n}{2\sigma^4}\left[1-\frac{\phi(\beta)\beta-\phi(\alpha)\alpha}{\Phi((a,b);\mu,\sigma^2)}-\frac{(\beta^3-\beta)\phi(\beta)-(\alpha^3-\alpha)\phi(\alpha)}{2\Phi((a,b);\mu,\sigma^2)}-\frac{[\beta\phi(\beta)-\alpha\phi(\alpha)]^2}{2\left[\Phi((a,b);\mu,\sigma^2)\right]^2}\right] \\
&= -\frac{1}{\sigma^6}\sum_{j=1}^{n}(x_j-\mu)^2+\frac{n}{\sigma^6}M_{TN}^{(2)}-\frac{n}{2\sigma^4}+n\frac{(\beta^3-\beta)\phi(\beta)-(\alpha^3-\alpha)\phi(\alpha)}{4\sigma^4\Phi((a,b);\mu,\sigma^2)}+\frac{n}{4\sigma^4}\frac{[\beta\phi(\beta)-\alpha\phi(\alpha)]^2}{[\Phi((a,b);\mu,\sigma^2)]^2}.
\end{aligned}
\tag{C.12}
$$

Appendix D Proofs of Equations (6) and (7)

(a) To derive the first moment of $Y\sim Tt(\mu,\sigma^2,\nu;(a,b))$, we proceed with the following expression for the expectation:

$$
E(Y-\mu) = \int_a^b (y-\mu)f(y)dy=\frac{C}{\sigma}\int_a^b(y-\mu)\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-(\nu+1)/2}dy, \tag{D.1}
$$

where

$$
C=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}.
$$

Next, we perform a change of variable by letting the term $1+\frac{(y-\mu)^2}{\nu\sigma^2}$ be equal to z, which gives the differential relation $dz=\frac{2(y-\mu)}{\nu\sigma^2}dy$. Therefore, we can express $(y-\mu)dy$ as $\frac{\nu\sigma^2}{2}dz$. The integration limits change to $a'=1+\frac{(a-\mu)^2}{\nu\sigma^2}$ and $b'=1+\frac{(b-\mu)^2}{\nu\sigma^2}$. Rearranging Eq. (D.1), we have

$$
\begin{aligned}
E(Y) &= \mu + C\int_{a'}^{b'} z^{-\frac{(\nu+1)}{2}}\frac{\nu\sigma}{2}dz \\
&= \mu+\frac{C\nu\sigma}{2}\left\{\frac{1}{-\frac{(\nu+1)}{2}+1}z^{-\frac{(\nu+1)}{2}+1}\Big|_{a'}^{b'}\right\} \\
&= \mu+C\frac{\nu\sigma}{\nu-1}\left\{\left[1+\frac{(a-\mu)^2}{\nu\sigma^2}\right]^{-\frac{(\nu-1)}{2}}-\left[1+\frac{(b-\mu)^2}{\nu\sigma^2}\right]^{-\frac{(\nu-1)}{2}}\right\}.
\end{aligned}
$$

(b) Since $(y-\mu)^2=\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}-1\right]\nu\sigma^2$, it follows that

$$
\begin{aligned}
\mathrm{Var}(Y) &= E[(Y-\mu)^2] \\
&= C\int_a^b\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-(\nu+1)/2}\left\{\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]-1\right\}\nu\sigma\, dy \\
&= C\nu\sigma\left\{\int_a^b\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-(\nu+1)/2}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]dy-\int_a^b\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-(\nu+1)/2}dy\right\} \\
&= \nu\sigma^2\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_a^b\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-\frac{(\nu+1)}{2}+1}dy \\
&\quad -\nu\sigma^2\int_a^b\frac{C}{\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-(\nu+1)/2}dy.
\end{aligned}
\tag{D.2}
$$

Note that $-\frac{(\nu+1)}{2}+1=-\frac{(\nu-2)+1}{2}$, which shows that the DOF changes from ν to $\nu-2$. Consequently, the scale parameter also adjusts, changing from σ^2 to $\frac{\nu}{\nu-2}\sigma^2$. Furthermore, the second term in Eq. (D.2) simplifies to $\nu\sigma^2$, as the pdf of $Tt(\mu,\sigma^2,\nu;(a,b))$ is $\frac{C}{\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-\frac{(\nu+1)}{2}}$. Therefore, the variance of Y can be expressed as

$$
\begin{aligned}
\mathrm{Var}(Y) &= \frac{\nu\sigma^2}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_a^b \frac{\Gamma\left(\frac{\nu-1}{2}\right)\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu-2}{2}\right)\left(\frac{\nu-2}{2}\right)\sqrt{\pi(\nu-2)\left(\frac{\nu\sigma^2}{\nu-2}\right)}} \\
&\quad\times\left[1+\frac{(y-\mu)^2}{(\nu-2)\left(\frac{\nu\sigma^2}{\nu-2}\right)}\right]^{-\frac{(\nu-2)+1}{2}} dy-\nu\sigma^2 \\
&= \frac{\nu\sigma^2}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\left(\frac{\nu-1}{\nu-2}\right)\int_a^b \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu-2}{2}\right)\sqrt{\pi(\nu-2)\frac{\nu}{\nu-2}\sigma^2}} \\
&\quad\times\left[1+\frac{(y-\mu)^2}{(\nu-2)\left(\frac{\nu\sigma^2}{\nu-2}\right)}\right]^{-\frac{(\nu-2)+1}{2}} dy-\nu\sigma^2 \\
&= \frac{\nu\sigma^2}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\left(\frac{\nu-1}{\nu-2}\right)\mathscr{T}((a,b);\mu,\frac{\nu}{\nu-2}\sigma^2,\nu-2) \\
&\quad\times\int_a^b \frac{1}{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu-2}\sigma^2,\nu-2)}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu-2}{2}\right)\sqrt{\pi(\nu-2)\frac{\nu}{\nu-2}\sigma^2}} \\
&\quad\times\left[1+\frac{(y-\mu)^2}{(\nu-2)\left(\frac{\nu\sigma^2}{\nu-2}\right)}\right]^{-\frac{(\nu-2)+1}{2}} dy-\nu\sigma^2, \\
&= \nu\sigma^2\left(\frac{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu-2}\sigma^2,\nu-2)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\frac{\nu-1}{\nu-2}-1\right). \qquad \text{(D.3)}
\end{aligned}
$$

Note that the integrand in Eq. (D.3) is the pdf of a TT distribution $Tt(\mu,\frac{\nu}{\nu-2}\sigma^2,\nu-2;(a,b))$. Therefore, the integral over the interval (a,b) equals 1. Using the identity $\mathrm{Var}(Y)=E(Y^2)-2\mu E(Y)+\mu^2$, we can directly derive the second moment of Y as follows:

$$
E(Y^2)=2\mu E(Y)-\mu^2+\nu\sigma^2\left(\frac{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu-2}\sigma^2,\nu-2)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\frac{\nu-1}{\nu-2}-1\right).
$$

Appendix E Proofs of Equations (8), (9) and (10)

(a) Recall that $\log\mathscr{T}((a,b);\mu,\sigma^2,\nu)=\log\int_a^b t(y;\mu,\sigma^2,\nu)dy$. By the chain rule and Leibniz's rule for differentiation under the integral sign, the partial derivative of $\log\mathscr{T}((a,b);\mu,\sigma^2,\nu)$ with respect to μ is given by

$$
\begin{aligned}
&\frac{\partial\log\mathscr{T}((a,b);\mu,\sigma^2,\nu)}{\partial\mu} \\
&= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_a^b \frac{\partial}{\partial\mu}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma}\left\{1+\frac{(y-\mu)^2}{\nu\sigma^2}\right\}^{-\frac{\nu+1}{2}} dy \\
&= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\frac{\nu+1}{\nu\sigma^2}\int_a^b (y-\mu)\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma}\left\{1+\frac{(y-\mu)^2}{\nu\sigma^2}\right\}^{-\frac{\nu+1}{2}-1} dy.
\end{aligned}
$$

With the change of variable $x = y-\mu$, it follows that $dx = dy$ and the interval transforms to $x \in (a-\mu, b-\mu)$. Note that the gamma function satisfies the recurrence relation $\Gamma(x+1) = x\Gamma(x)$. Hence, the above integral can be simplified as

$$\begin{aligned}
&\frac{1}{\sigma^2}\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_{a-\mu}^{b-\mu} x\frac{\Gamma(\frac{\nu+2+1}{2})}{\Gamma(\frac{\nu+2}{2})\sqrt{\pi(\nu+2)}\left[(\frac{\nu}{\nu+2})\sigma^2\right]^{1/2}}\left\{1+\frac{x^2}{(\nu+2)(\frac{\nu}{\nu+2}\sigma^2)}\right\}^{-\frac{\nu+2+1}{2}} dx\\
=&\frac{1}{\sigma^2}\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_{a-\mu}^{b-\mu} xt\left(x;0,\frac{\nu}{\nu+2}\sigma^2,\nu+2\right)dx.
\end{aligned}$$

To express the derivative in the form of an expectation, we multiply both the numerator and the denominator by

$$\int_{a-\mu}^{b-\mu} t(x;0,\frac{\nu}{\nu+2}\sigma^2,\nu+2)\,dx = \mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2),$$

which enables us to interpret the integral as $M_{TT}^{(1)}$, representing the first moment of $Tt\left(0,\frac{\nu}{\nu+2}\sigma^2,\nu+2;(a-\mu,b-\mu)\right)$. Putting it all together, we have

$$\begin{aligned}
\frac{\partial}{\partial\mu}\log\mathscr{T}((a,b);\mu,\sigma^2,\nu) &= \frac{1}{\sigma^2}\frac{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_{a-\mu}^{b-\mu} x\frac{t(x;0,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}dx\\
&= \frac{1}{\sigma^2}\frac{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}M_{TT}^{(1)}.
\end{aligned}$$

(b) With similar arguments, the partial derivative of $\log\mathscr{T}((a,b);\mu,\sigma^2,\nu)$ with respect to σ^2 is given by

$$\begin{aligned}
&\frac{\partial\log\mathscr{T}((a,b);\mu,\sigma^2,\nu)}{\partial\sigma^2}\\
=&\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_a^b\frac{\partial}{\partial\sigma^2}\left\{\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-\frac{\nu+1}{2}}\right\}dy\\
=&\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\left\{(-\frac{1}{2\sigma^2})\int_a^b\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-\frac{\nu+1}{2}}dy\right.\\
&\left.+\frac{1}{2\sigma^4}\int_a^b\frac{(y-\mu)^2(\frac{\nu+1}{2})\Gamma(\frac{\nu+1}{2})}{(\frac{\nu}{2})\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu+2)}[(\frac{\nu}{\nu+2})\sigma^2]^{1/2}}\left[1+\frac{(y-\mu)^2}{(\nu+2)(\frac{\nu}{\nu+2}\sigma^2)}\right]^{-\frac{\nu+2+1}{2}}dy\right\}\\
=&\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\left\{(-\frac{1}{2\sigma^2})\int_a^b\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-\frac{\nu+1}{2}}dy\right.\\
&\left.+\frac{1}{2\sigma^4}\int_{a-\mu}^{b-\mu}x^2\frac{\Gamma(\frac{\nu+2+1}{2})}{\Gamma(\frac{\nu+2}{2})\sqrt{\pi(\nu+2)}[(\frac{\nu}{\nu+2})\sigma^2]^{1/2}}\left[1+\frac{x^2}{(\nu+2)(\frac{\nu}{\nu+2}\sigma^2)}\right]^{-\frac{(\nu+2)+1}{2}}dx\right\}\\
=&(-\frac{1}{2\sigma^2})\frac{\int_a^b t(y;\mu,\sigma^2,\nu)dy}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}+\frac{1}{2\sigma^4}\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\\
&\times\int_{a-\mu}^{b-\mu}x^2\frac{\Gamma(\frac{\nu+2+1}{2})}{\Gamma(\frac{\nu+2}{2})\sqrt{\pi(\nu+2)}[(\frac{\nu}{\nu+2})\sigma^2]^{1/2}}\left[1+\frac{x^2}{(\nu+2)(\frac{\nu}{\nu+2}\sigma^2)}\right]^{-\frac{(\nu+2)+1}{2}}dx\\
=&-\frac{1}{2\sigma^2}+\frac{1}{2\sigma^4}\frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_{a-\mu}^{b-\mu}x^2t\left(x;0,\frac{\nu}{\nu+2}\sigma^2,\nu+2\right)dx\\
=&-\frac{1}{2\sigma^2}+\frac{1}{2\sigma^4}\frac{\int_{a-\mu}^{b-\mu}t(x;0,\frac{\nu}{\nu+2}\sigma^2,\nu+2)\,dx}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_{a-\mu}^{b-\mu}x^2\frac{t(x;0,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\int_{a-\mu}^{b-\mu}t(x;0,\frac{\nu}{\nu+2}\sigma^2,\nu+2)\,dx}dx\\
=&-\frac{1}{2\sigma^2}+\frac{1}{2\sigma^4}\frac{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}M_{TT}^{(2)},
\end{aligned}$$

where $M_{TT}^{(2)}$ denotes the second moment of $Tt\left(0, \frac{\nu}{\nu+2}\sigma^2, \nu+2; (a-\mu, b-\mu)\right)$.

(c) The partial derivative of $\log \mathscr{T}((a,b);\mu,\sigma^2,\nu)$ with respect to ν is derived as follows:

$$\frac{\partial \log \mathscr{T}((a,b);\mu,\sigma^2,\nu)}{\partial \nu} = \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)} \int_a^b \frac{\partial}{\partial \nu} t(y;\mu,\sigma^2,\nu)\, dy, \tag{E.1}$$

where $\frac{\partial}{\partial \nu} t(y;\mu,\sigma^2,\nu)$ can be alternatively rewritten as

$$t(y;\mu,\sigma^2,\nu)\frac{\partial}{\partial \nu}\log t(y;\mu,\sigma^2,\nu).$$

Notice that the logarithm of $t(y;\mu,\sigma^2,\nu)$ is given by

$$\begin{aligned}\log t(y;\mu,\sigma^2,\nu) &= \log \Gamma\left(\frac{\nu+1}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2}\log(\pi\nu) \\ &\quad -\log\sigma - \frac{\nu+1}{2}\log\left(1+\frac{(y-\mu)^2}{\nu\sigma^2}\right).\end{aligned} \tag{E.2}$$

We now consider the partial derivative of Eq. (E.2) with respect to ν, where δ denotes the standardized form of y, defined as $\delta = (y-\mu)/\sigma$. Furthermore, we make use of the identity $\frac{d}{dx}\log\Gamma(g(x)) = \mathrm{DG}(g(x))\frac{dg(x)}{dx}$, where $\mathrm{DG}(\cdot)$ denotes the digamma function. This yields

$$\begin{aligned}\frac{\partial}{\partial \nu}\log t(y;\mu,\sigma^2,\nu) &= \frac{1}{2}\mathrm{DG}\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\mathrm{DG}\left(\frac{\nu}{2}\right) - \frac{1}{2\nu} - \frac{1}{2}\log\left(1+\frac{\delta^2}{\nu}\right) \\ &\quad - \frac{\nu+1}{2(\nu+\delta^2)} + \frac{\nu+1}{2\nu}.\end{aligned} \tag{E.3}$$

Incorporating Eq. (E.3) into Eq. (E.1), we obtain

$$\begin{aligned}\frac{\partial \log \mathscr{T}((a,b);\mu,\sigma^2,\nu)}{\partial \nu} &= \int_a^b \frac{t(y;\mu,\sigma^2,\nu)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\left\{\frac{1}{2}\mathrm{DG}\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\mathrm{DG}\left(\frac{\nu}{2}\right) - \frac{1}{2\nu}\right. \\ &\qquad \left. -\frac{1}{2}\log\left(1+\frac{\delta^2}{\nu}\right) - \frac{\nu+1}{2(\nu+\delta^2)} + \frac{\nu+1}{2\nu}\right\} dy, \\ &= \left[\frac{1}{2}\mathrm{DG}\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\mathrm{DG}\left(\frac{\nu}{2}\right) - \frac{1}{2\nu} + \frac{\nu+1}{2\nu}\right] + \int_a^b \frac{1}{2}\log\left(1+\frac{\delta^2}{\nu}\right) \\ &\qquad \times \frac{t(y;\mu,\sigma^2,\nu)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}dy - \int_a^b \frac{\nu+1}{2(\nu+\delta^2)}\frac{t(y;\mu,\sigma^2,\nu)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}dy \\ &= \frac{1}{2}\mathrm{DG}\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\mathrm{DG}\left(\frac{\nu}{2}\right) + \frac{1}{2} - \frac{1}{2}E\left[\log\left(1+\frac{\delta^2}{\nu}\right)\right] - \frac{1}{2}E\left[\frac{\nu+1}{\nu+\delta^2}\right] \\ &= \frac{1}{2}\left\{\mathrm{DG}\left(\frac{\nu+1}{2}\right) - \mathrm{DG}\left(\frac{\nu}{2}\right) + 1 - E\left[\log\left(1+\frac{\delta^2}{\nu}\right)\right] - \eta\right\},\end{aligned}$$

where

$$
\begin{aligned}
\eta &= E\left[\frac{\nu+1}{\nu+\delta^2}\right] \\
&= \int_a^b \frac{\nu+1}{\nu+\delta^2}\frac{t(y;\mu,\sigma^2,\nu)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\,dy \\
&= \frac{\nu+1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_a^b \frac{\frac{1}{\nu}}{1+\frac{(y-\mu)^2}{\nu\sigma^2}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-\frac{(\nu+1)}{2}}\,dy \\
&= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_a^b \left[1+\frac{(y-\mu)^2}{(\nu+2)\frac{\nu\sigma^2}{\nu+2}}\right]^{-\frac{(\nu+1)}{2}-1}\frac{\Gamma\left(\frac{\nu+3}{2}\right)}{\Gamma\left(\frac{\nu+2}{2}\right)\sqrt{\pi(\nu+2)\frac{\nu\sigma^2}{\nu+2}}}\,dy \\
&= \frac{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\int_a^b \frac{t(y;\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}\,dy \\
&= \frac{\mathscr{T}((a,b);\mu,\frac{\nu}{\nu+2}\sigma^2,\nu+2)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}.
\end{aligned}
$$

Appendix F Proof of Proportion 3

Marginalizing the joint pdf $f(y,\tau)=f(y\mid\tau)h(\tau)$ over τ yields

$$
\begin{aligned}
f(y) &= \int_0^\infty f(y\mid\tau)h(\tau)\,d\tau \\
&= \int_0^\infty \frac{\phi(y;\mu,\sigma^2,\sigma^2/\tau)}{\Phi((a,b);\mu,\sigma^2/\tau)}g(\tau;\frac{\nu}{2},\frac{\nu}{2})\frac{\Phi((a,b);\mu,\sigma^2/\tau)}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\,d\tau \\
&= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)} \\
&\quad\times\int_0^\infty \frac{\sqrt{\tau}}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{\tau(y-\mu)^2}{2\sigma^2}\right\}\frac{\nu^{\nu/2}\tau^{\nu/2-1}}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}\exp\left\{\frac{\nu}{2}\tau\right\}}\,d\tau \\
&= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)} \\
&\quad\times\int_0^\infty \frac{\nu^{\nu/2}}{\sqrt{\pi}2^{(\nu+1)/2}\sigma\Gamma\left(\frac{\nu}{2}\right)}\tau^{\frac{\nu+1}{2}-1}\exp\left\{-\frac{\tau}{2}\left[\left(\frac{y-\mu}{\sigma}\right)^2+\nu\right]\right\}\,d\tau. \qquad \text{(F.1)}
\end{aligned}
$$

Factoring out the terms that are independent of τ from the integral, Eq. (F.1) can be rewritten as

$$
\begin{aligned}
f(y) &= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\frac{\nu^{\nu/2}}{\sqrt{\pi}2^{(\nu+1)/2}\sigma\Gamma\left(\frac{\nu}{2}\right)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{1}{2}\left[\left(\frac{y-\mu}{\sigma}\right)^2+\nu\right]\right)^{\frac{\nu+1}{2}}} \\
&\quad\times\int_0^\infty \frac{\left(\frac{1}{2}\left[\left(\frac{y-\mu}{\sigma}\right)^2+\nu\right]\right)^{\frac{\nu+1}{2}}}{\Gamma\left(\frac{\nu+1}{2}\right)}\tau^{\frac{\nu+1}{2}-1}\exp\left\{-\frac{\tau}{2}\left[\left(\frac{y-\mu}{\sigma}\right)^2+\nu\right]\right\}\,d\tau \\
&= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left[1+\frac{(y-\mu)^2}{\nu\sigma^2}\right]^{-\frac{\nu+1}{2}} \\
&= \frac{1}{\mathscr{T}((a,b);\mu,\sigma^2,\nu)}t(y;\mu,\sigma^2,\nu), \quad a<y<b.
\end{aligned}
$$

Appendix G Further detailed derivations of the ECME algorithm

As previously defined, the complete data are given by $\boldsymbol{y}_c = \{y_j, \tau_j, \boldsymbol{Z}_j\}_{j=1}^n$, where $\boldsymbol{Z}_j = (Z_{1j}, \ldots, Z_{gj})^\top$ is a $g \times 1$ vector indicating the component membership for each observation. Based on the hierarchical structure in Eq. (12), we have

$$f(y_j \mid \tau_j, Z_{ij} = 1) = \frac{\phi(y_j; \mu_i, \sigma_i^2/\tau_j)}{\Phi((a,b); \mu_i, \sigma_i^2/\tau_j)} I\{a < y_j < b\}, \tag{G.1}$$

and

$$f(\tau_j \mid Z_{ij} = 1) = g(\tau; \nu/2, \nu/2) \frac{\Phi((a,b); \mu, \sigma^2/\tau)}{\mathscr{T}((a,b); \mu, \sigma^2, \nu)}. \tag{G.2}$$

The likelihood function of $\boldsymbol{\Theta}$ for the complete data $\boldsymbol{y}_c$ is then expressed as

$$L_c(\boldsymbol{\Theta}|\boldsymbol{y}_c) = \prod_{i=1}^{g} \prod_{j=1}^{n} \left[f(y_j \mid \tau_j, Z_{ij} = 1) f(\tau_j \mid Z_{ij} = 1) \right]^{Z_{ij}}. \tag{G.3}$$

Based on Eqs. (G.1) and (G.2), the complete-data log-likelihood function can be explicitly derived as follows:

$$\begin{aligned}
\ell_c(\boldsymbol{\Theta}|\boldsymbol{y}_c) &= \sum_{i=1}^{g} \sum_{j=1}^{n} Z_{ij} \Big[\log \pi_i + \log f(y_j \mid \tau_j, Z_{ij} = 1) + \log f(\tau_j \mid Z_{ij} = 1) \Big] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{g} Z_{ij} \Big[\log \pi_i - \frac{1}{2} \log \tau_j - \frac{1}{2} \log \sigma_i^2 - \frac{1}{2} \log(2\pi) - \frac{\tau_j}{2} \left(\frac{y_j - \mu_i}{\sigma_i} \right)^2 \\
&\quad + \frac{\nu_i}{2} \log \left(\frac{\nu_i}{2} \right) - \log \Gamma \left(\frac{\nu_i}{2} \right) + \frac{\nu_i}{2} (\log \tau_j - \tau_j) - \log \mathscr{T}((a,b); \mu_i, \sigma_i^2, \nu_i) \Big].
\end{aligned}$$

Neglecting terms that are constant with respect to $\boldsymbol{\Theta}$, the Q-function can be expressed as

$$\begin{aligned}
Q(\boldsymbol{\Theta}|\widehat{\boldsymbol{\Theta}}^{(k)}) &= E\left(\ell_c(\boldsymbol{\Theta} \mid \boldsymbol{y}_c) \mid \boldsymbol{y}, \widehat{\boldsymbol{\Theta}}^{(k)} \right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{g} E\left(Z_{ij} \mid y_j, \widehat{\boldsymbol{\Theta}}^{(k)} \right) \Big[\log \pi_i - \frac{1}{2} \log \sigma_i^2 \\
&\quad - \frac{1}{2} E\left(\tau_j \mid y_j, Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)} \right) \left(\frac{y_j - \mu_i}{\sigma_i} \right)^2 + \frac{\nu_i}{2} \log(\frac{\nu_i}{2}) - \log \Gamma(\frac{\nu_i}{2}) \\
&\quad + \frac{\nu_i}{2} \left(E\left(\log \tau_j \mid y_j, Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)} \right) - E\left(\tau_j \mid y_j, Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)} \right) \right) \\
&\quad - \log \mathscr{T}((a,b); \mu_i, \sigma_i^2, \nu_i) \Big] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{g} \widehat{z}_{ij}^{(k)} \Big[\log \pi_i - \frac{1}{2} \log \sigma_i^2 - \frac{\widehat{\tau}_{ij}^{(k)}}{2} \left(\frac{y_j - \mu_i}{\sigma_i} \right)^2 + \frac{\nu_i}{2} \log(\frac{\nu_i}{2}) \\
&\quad - \log \Gamma(\frac{\nu_i}{2}) + \frac{\nu_i}{2} \left(\widehat{\kappa}_{ij}^{(k)} - \widehat{\tau}_{ij}^{(k)} \right) - \log \mathscr{T}((a,b); \mu_i, \sigma_i^2, \nu_i) \Big],
\end{aligned}$$

where $\widehat{z}_{ij}^{(k)} = E(Z_{ij} \mid y_j, \widehat{\boldsymbol{\Theta}}^{(k)})$, $\widehat{\tau}_{ij}^{(k)} = E(\tau_j \mid y_j, Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)})$ and $\widehat{\kappa}_{ij}^{(k)} = E(\log \tau_j \mid y_j, Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)})$. Since Z_{ij} follows a Bernoulli distribution, its expectation equals the probability of success. Hence,

$$\begin{aligned}
\widehat{z}_{ij}^{(k)} &= \Pr(Z_{ij} = 1 \mid y_j, \widehat{\boldsymbol{\Theta}}^{(k)}) = \frac{\Pr(Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)}) f(y_j \mid Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)})}{f(y_j; \widehat{\boldsymbol{\Theta}}^{(k)})} \\
&= \frac{\widehat{\pi}_i^{(k)} \dfrac{t\left(y_j; \widehat{\mu}_i^{(k)}, \widehat{\sigma}_i^{2(k)}, \widehat{\nu}_i^{(k)}\right)}{\mathscr{T}\left((a,b); \widehat{\mu}_i^{(k)}, \widehat{\sigma}_i^{2(k)}, \widehat{\nu}_i^{(k)}\right)}}{\sum_{h=1}^{g} \widehat{\pi}_h^{(k)} \dfrac{t\left(y_j; \widehat{\mu}_h^{(k)}, \widehat{\sigma}_h^{2(k)}, \widehat{\nu}_h^{(k)}\right)}{\mathscr{T}\left((a,b); \widehat{\mu}_h^{(k)}, \widehat{\sigma}_h^{2(k)}, \widehat{\nu}_h^{(k)}\right)}}.
\end{aligned}$$

By applying Proposition 3 in conjunction with Bayes' theorem, we obtain

$$f(\tau_j \mid y_j, Z_{ij} = 1) \propto f(\tau_j, y_j \mid Z_{ij} = 1) \propto \tau_j^{\frac{\nu_i+1}{2}-1} \exp\left\{ -\frac{\nu_i + \delta_{ij}^2}{2} \tau_j \right\}, \tag{G.4}$$

which represents the kernel of a Gamma distribution. Thus, we have

$$\tau_j \mid (y_j, Z_{ij} = 1) \sim \text{Gamma}\left(\frac{\nu_i + 1}{2}, \frac{\nu_i + \delta_{ij}^2}{2} \right).$$

As a consequence, we have

$$\widehat{\tau}_{ij}^{(k)} = E(\tau_j \mid y_j, Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)}) = \frac{\widehat{\nu}_i^{(k)} + 1}{\widehat{\nu}_i^{(k)} + \widehat{\delta}_{ij}^{2(k)}}$$

and

$$\widehat{\kappa}_{ij}^{(k)} = E(\log \tau_j \mid y_j, Z_{ij} = 1, \widehat{\boldsymbol{\Theta}}^{(k)}) = \text{DG}\left(\frac{\widehat{\nu}_i^{(k)} + 1}{2} \right) - \log\left(\frac{\widehat{\nu}_i^{(k)} + \widehat{\delta}_{ij}^{2(k)}}{2} \right),$$

where $\widehat{\delta}_{ij}^{2(k)} = (y_j - \widehat{\mu}_i^{(k)})^2 / \widehat{\sigma}_i^{2(k)}$.

Since π_i represents the mixing proportion, it must satisfy $\sum_{i=1}^{g} \pi_i = 1$. To enforce this constraint, we apply the method of Lagrange multipliers by introducing the following augmented objective function:

$$L(\pi_i, \lambda) = Q(\boldsymbol{\Theta} | \widehat{\boldsymbol{\Theta}}^{(k)}) - \lambda \left(\sum_{i=1}^{g} \pi_i - 1 \right). \tag{G.5}$$

Taking the partial derivatives of Eq. (G.5) with respect to π_i and λ, and setting them to zero, we obtain

$$\begin{aligned}
\frac{\partial L(\pi_i, \lambda)}{\partial \pi_i} &= \sum_{j=1}^{n} \widehat{z}_{ij}^{(k)} \frac{1}{\pi_i} - \lambda = 0 \Rightarrow \pi_i = \frac{\sum_{j=1}^{n} \widehat{z}_{ij}^{(k)}}{\lambda}, \\
\frac{\partial L(\pi_i, \lambda)}{\partial \lambda} &= -\left(\sum_{i=1}^{g} \pi_i - 1 \right) = 0 \Rightarrow 1 = \sum_{i=1}^{g} \pi_i = \frac{\sum_{j=1}^{n} \sum_{i=1}^{g} \widehat{z}_{ij}^{(k)}}{\lambda} = \frac{n}{\lambda}.
\end{aligned}$$

Solving the above yields $\lambda = n$, and hence the updated mixing proportions are $\widehat{\pi}_i^{(k+1)} = \widehat{n}_i^{(k)}/n$, for $i = 1, \ldots, g$, where $\widehat{n}_i^{(k)} = \sum_{j=1}^{n} \widehat{z}_{ij}^{(k)}$.

Taking the partial derivative of $Q(\boldsymbol{\Theta} \mid \widehat{\boldsymbol{\Theta}}^{(k)})$ with respect to μ_i and setting it equal to zero yields

$$\begin{aligned}\frac{\partial Q(\boldsymbol{\Theta} \mid \widehat{\boldsymbol{\Theta}}^{(k)})}{\partial \mu_i} &= -\frac{1}{2}\frac{\partial}{\partial \mu_i}\sum_{j=1}^{n} \widehat{z}_{ij}^{(k)}\widehat{\tau}_{ij}^{(k)}\left(\frac{y_j-\mu_i}{\sigma_i}\right)^2 - \sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\frac{1}{\sigma_i^2}\widehat{\eta}_i^{(k)}\hat{M}_{TT}^{(1)(k)} \\ &= \frac{1}{\sigma_i^2}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\tau}_{ij}^{(k)}(y_j-\mu_i) - \frac{1}{\sigma_i^2}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\eta}_i^{(k)}\hat{M}_{TT}^{(1)(k)} = 0.\end{aligned}$$

Thus, the update for μ_i is

$$\widehat{\mu}_i^{(k+1)} = \frac{\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\tau}_{ij}^{(k)}y_j - \sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\eta}_i^{(k)}\hat{M}_{TT}^{(1)(k)}}{\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\tau}_{ij}^{(k)}}.$$

Fixing $\mu_i = \widehat{\mu}_i^{(k+1)}$, we update $\widehat{\sigma}_i^{2(k)}$ by maximizing the Q-function with respect to σ_i^2. The derivative of Eq. (17) with respect to σ_i^2 is

$$\begin{aligned}\frac{\partial Q(\boldsymbol{\Theta}|\widehat{\boldsymbol{\Theta}}^{(k)})}{\partial \sigma_i^2} &= -\frac{1}{2\sigma_i^2}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)} - \frac{1}{2}\frac{\partial}{\partial \sigma_i^2}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\left[\frac{\widehat{\tau}_{ij}^{(k)}}{\sigma_i^2}(y_j-\widehat{\mu}_i^{(k+1)})^2\right] + \frac{1}{2\sigma_i^2}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)} - \frac{1}{2\sigma_i^4}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\eta}_i^{(k)}\hat{M}_{TT}^{(2)(k)} \\ &= -\frac{1}{2\sigma_i^2}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)} + \frac{1}{2\sigma_i^4}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\tau}_{ij}^{(k)}(y_j-\widehat{\mu}_i^{(k+1)})^2 + \frac{1}{2\sigma_i^2}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)} - \frac{1}{2\sigma_i^4}\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\eta}_i^{(k)}\hat{M}_{TT}^{(2)(k)}.\end{aligned} \tag{G.6}$$

Setting Eq. (G.6) to zero yields

$$\sigma_i^2 = \frac{\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\tau}_{ij}^{(k)}(y_j-\widehat{\mu}_i^{(k+1)})^2}{\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}} + \sigma_i^2 - \widehat{\eta}_i^{(k)}\hat{M}_{TT}^{(2)(k)}.$$

To solve this equation, we apply the fixed-point iteration method, which reformulates a root-finding problem of the form $f(x)=0$ into a fixed-point equation

$$f(x)=0 \quad \Leftrightarrow \quad x=g(x). \tag{G.7}$$

Starting with an initial guess $x=x_0$, the estimate is updated iteratively as

$$x_{n+1}=g(x_n), \quad n \geq 0.$$

If the sequence $\{x_n\}$ converges, its limit is taken as the solution, and the iteration terminates. Accordingly, the updating equation for σ_i^2 can be solved as

$$\widehat{\sigma}_i^{2(k+1)} = \frac{\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}\widehat{\tau}_{ij}^{(k)}(y_j-\widehat{\mu}_i^{(k+1)})^2}{\sum_{j=1}^{n}\widehat{z}_{ij}^{(k)}} + \widehat{\sigma}_i^{2(k)} - \widehat{\eta}_i^{(k)}\hat{M}_{TT}^{(2)(k)}.$$

Appendix H The detailed derivations of Equation (20)

The individual complete-data log-likelihood function of $\boldsymbol{\Theta}$ on the basis of $\boldsymbol{y}_{cj}=\{y_j,\tau_j,\boldsymbol{Z}_j\}$ is given by

$$\begin{aligned}\ell_{cj}(\boldsymbol{\Theta} \mid \boldsymbol{y}_{cj}) = \sum_{r=1}^{g} Z_{rj}\Bigg[&\log\pi_r - \frac{1}{2}\log\tau_j - \frac{1}{2}\log\sigma_r^2 - \frac{1}{2}\log(2\pi) - \frac{\tau_j}{2}\left(\frac{y_j-\mu_r}{\sigma_r}\right)^2 \\ &+\frac{\nu_r}{2}\log\left(\frac{\nu_r}{2}\right) - \log\Gamma\left(\frac{\nu_r}{2}\right) + \frac{\nu_r}{2}(\log\tau_j-\tau_j) - \log\mathscr{T}\left((a,b);\mu_r,\sigma_r^2,\nu_r\right)\Bigg].\end{aligned}$$

The score function for π_r, for $r = 1, \cdots, g-1$, can be derived as follows:

$$\hat{s}_{j,\pi_r} = E\left[\frac{\partial}{\partial \pi_r}\ell_{cj}(\boldsymbol{\Theta} \mid \boldsymbol{y}_c) \mid y_j, \hat{\boldsymbol{\Theta}}\right] = E\left[Z_{rj}\left(\frac{1}{\pi_r}\right) + \left(-\frac{Z_{gj}}{\pi_g}\right) \mid y_j, \hat{\boldsymbol{\Theta}}\right] = \frac{\hat{z}_{rj}}{\hat{\pi}_r} - \frac{\hat{z}_{gj}}{\hat{\pi}_g}.$$

Using Eq. (8), the score function for μ_r is given by

$$\begin{aligned}
\hat{s}_{j,\mu_r} &= E\left[\frac{\partial}{\partial \mu_r}\ell_{cj}(\boldsymbol{\Theta} \mid \boldsymbol{y}_c) \mid y_j, \hat{\boldsymbol{\Theta}}\right] \\
&= E\left\{Z_{rj}\tau_j \frac{y_j - \mu_r}{\sigma_r^2} - Z_{rj}\frac{\partial}{\partial \mu_r}\left[\log \mathscr{T}\left((a,b); \mu_r, \sigma_r^2, \nu_r\right)\right] \mid y_j, \hat{\boldsymbol{\Theta}}\right\} \\
&= \hat{z}_{rj}\hat{\tau}_{rj}\frac{\hat{\delta}_{rj}}{\hat{\sigma}_r} - \hat{z}_{rj}\hat{\eta}_r \frac{1}{\hat{\sigma}_r^2}\hat{M}_{TT}^{(1)},
\end{aligned}$$

where the last equality follows from the identity

$$E(Z_{rj}\tau_j \mid y_j, \hat{\boldsymbol{\Theta}}) = E[Z_{rj}E(\tau_j \mid Z_{rj} = 1, y_j, \hat{\boldsymbol{\Theta}}) \mid y_j, \hat{\boldsymbol{\Theta}}] = \hat{z}_{rj}\hat{\tau}_{rj}. \tag{H.1}$$

Using Eq. (9), it directly leads to the score function for σ_r^2 as follows:

$$\begin{aligned}
\hat{s}_{j,\sigma_r^2} &= E\left[\frac{\partial}{\partial \sigma_r^2}\ell_{cj}(\boldsymbol{\Theta} \mid \boldsymbol{y}_c) \mid y_j, \hat{\boldsymbol{\Theta}}\right] \\
&= E\left\{Z_{rj}\left[-\frac{1}{2}\log \sigma_r^2 + \frac{\tau_j}{2}\left(\frac{y_j - \mu_r}{\sigma_r}\right)^2 - \frac{\partial}{\partial \sigma_r^2}\log \mathscr{T}\left((a,b); \mu_r, \sigma_r^2, \nu_r\right)\right] \mid y_j, \hat{\boldsymbol{\Theta}}\right\} \\
&= \frac{\hat{\delta}_{rj}^2}{2\hat{\sigma}_r^2}\hat{z}_{rj}\hat{\tau}_{rj} - \frac{1}{2\hat{\sigma}_r^4}\hat{z}_{rj}\hat{\eta}_r\hat{M}_{TT}^{(2)}.
\end{aligned}$$

Using Eq. (10), it is straightforward to derive the score function for ν_r as follows:

$$\begin{aligned}
\hat{s}_{j,\nu_r} &= E\left[\frac{\partial}{\partial \nu_r}\ell_{cj}(\boldsymbol{\Theta} \mid \boldsymbol{y}_c) \mid y_j, \hat{\boldsymbol{\Theta}}\right] \\
&= E\left\{Z_{rj}\left[\frac{1}{2}\log\left(\frac{\nu_r}{2}\right) + \frac{\nu_r}{2}\frac{\frac{1}{2}}{\frac{\nu_r}{2}} - \frac{1}{2}\mathrm{DG}\left(\frac{\nu_r}{2}\right) + \frac{1}{2}\left(\log \tau_j - \tau_j\right)\right.\right. \\
&\qquad \left.\left. - \frac{\partial}{\partial \nu_r}\log \mathscr{T}\left((a,b); \mu_r, \sigma_r^2, \nu_r\right)\right] \mid y_j, \hat{\boldsymbol{\Theta}}\right\} \\
&= E\left\{\frac{1}{2}Z_{rj}\left[\log\left(\frac{\nu_r}{2}\right) + \log \tau_j - \tau_j - \mathrm{DG}\left(\frac{\nu_r + 1}{2}\right)\right.\right. \\
&\qquad \left.\left. + E\left[\log\left(1 + \frac{\delta_{rj}^2}{\nu_r}\right)\right] + \hat{\eta}_r\right] \mid y_j, \hat{\boldsymbol{\Theta}}\right\} \\
&= \frac{1}{2}\hat{z}_{rj}\left\{\log\left(\frac{\hat{\nu}_r}{2}\right) - \mathrm{DG}\left(\frac{\hat{\nu}_r + 1}{2}\right) + \hat{\kappa}_{rj} - \hat{\tau}_{rj} + \hat{\eta}_r + E\left[\log\left(1 + \frac{\hat{\delta}_{rj}^2}{\hat{\nu}_r}\right)\right]\right\},
\end{aligned}$$

where the last equality follows from Eq. (H.1) and the following identity

$$E(Z_{rj}\log \tau_j \mid y_j, \hat{\boldsymbol{\Theta}}) = E[Z_{rj}E(\log \tau_j \mid Z_{rj} = 1, y_j, \hat{\boldsymbol{\Theta}}) \mid y_j, \hat{\boldsymbol{\Theta}}] = \hat{z}_{rj}\hat{\kappa}_{rj}.$$

When $\nu_1 = \cdots = \nu_g = \nu$, the score function for ν is given by the sum of $\hat{s}_{j,\nu_r}$ over all components, that is,

$$\hat{s}_{j,\nu} = \sum_{r=1}^{g}\left\{\frac{1}{2}\hat{z}_{rj}\left\{\log\left(\frac{\hat{\nu}_r}{2}\right) - \mathrm{DG}\left(\frac{\hat{\nu}_r + 1}{2}\right) + \hat{\kappa}_{rj} - \hat{\tau}_{rj} + \hat{\eta}_r + E\left[\log\left(1 + \frac{\hat{\delta}_{rj}^2}{\hat{\nu}_r}\right)\right]\right\}\right\}.$$

Appendix I Additional figures

This appendix provides supplementary visualizations supporting the empirical analyses in the main text. Figure S.1 shows histograms of three biomarkers (CD3, CD5, and CD19) from the DLBCL dataset in Section 6.2, illustrating their empirical distributions. Figure S.2 displays pixel intensity histograms of the melanoma image data from Section 6.3, overlaid with fitted density curves of the four models for visual comparison of their performance.

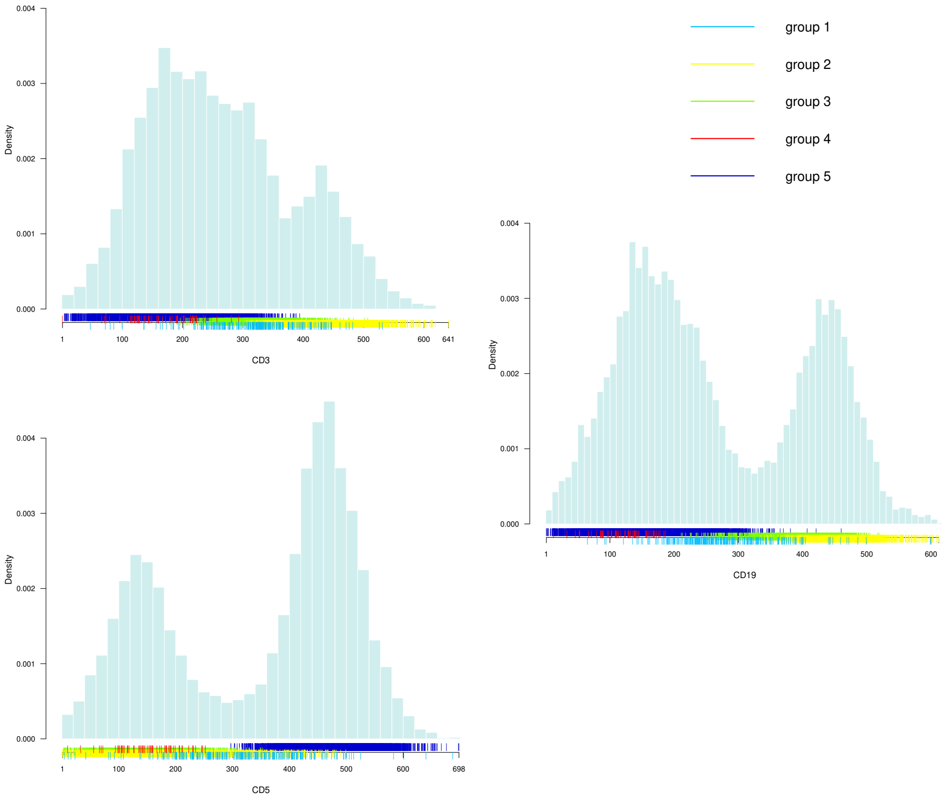

Fig. S.1 Histograms of three variables from the DLBCL dataset, accompanied by rug plots at the bottom that denote the true underlying cluster memberships.

Appendix J Additional tables

This appendix provides detailed numerical results for the simulations and applications in the main text. Table S.1 compares model fits for four models applied to FM-TN data (Section 5.2). Table S.2 reports parameter estimates and standard errors for the FM-TT(UE) model fitted to the DLBCL dataset (Section 6.2). Table S.3 presents parameter estimates and standard errors for six models fitted to the melanoma image data (Section 6.3).

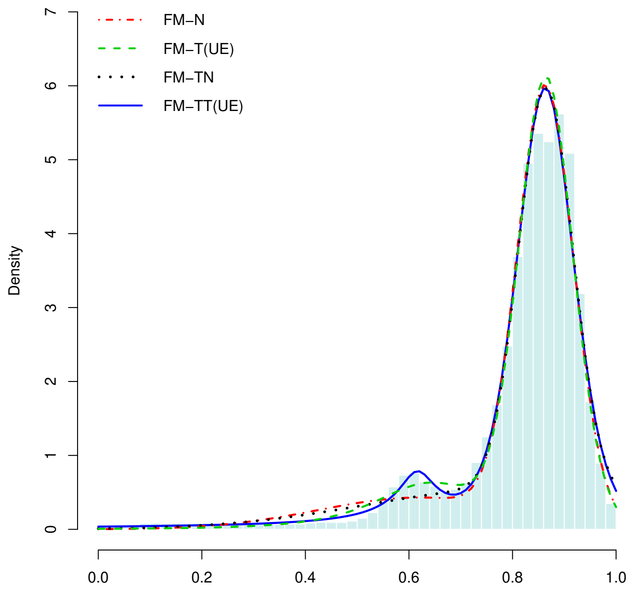

Fig. S.2 Comparisons of fitted FM-N, FM-T(UE), FM-TN, FM-TT(UE) models for the melanoma image data.

Table S.1 Performance comparisons of four mixture models based on 5000 replications, assessing model fit and classification accuracy for data generated from a 3-component FM-TN model.

n	Criterion	FM-N			FM-T			FM-TN			FM-TT		
		Mean	Std	Freq	Mean	Std	Freq	Mean	Std	Freq	Mean	Std	Freq
300	BIC	1703	22.0649	1	1708	22.0572	0	1695	21.0051	4969	1701	20.9986	30
	MISE $\times 10^3$	1.77	0.95	387	1.81	0.96	109	1.44	0.91	3064	1.49	0.92	1440
	ARI	0.690	0.0735	2431	0.694	0.0720	852	0.696	0.0705	670	0.700	0.0695	1047
	CCR	0.884	0.0364	2496	0.886	0.0347	885	0.887	0.0343	594	0.889	0.0327	1025
600	BIC	3374	30.8325	0	3381	30.8371	0	3359	29.4755	4960	3365	29.4719	40
	MISE $\times 10^3$	1.04	0.45	149	1.05	0.45	47	0.68	0.41	2965	0.70	0.41	1839
	ARI	0.703	0.0517	1615	0.707	0.0502	853	0.709	0.0479	978	0.715	0.0464	1554
	CCR	0.890	0.0250	1695	0.893	0.0233	929	0.894	0.0225	847	0.897	0.0207	1529
1200	BIC	6711	44.0584	0	6719	44.0613	0	6682	41.9504	4958	6688	41.9412	42
	MISE $\times 10^3$	0.70	0.24	26	0.70	0.24	15	0.34	0.19	2744	0.34	0.20	2215
	ARI	0.713	0.0352	1097	0.719	0.0321	772	0.721	0.0308	1177	0.726	0.0284	1954
	CCR	0.896	0.0163	1197	0.899	0.0141	876	0.899	0.0137	990	0.902	0.0118	1937
1800	BIC	10047	54.2681	0	10054	54.2834	0	10002	51.5959	4946	10009	51.6132	54
	MISE $\times 10^3$	0.60	0.17	6	0.59	0.17	2	0.23	0.13	2595	0.23	0.13	2397
	ARI	0.717	0.0294	922	0.723	0.0263	702	0.724	0.0252	1243	0.729	0.0223	2133
	CCR	0.897	0.0137	996	0.900	0.0116	835	0.901	0.0112	1033	0.903	0.0092	2136

Table S.2 ML estimates (EST) and standard errors (SE) for CD3, CD5, and CD19 markers from the fitted FM-TT(UE) model.

Parameter	CD3		CD5		CD19	
	EST	SE	EST	SE	EST	SE
π_1	0.071	0.539	0.031	0.119	0.057	0.125
π_2	0.225	0.040	0.179	31.039	0.179	0.554
π_3	0.171	0.157	0.162	31.306	0.192	0.527
π_4	0.010	0.020	0.010	0.580	0.009	0.018
π_5	0.524	—	0.618	—	0.563	—
μ_1	254.534	12.333	284.990	79.708	242.506	12.439
μ_2	437.585	8.758	130.156	1579.487	453.161	45.249
μ_3	314.673	13.049	144.195	1388.413	413.908	90.580
μ_4	112.889	6.289	133.338	22.854	136.103	3.384
μ_5	176.198	49.140	466.851	2.363	160.528	17.840
σ_1^2	732.885	4731.415	2382.411	6821.221	903.207	1373.357
σ_2^2	2790.692	553.497	3717.764	63356.749	1332.539	1610.309
σ_3^2	1003.074	504.734	2950.063	85585.996	2412.862	2175.042
σ_4^2	119.733	205.272	251.922	10734.252	40.665	68.972
σ_5^2	3905.765	2307.931	3066.665	208.465	4392.565	1647.116
ν_1	6.272	178.205	6.418	142.828	4.426	16.766
ν_2	6.885	3.294	5.104	208.948	6.942	11.466
ν_3	5.401	19.568	3.833	87.502	1.989	6.797
ν_4	7.303	116.489	4.833	625.088	4.022	46.957
ν_5	7.841	10.127	9.042	4.546	6.677	8.818

Table S.3 ML estimates (EST) and standard errors (SE) for six comparative models with $g = 2$ fitted to the melanoma image dataset.

Parameter	FM-N		FM-T (EQ)		FM-T (UE)		FM-TN		FM-TT (EQ)		FM-TT (UE)	
	EST	SE	EST	SE	EST	SE	EST	SE	EST	SE	EST	SE
π_1	0.2097	0.0016	0.2096	0.0037	0.2016	0.0020	0.2913	0.0031	0.2305	0.0179	0.1693	0.0011
π_2	0.7903	—	0.7904	—	0.7984	—	0.7087	—	0.7695	—	0.8307	—
μ_1	0.6196	0.0021	0.6213	0.0051	0.6419	0.0016	0.7939	0.0074	0.7042	0.0182	0.6162	0.0006
μ_2	0.8635	0.0002	0.8637	0.0001	0.8645	0.0001	0.8642	0.0002	0.8672	0.0002	0.8635	0.0001
σ_1^2	0.0379	0.0003	0.0368	0.0012	0.0133	0.0003	0.0762	0.0015	0.0366	0.0024	0.0182	0.0024
σ_2^2	0.0029	0.0000	0.0029	0.0000	0.0028	0.0000	0.0027	0.0000	0.0025	0.0001	0.0031	0.0000
ν_1	—	—	100.0000	54.7386	2.4491	0.0700	—	—	3.8366	0.3874	0.0705	0.0126
ν_2	—	—	100.0000	54.7386	40.7450	9.3597	—	—	3.8366	0.3874	9.2370	0.5535