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Supplementary Material: Bandwidth selection for statistical matching and prediction

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Contents

1 Proof of the results in Section 1: The bandwidth selection method

1.1 Closed-expression for the criterion functions

Theorem 1 *If assumption (A1) from the paper holds, x is an interior point of the support of X, $(X_1,\ldots,X_n)$ an i.i.d. sample, and $f(x)\neq 0$, then the MSE_x of $\tilde{m}_h$ can be expressed as*

$$MSE_x(h)=\frac{n-1}{nf(x)^2}(K_h*q_x)^2(x)+\frac{1}{nf(x)^2}\left[(K_h)^2*p_x\right](x), \tag{1}$$

where $p_x(z)=\left(\sigma^2(z)+(m(z)-m(x))^2\right)f(z)$, $q_x(z)=(m(z)-m(x))f(z)$ and $\sigma^2(x):=Var\left(Y|X=x\right)$ stands for the volatility function.

Similarly, if $\hat{f}_{g_X}(x)\neq 0$, then the smoothed bootstrap version of the $MSE_x(h)$ is

$$\begin{aligned}MSE_x^*(h)=&\frac{1}{n\hat{f}_{g_X}^2(x)}\left[\frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\left[(K_h)^2*K_{g_X}\right](x-X_i)+\left[(K_h)^2*\hat{p}_{x,g_X}\right](x)\right]\\&+\frac{n-1}{n\hat{f}_{g_X}^2(x)}(K_h*\hat{q}_{x,g_X})^2(x),\end{aligned} \tag{2}$$

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where $\hat{p}_{x,g_X}(z) = \left(\hat{\sigma}^2_{g_X}(z) + (\tilde{m}_{g_X}(z) - \tilde{m}_{g_X}(x))^2\right)\hat{f}_{g_X}(z)$, $\hat{q}_{x,g_X}(z) = (\tilde{m}_{g_X}(z) - \tilde{m}_{g_X}(x))\hat{f}_{g_X}(z)$ *and* $\mu_r(K) = \int t^r K(t)\,dt$, $\hat{\sigma}^2_{g_X}(z) = \tilde{m}_{2,g_X}(z) - \tilde{m}^2_{g_X}(z)$, $\tilde{m}_{2,g_X}(z) = \dfrac{\sum_{i=1}^n K_{g_X}(z - X_i)\,Y_i^2}{\sum_{i=1}^n K_{g_X}(z - X_i)}$.

Proof Consider the following expression:

$$MSE_x(h) = \mathbb{E}\left[\left(\tilde{m}_h^{NW}(x) - m(x)\right)^2\right] = (\mathbb{E}[A_1])^2 + Var[A_1], \tag{3}$$

where $A_1 = \tilde{m}_h^{NW}(x) - m(x)$. Focusing first on the bias term:

$$\begin{aligned}
\mathbb{E}[A_1] &= \frac{1}{nf(x)}\sum_{i=1}^n \mathbb{E}[K_h(x - X_i)(Y_i - m(x))] \\
&= \frac{1}{f(x)}\mathbb{E}[K_h(x - X_1)(Y_1 - m(x))] \\
&= \frac{1}{f(x)}\mathbb{E}[\mathbb{E}[(K_h(x - X_1)(Y_1 - m(x))\,|X_1]] \\
&= \frac{1}{f(x)}\mathbb{E}[K_h(x - X_1)(\mathbb{E}[Y_1\,|X_1] - m(x))] \\
&= \frac{1}{f(x)}\mathbb{E}[K_h(x - X_1)(m(X_1) - m(x))] \\
&= \frac{1}{f(x)}\int K_h(x - y)(m(y) - m(x))f(y)\,dy \\
&= \frac{1}{f(x)}\int K_h(x - y)q_x(y)\,dy = \frac{1}{f(x)}[K_h * q_x](x),
\end{aligned} \tag{4}$$

where $q_x(z) = (m(z) - m(x))f(z)$. On the other hand,

$$\begin{aligned}
Var[A_1] &= \frac{1}{n^2 f(x)^2}\sum_{i=1}^n Var[K_h(x - X_i)(Y_i - m(x))] \\
&= \frac{1}{nf(x)^2}Var[K_h(x - X_1)(Y_1 - m(x))] \\
&= \frac{1}{nf(x)^2}\mathbb{E}\left[K_h(x - X_1)^2(Y_1 - m(x))^2\right] \\
&\quad - \frac{1}{nf(x)^2}\left(\mathbb{E}[K_h(x - X_1)(Y_1 - m(x))]\right)^2.
\end{aligned} \tag{5}$$

The first term of (5) turns out to be:

$$\begin{aligned}
\mathbb{E}\left[K_h(x - X_1)^2(Y_1 - m(x))^2\right] &= \mathbb{E}\left[\mathbb{E}\left[K_h(x - X_1)^2(Y_1 - m(x))^2\,|X_1\right]\right] \\
&= \mathbb{E}\left[K_h(x - X_1)^2\mathbb{E}\left[(Y_1 - m(x))^2\,|X_1\right]\right] \\
&= \mathbb{E}\left[K_h(x - X_1)^2\left[Var(Y_1 - m(x)\,|X_1)\right.\right. \\
&\quad \left.\left. + (\mathbb{E}[Y_1 - m(x)\,|X_1])^2\right]\right] \\
&= \mathbb{E}\left[K_h(x - X_1)^2\right. \\
&\quad \left.\cdot\left[Var(Y_1\,|X_1) + (m(X_1) - m(x))^2\right]\right] \\
&= \int K_h(x - y)^2\left(\sigma^2(y) + (m(y) - m(x))^2\right)f(y)\,dy \\
&= \left[(K_h)^2 * p_x\right](x),
\end{aligned} \tag{6}$$

where $p_x(z) = \left(\sigma^2(z) + (m(z) - m(x))^2\right) f(z)$. Collecting terms (4) and (6) and plugging them into (5), and then plugging (4) and (5) into (3), the first part of Theorem 1 is proven.

For the proving (2) for the bootstrap analogue, consider the following expression:

$$MSE_x^*(h) = \mathbb{E}^*\left[\left(\tilde{m}_h^{NW*}(x) - \tilde{m}_{g_X}^{NW}(x)\right)^2\right] = (\mathbb{E}^*[A_1^*])^2 + Var^*[A_1^*], \tag{7}$$

where $A_1^* = \tilde{m}_h^{NW*}(x) - \hat{m}_{g_X}(x)$. Focusing on the bootstrap version of the bias term,

$$\begin{aligned}
\mathbb{E}^*[A_1^*] &= \frac{1}{n\hat{f}_{g_X}(x)} \sum_{i=1}^{n} \mathbb{E}^*\left[K_h(x - X_i^*)(Y_i^* - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)} \mathbb{E}^*\left[K_h(x - X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)} \mathbb{E}^*\left[\mathbb{E}^*\left[(K_h(x - X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x)) \mid X_1^*\right]\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)} \mathbb{E}^*\left[K_h(x - X_1^*)(\mathbb{E}^*[Y_1^* \mid X_1^*] - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)} \mathbb{E}^*\left[K_h(x - X_1^*)(\hat{m}_{g_X}^{NW}(X_1^*) - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{\hat{f}_{g_X}(x)} \int K_h(x - y)(\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}(y)\, dy \\
&= \frac{1}{\hat{f}_{g_X}(x)} \int K_h(x - y)\hat{q}_{x,g}(y)\, dy = \frac{1}{\hat{f}_{g_X}(x)} \left[K_h * \hat{q}_{x,g}\right](x),
\end{aligned} \tag{8}$$

where $\hat{q}_{x,g}(z) = (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}(z)$. On the other hand,

$$\begin{aligned}
Var^*[A_1^*] &= \frac{1}{n^2\hat{f}_{g_X}(x)^2} \sum_{i=1}^{n} Var^*\left[K_h(x - X_i^*)(Y_i^* - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{n\hat{f}_{g_X}(x)^2} Var^*\left[K_h(x - X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x))\right] \\
&= \frac{1}{n\hat{f}_{g_X}(x)^2} \mathbb{E}^*\left[K_h(x - X_1^*)^2(Y_1^* - \hat{m}_{g_X}^{NW}(x))^2\right]
\end{aligned}$$

$$- \frac{1}{n\hat{f}_{g_X}(x)^2} \left(\mathbb{E}^*\left[K_h(x - X_1^*)(Y_1^* - \hat{m}_{g_X}^{NW}(x))\right]\right)^2. \tag{9}$$

The first term of (9) turns out to be:

$$
\begin{aligned}
&\mathbb{E}^*\left[K_h(x-X_1^*)^2(Y_1^*-\hat{m}_{g_X}^{NW}(x))^2\right]\\
&=\mathbb{E}^*\left[\mathbb{E}^*\left[K_h(x-X_1^*)^2(Y_1^*-\hat{m}_{g_X}^{NW}(x))^2\,|X_1^*\right]\right]\\
&=\mathbb{E}^*\left[K_h(x-X_1^*)^2\mathbb{E}^*\left[(Y_1^*-\hat{m}_{g_X}^{NW}(x))^2\,|X_1^*\right]\right]\\
&=\mathbb{E}^*\left[K_h(x-X_1^*)^2\left[Var^*(Y_1^*-\hat{m}_{g_X}^{NW}(x)\,|X_1^*)+\left(\mathbb{E}^*\left[Y_1^*-\hat{m}_{g_X}^{NW}(x)\,|X_1^*\right]\right)^2\right]\right]\\
&=\mathbb{E}^*\left[K_h(x-X_1^*)^2\left[Var^*(Y_1^*\,|X_1^*)+\left(\hat{m}_{g_X}^{NW}(X_1^*)-\hat{m}_{g_X}^{NW}(x)\right)^2\right]\right]\\
&=\mathbb{E}^*\left[K_h(x-X_1^*)^2\left[\sigma^{*2}(X_1^*)+\left(\hat{m}_{g_X}^{NW}(X_1^*)-\hat{m}_{g_X}^{NW}(x)\right)^2\right]\right]\\
&=\mathbb{E}^*\left[K_h(x-X_1^*)^2\left[\hat{\sigma}_{g_X}^2(X_1^*)+g_Y^2\mu_2(K)+\left(\hat{m}_{g_X}^{NW}(X_1^*)-\hat{m}_{g_X}^{NW}(x)\right)^2\right]\right]\\
&=\int K_h(x-y)^2\left(\hat{\sigma}_{g_X}^2(y)+g_Y^2\mu_2(K)+(\hat{m}_{g_X}^{NW}(y)-\hat{m}_{g_X}^{NW}(x))^2\right)\hat{f}_{g_X}(y)\,dy\\
&=g_Y^2\mu_2(K)\int K_h(x-y)^2\hat{f}_{g_X}(y)dy\\
&+\int K_h(x-y)^2\left(\hat{\sigma}_{g_X}^2(y)+(\hat{m}_{g_X}^{NW}(y)-\hat{m}_{g_X}^{NW}(x))^2\right)\hat{f}_{g_X}(y)\,dy\\
&=\frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\int K_h(x-y)^2K_{g_X}(y-X_i)dy+\int K_h(x-y)^2\hat{p}_{x,g}(y)\,dy\\
&=\frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\left[(K_h)^2*K_{g_X}\right](x-X_i)+\int K_h(x-y)^2\hat{p}_{x,g}(y)\,dy\\
&=\frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\left[(K_h)^2*K_{g_X}\right](x-X_i)+\left[(K_h)^2*\hat{p}_{x,g}\right](x),
\end{aligned}
\tag{10}
$$

where $\hat{p}_{x,g}(y)=\left(\hat{\sigma}_{g_X}^2(y)+(\hat{m}_{g_X}^{NW}(y)-\hat{m}_{g_X}^{NW}(x))^2\right)\hat{f}_{g_X}(y)$ and $\mu_2(K)=\int t^2K(t)\,dt$.

Collecting terms (8) and (10), and plugging them into (9), and then plugging (8) and (9) into (7), Theorem 1 is proven.

Proposition 1 *If F_1 is the distribution function of the target population and $\hat{m}_h$ the estimated regression function. Then, an upper bound for expression (6) in the paper is given by*

$$\mathbb{E}\left[\left(\int(\hat{m}_h(x)-m(x))\,dF_1(x)\right)^2\right]\leq\mathbb{E}\left[\int(\hat{m}_h(x)-m(x))^2\,dF_1(x)\right]. \tag{11}$$

On the other hand, the average prediction error is given by

$$\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1-\hat{m}_h\left(X_i^1\right)\right)^2\right]=\int\sigma^2(x)\,dF_1(x)+\mathbb{E}\left[\int(\hat{m}_h(x)-m(x))^2\,dF_1(x)\right]. \tag{12}$$

Proof On the one hand, consider Jensen's inequality and the convex function $\psi(z) = z^2$. Then,

$$\begin{aligned}
\int \left(\hat{m}_h(x) - m(x)\right)^2 dF_1(x) &= \mathbb{E}\left[\psi\left(\hat{m}_h\left(X^1\right) - m\left(X^1\right)\right)\big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right)}\right] \\
&\geq \psi\left(\mathbb{E}\left[\hat{m}_h\left(X^1\right) - m\left(X^1\right)\big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right)}\right]\right) \\
&= \left(\int \left(\hat{m}_h(x) - m(x)\right) dF_1(x)\right)^2.
\end{aligned}$$

Then, expression (11) holds.

On the other hand, the average prediction error introduced in Definition 1 in the paper can be expressed as:

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] = \frac{1}{n_1}\sum_{i=1}^{n_1}\mathbb{E}\left[\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] \\
&= \mathbb{E}\left[\left(Y_1^1 - \hat{m}_h\left(X_1^1\right)\right)^2\right] = \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right) + m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right] \\
&= A_1 + A_2 + 2A_3,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &:= \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)^2\right], \\
A_2 &:= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right], \\
A_3 &:= \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)\cdot\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)\right].
\end{aligned}$$

Term A_1 does not depend on h. In fact,

$$\begin{aligned}
A_1 &= \mathbb{E}\left[\mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)^2\Big|_{X_1^1}\right]\right] = \mathbb{E}\left[Var\left(Y_1^1\big|_{X_1^1}\right)\right] \\
&= \mathbb{E}\left[\sigma^2\left(X_1^1\right)\right] = \int \sigma^2(x)\,dF_1(x).
\end{aligned}$$

Moreover, carrying on with further computations in term A_2:

$$\begin{aligned}
A_2 &= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\Big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right)}\right]\right] \\
&= \mathbb{E}\left[\int \left(\hat{m}_h(x) - m(x)\right)^2 dF_1(x)\right].
\end{aligned}$$

Finally, term A_3 results in:

$$\begin{aligned}
A_3 &= \mathbb{E}\left[\mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)\cdot\left(Y_1^1 - m\left(X_1^1\right)\right)\big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right),X_1^1}\right]\right] \\
&= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)\cdot\left(\mathbb{E}\left[Y_1^1\big|_{X_1^1}\right] - m\left(X_1^1\right)\right)\right] = 0,
\end{aligned}$$

as a consequence of $\mathbb{E}\left[Y_1^1\big|_{X_1^1}\right] = m\left(X_1^1\right)$. Then, expression (12) holds.

Theorem 2 *Assume (A1) from the paper, let $\{(X_1^0,Y_1^0),\ldots,(X_{n_0}^0,Y_{n_0}^0)\}$ be a simple random sample coming from the source population, and $\{X_1^1,\ldots,X_{n_1}^1\}$ a simple random sample coming from the target population. Then the MASE prediction error is*

$$MASE_{\tilde{m}_h,X^1}(h)=\frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{f^0(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left[K_h*q^0_{X_j^1}\right]^2(X_j^1)+\frac{1}{n_0}\left[(K_h)^2*p^0_{X_j^1}\right](X_j^1)\right], \tag{13}$$

where $q_x^0(z)=(m(z)-m(x))f^0(z)$ and $p_x^0(y)=\left(\sigma_0^2(y)+(m(y)-m(x))^2\right)f^0(y)$. Similarly

$$\begin{aligned}MASE^*_{\tilde{m}_h,X^1}(h)=&\frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}^0_{g_X}(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left[K_h*\hat{q}^0_{X_j^1,g_X}\right]^2(X_j^1)\right.\\&\left.+\frac{1}{n_0}\left[(K_h)^2*\hat{p}^0_{X_j^1,g_X}\right](X_j^1)+\frac{g_Y^2\mu_2(K)}{n_0}\left[(K_h)^2*\hat{f}^0_{g_X}\right](X_j^1)\right],\end{aligned} \tag{14}$$

where $\hat{p}^0_{x,g_X}(y)=\left(\hat{\sigma}^2_{0,g_X}(y)+(\hat{m}_{g_X}(y)-\hat{m}_{g_X}(x))^2\right)\hat{f}^0_{g_X}(y)$ and $\hat{q}^0_{x,g_X}(z)=(\hat{m}_{g_X}(z)-\hat{m}_{g_X}(x))\hat{f}^0_{g_X}(z)$, and $\hat{\sigma}^2_{0,g_X}(z)=\tilde{m}_{2,g_X}(z)-\tilde{m}^2_{g_X}(z)$ with $\tilde{m}_{2,g_X}(z)=\dfrac{\sum_{i=1}^n K_{g_X}\left(z-X_i^0\right)\left(Y_i^0\right)^2}{\sum_{i=1}^n K_{g_X}\left(z-X_i^0\right)}$.

Proof The target is to work out an explicit expression for the prediction error MASE, given by:

$$\begin{aligned}MASE_{\tilde{m}_h^{NW},X^1}(h)&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[\mathbb{E}_0\left[\left(\tilde{m}_h^{NW}(X_j^1)-m(X_j^1)\right)^2\right]\right]=\\&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0\left[\frac{1}{n_0f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right.\\&\left.+\left(\mathbb{E}_0\left[\frac{1}{n_0f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right)^2\right]=\\&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0[A_1^0]+\left(\mathbb{E}_0\left[A_1^0\right]\right)^2\right]\\&=\frac{1}{n_1}\sum_{j=1}^{n_1}Var_0[A_1^0]+\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0\left[A_1^0\right]\right)^2.\end{aligned} \tag{15}$$

Focusing now on the second term of (15):

$$
\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0\left[A_1^0\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}\mathbb{E}_0\left[K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\,|X_1^0\right]\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)\left(\mathbb{E}_0\left[Y_1^0\,|X_1^0\right]-m(X_j^1)\right)\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)\left(m(X_1^0)-m(X_j^1)\right)\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\int K_h(X_j^1-y)\left(m(y)-m(X_j^1)\right)f^0(y)\,dy\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\int K_h(X_j^1-y)q^0_{X_j^1}(y)\,dy\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\left[K_h*q^0_{X_j^1}\right](X_j^1)\right)^2
\end{aligned}
\tag{16}
$$

where $q_x^0(z)=(m(z)-m(x))f^0(z)$. On the other hand,

$$
\begin{aligned}
\frac{1}{n_1}\sum_{j=1}^{n_1}Var_0[A_1^0]&=\frac{1}{n_1}\sum_{j=1}^{n_1}Var_0\left[\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[\frac{1}{n_0 f^0(X_j^1)^2}Var_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\right]\right]\\
&=\frac{1}{n_0 n_1}\sum_{j=1}^{n_1}\frac{1}{f^0(X_j^1)^2}\left[\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2(Y_1^0-m(X_j^1))^2\right]-\right.\\
&\left.\left(\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\right]\right)^2\right].
\end{aligned}
\tag{17}
$$

The first term of (17) turns out to be:

$$\begin{aligned}
&\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2(Y_1^0-m(X_j^1))^2\right]\\
&=\mathbb{E}_0\left[\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2(Y_1^0-m(X_j^1))^2\,|X_1^0\right]\right]\\
&=\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2\mathbb{E}_0\left[(Y_1^0-m(X_j^1))^2\,|X_1^0\right]\right]\\
&=\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2\left[Var_0(Y_1^0-m(X_j^1)\,|X_1^0)+\left(\mathbb{E}_0\left[Y_1^0-m(X_j^1)\,|X_1^0\right]\right)^2\right]\right]\\
&=\mathbb{E}_0\left[K_h(X_j^1-X_1^0)^2\left[Var_0(Y_1^0\,|X_1^0)+\left(m(X_1^0)-m(X_j^1)\right)^2\right]\right]\\
&=\int K_h(X_j^1-y)^2\left(\sigma_0^2(y)+(m(y)-m(X_j^1))^2\right)f^0(y)\,dy\\
&=\left[(K_h)^2*p^0_{X_j^1}\right](X_j^1),
\end{aligned}\tag{18}$$

where $p_x^0(y)=\left(\sigma_0^2(y)+(m(y)-m(x))^2\right)f^0(y)$.

Collecting terms (16) and (18) and plugging them into (17), and then plugging (16) and (17) into (15), the first part of Theorem 2 holds.

For proving the second part, the aim is to compute a closed-form expression for the bootstrap version of the MASE:

$$\begin{aligned}
MASE^*_{\tilde{m}_h^{NW},X^1}(h)&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[\mathbb{E}_0^*\left[\left(\tilde{m}_h^{NW*}(X_j^1)-\hat{m}_{g_X}^{NW}(X_j^1)\right)^2\right]\right]=\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0^*\left[\frac{1}{n_0\hat{f}^0_{g_X}(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^{0*})\right.\right.\\
&\left.\cdot(Y_i^{0*}-\hat{m}_{g_X}^{NW}(X_j^1))\right]+\left(\mathbb{E}_0^*\left[\frac{1}{n_0\hat{f}^0_{g_X}(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^{0*})\right.\right.\\
&\left.\left.\left.\cdot(Y_i^{0*}-\hat{m}_{g_X}^{NW}(X_j^1))\right]\right)^2\right]\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0^*[A_1^{0*}]+\left(\mathbb{E}_0^*\left[A_1^{0*}\right]\right)^2\right]\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}Var_0^*[A_1^{0*}]+\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0^*\left[A_1^{0*}\right]\right)^2.
\end{aligned}\tag{19}$$

Focusing now on the second term of (19):

$$
\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0^*\left[A_1^{0*}\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}\mathbb{E}_0^*\left[K_h(X_j^1-X_i^{0*})(Y_i^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\mathbb{E}_0^*\left[\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\,|X_1^{0*}\right]\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^{0*}(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})\left(\mathbb{E}_0^*\left[Y_1^{0*}\,|X_1^{0*}\right]-\hat{m}_{gX}^{NW}(X_j^1)\right)\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})\left(\hat{m}_{gX}^{NW}(X_1^{0*})-\hat{m}_{gX}^{NW}(X_j^1)\right)\right]\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\int K_h(X_j^1-y)\left(\hat{m}_{gX}^{NW}(y)-\hat{m}_{gX}^{NW}(X_j^1)\right)\hat{f}_{gX}^0(y)\,dy\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\int K_h(X_j^1-y)\hat{q}_{X_j^1,g}^0(y)\,dy\right)^2\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\left[K_h*\hat{q}_{X_j^1,g}^0\right](X_j^1)\right)^2, \qquad (20)
\end{aligned}
$$

where $\hat{q}_x^0(z)=(\hat{m}_{gX}^{NW}(z)-\hat{m}_{gX}^{NW}(x))\hat{f}_{gX}^0(z)$. On the other hand,

$$
\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1}Var_0^*\left[A_1^{0*}\right]\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}Var_0^*\left[\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^{0*})(Y_i^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\\
&=\frac{1}{n_1}\sum_{j=1}^{n_1}\left[\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)^2}Var_0^*\left[K_h(X_j^1-X_1^{0*})(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\right]\\
&=\frac{1}{n_0n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}_{gX}^0(X_j^1)^2}\left[\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))^2\right]-\right.\\
&\left.\left(\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2\right]. \qquad (21)
\end{aligned}
$$

The first term of (21) turns out to be:

$$
\begin{aligned}
&\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))^2\right] \\
&=\mathbb{E}_0^*\left[\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))^2\mid X_1^{0*}\right]\right] \\
&=\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2\mathbb{E}_0^*\left[(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))^2\mid X_1^{0*}\right]\right] \\
&=\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2\left[Var_0^*(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1)\mid X_1^{0*})\right.\right. \\
&\left.\left.+\left(\mathbb{E}_0^*\left[Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1)\mid X_1^{0*}\right]\right)^2\right]\right] \\
&=\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2\left[Var_0^*(Y_1^{0*}\mid X_1^{0*})+\left(\hat{m}_{gX}^{NW}(X_1^{0*})-\hat{m}_{gX}^{NW}(X_j^1)\right)^2\right]\right] \\
&=\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2\left[\sigma_0^{*2}(X_1^{0*})+\left(\hat{m}_{gX}^{NW}(X_1^{0*})-\hat{m}_{gX}^{NW}(X_j^1)\right)^2\right]\right] \\
&=\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})^2\left[\hat{\sigma}_{0,g}^2(X_1^{0*})+g_Y^2\mu_2(K)+\left(\hat{m}_{gX}^{NW}(X_1^{0*})-\hat{m}_{gX}^{NW}(X_j^1)\right)^2\right]\right] \\
&=\int K_h(X_j^1-y)^2\left(\hat{\sigma}_{0,g}^2(y)+g_Y^2\mu_2(K)+(\hat{m}_{gX}^{NW}(y)-\hat{m}_{gX}^{NW}(X_j^1))^2\right)\hat{f}_{gX}^0(y)\,dy \\
&=g_Y^2\mu_2(K)\int K_h(X_j^1-y)^2\hat{f}_{gX}^0(y)\,dy \\
&+\int K_h(X_j^1-y)^2\left(\hat{\sigma}_{0,g}^2(y)+(\hat{m}_{gX}^{NW}(y)-\hat{m}_{gX}^{NW}(X_j^1))^2\right)\hat{f}_{gX}^0(y)\,dy \\
&=\frac{g_Y^2\mu_2(K)}{n_0}\sum_{i=1}^{n_0}\int K_h(X_j^1-y)^2K_{gX}(y-X_i^0)\,dy \\
&+\int K_h(X_j^1-y)^2\hat{p}_{X_j^1,g}^0(y)\,dy \\
&=\frac{g_Y^2\mu_2(K)}{n_0}\sum_{i=1}^{n_0}\left[(Kh)^2*K_{gX}\right](X_j^1-X_i^0)+\left[K_h*\hat{p}_{X_j^1,g}^0\right](X_j^1),
\end{aligned}
\tag{22}
$$

where $\hat{p}_{x,g}^0(y)=\left(\hat{\sigma}_{0,g}^2(y)+(\hat{m}_{gX}^{NW}(y)-\hat{m}_{gX}^{NW}(x))^2\right)\hat{f}_{gX}^0(y)$.

Collecting terms (20) and (22) and plugging them into (21), and then plugging (20) and (21) into (19), Theorem 2 holds.

Corollary 1 *If K is a Gaussian kernel, then expression (14) can be rewritten as follows:*

$$
\begin{aligned}
&MASE_{\tilde{m}_h,X^1}^*(h)=\frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}_g^0(X_j^1)^2}\left[\frac{n_0-1}{n_0^3}\cdot\left[\sum_{i=1}^{n_0}K_h*K_{gX}\left(X_j^1-X_i^0\right)\cdot\left(Y_i^0-\hat{m}_{gX}\left(X_j^1\right)\right)\right]^2+\right. \\
&\left.\frac{1}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2*K_{gX}\right](X_j^1-X_i^0)\cdot\left[Y_i^0-\hat{m}_{gX}\left(X_j^1\right)\right]^2+\frac{g^2\mu_2(K)}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2*K_g\right](X_j^1-X_i^0)\right].
\end{aligned}
\tag{23}
$$

Proof Carrying on with computations in expression (14), we have:

$$
\begin{aligned}
\hat{q}_{x,gX}^0(z)&=(\hat{m}_{gX}(z)-\hat{m}_{gX}(x))\,\hat{f}_{gX}^0(z)=\frac{1}{n_0}\sum_{i=1}^{n_0}\left(K_{gX}(z-X_i^0)Y_i^0-\hat{m}_{gX}(x)\sum_{i=1}^{n_0}K_{gX}\left(z-X_i^0\right)\right) \\
&=\frac{1}{n_0}\sum_{i=1}^{n_0}K_{gX}\left(z-X_i^0\right)\left(Y_i^0-\hat{m}_{gX}(x)\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left[K_h * \hat{q}^0_{X_j^1,g_X}\right]^2(X_j^1) &= K_h * \hat{q}^0_{X_j^1,g_X}(X_j^1)^2 \\
&= \left[K_h * \frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}\left(\cdot - X_i^0\right)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)\right](X_j^1)^2 \\
&= \left[\frac{1}{n_0}\sum_{i=1}^{n_0}\left[K_h * K_{g_X}\left(\cdot - X_i^0\right)\right](X_j^1)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)\right]^2 \\
&= \left[\frac{1}{n_0}\sum_{i=1}^{n_0} K_h * K_{g_X}\left(X_j^1 - X_i^0\right)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)\right]^2. \qquad (24)
\end{aligned}
$$

On the other hand, using that $\hat{\sigma}^2_{0,g_X}(z) := \hat{m}_{2,g_X}(z) - \hat{m}_{g_X}(z)^2$, where $\hat{m}_{k,g_X}(z)$ is defined as $\hat{m}_{k,g_X}(z) = \frac{\sum_{i=1}^{n_0} K_{g_X}(z-X_i^0)(Y_i^0)^k}{\sum_{i=1}^{n_0} K_{g_X}(z-X_i^0)}$, then:

$$
\begin{aligned}
\hat{p}^0_{x,g_X}(z) &= \left[\hat{\sigma}^2_{0,g_X}(z) + \left(\hat{m}_{g_X}(z) - \hat{m}_{g_X}(x)\right)^2\right]\hat{f}^0_{g_X}(z) \\
&= \left[\hat{m}_{2,g_X}(z) - 2\hat{m}_{g_X}(z)\hat{m}_{g_X}(x) + \hat{m}_{g_X}(x)^2\right]\hat{f}_{g_X}(z) = \frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(z-X_i^0)(Y_i^0)^2 \\
&- 2\hat{m}_g(x)\frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(z-X_i^0)Y_i^0 + \hat{m}_{g_X}(x)^2\frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(y-X_i^0) \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0} K_{g_X}(z-X_i^0)\left[\left(Y_i^0\right)^2 - 2\hat{m}_{g_X}(x)Y_i^0 + \hat{m}_{g_X}(x)^2\right].
\end{aligned}
$$

Therefore, straightforward calculations lead to:

$$
\begin{aligned}
(K_h)^2 * \hat{p}^0_{X_j^1,g_X}(X_j^1) &= \frac{1}{n_0}\sum_{i=1}^{n_0}(K_h)^2 * K_{g_X}(X_j^1 - X_i^0)\cdot\left[\left(Y_i^0\right)^2 - 2\hat{m}_{g_X}(X_j^1)Y_i^0 + \hat{m}_{g_X}(X_j^1)^2\right] \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}(K_h)^2 * K_{g_X}(X_j^1 - X_i^0)\cdot\left(Y_i^0 - \hat{m}_{g_X}(X_j^1)\right)^2. \qquad (25)
\end{aligned}
$$

Finally, collecting expressions (24) and (25) and plugging them in expression (14), Corollary 1 is concluded.

1.2 Asymptotic theory

1.2.1 Asymptotic expression for the criterion functions

Lemma 1 *Under regularity conditions (B1)-(B4) from the paper, the function $MISE^a$ can be expressed as*

$$
MISE^a(h) = \frac{R(K)}{n_0 h}\int \gamma(x)dx + \frac{h^4\mu_2(K)^2}{4}\int \beta(x)\,dx + \mathscr{O}(h^6) + \mathscr{O}\left(\frac{h}{n_0}\right). \qquad (26)
$$

where $\beta(x) = \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)$ *and* $\gamma(x) = \frac{\sigma^2(x)f^1(x)}{f^0(x)}$.
The asymptotic version of expression (26) is given by:

$$
AMISE^a(h) = \frac{R(K)}{n_0 h}\int \gamma(x)dx + \frac{h^4}{4}\mu_2(K)^2\int \beta(x)\,dx. \qquad (27)
$$

Proof Using a Taylor expansion and a change of variable, we obtain:

$$\mathbb{E}\left[\int\left(\tilde{m}_h^{NW}(x)-m(x)\right)^2 dF_1(x)\right] \tag{28}$$
$$=\mathbb{E}\left[\int \frac{1}{n_0 f^0(x)}\sum_{i=1}^{n_0} K_h\left(x-X_i^0\right)\left(Y_i^0-m(x)\right)\right]^2 dF_1(x)$$
$$=\frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[\sum_{i=1}^{n_0} K_h\left(x-X_i^0\right)\left(Y_i^0-m(x)\right)\right]^2 dF_1(x)$$
$$=\frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[Var\left[\sum_{i=1}^{n_0} K_h\left(x-X_i^0\right)\left(Y_i^0-m(x)\right)\right]\right.$$
$$\left.+\left(\mathbb{E}\left[\sum_{i=1}^{n_0} K_h\left(x-X_i^0\right)\left(Y_i^0-m(x)\right)\right]\right)^2\right] dF_1(x)$$
$$=\frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[\sum_{i=1}^{n_0} Var\left[K_h\left(x-X_i^0\right)\left(Y_i^0-m(x)\right)\right]\right.$$
$$\left.+\left(\sum_{i=1}^{n_0}\mathbb{E}\left[K_h\left(x-X_i^0\right)\left(Y_i^0-m(x)\right)\right]\right)^2\right] dF_1(x)$$
$$=\frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[n_0 Var\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right)\right]\right.$$
$$\left.+n_0^2\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right)\right]\right)^2\right] dF_1(x)$$
$$=\frac{1}{n_0}\int \frac{1}{f^0(x)^2} Var\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right)\right] dF_1(x)$$
$$+\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right)\right]\right)^2 dF_1(x)$$
$$=\frac{1}{n_0}\int \frac{1}{f^0(x)^2}\left[\mathbb{E}\left[K_h\left(x-X_1^0\right)^2\left(Y_1^0-m(x)\right)^2\right]\right.$$
$$\left.-\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right)\right]\right)^2\right] dF_1(x)$$
$$+\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right)\right]\right)^2 dF_1(x)$$
$$=\frac{1}{n_0}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x-X_1^0\right)^2\left(Y_1^0-m(x)\right)^2\right] dF_1(x)$$
$$+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right)\right]\right)^2 dF_1(x)$$
$$=\frac{1}{n_0}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[\mathbb{E}\left[K_h\left(x-X_1^0\right)^2\left(Y_1^0-m(x)\right)^2 | X_1^0\right]\right] dF_1(x)$$
$$+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(Y_1^0-m(x)\right) | X_1^0\right]\right]\right)^2 dF_1(x)$$
$$=\frac{1}{n_0}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x-X_1^0\right)^2\mathbb{E}\left[\left(Y_1^0-m(x)\right)^2 | X_1^0\right]\right] dF_1(x)$$
$$+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(\mathbb{E}\left[Y_1^0 | X_1^0\right]-m(x)\right)\right]\right)^2 dF_1(x)$$
$$=\frac{1}{n_0}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x-X_1^0\right)^2\left(Var\left(Y_1^0-m(x) | X_1^0\right)\right.\right.$$
$$\left.\left.+\left(\mathbb{E}\left[Y_1^0-m(x) | X_1^0\right]\right)^2\right)\right] dF_1(x)$$
$$+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(m(X_1^0)-m(x)\right)\right]\right)^2 dF_1(x)$$

$$
\begin{aligned}
&= \frac{1}{n_0}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x-X_1^0\right)^2\left(Var\left(Y_1^0\mid X_1^0\right)+\left(\mathbb{E}\left[Y_1^0\mid X_1^0\right]-m(x)\right)^2\right)\right]dF_1(x)\\
&+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(m(X_1^0)-m(x)\right)\right]\right)^2 dF_1(x)\\
&= \frac{1}{n_0}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x-X_1^0\right)^2\left(\sigma^2(X_1^0)+\left(m(X_1^0)-m(x)\right)^2\right)\right]dF_1(x)\\
&+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x-X_1^0\right)\left(m(X_1^0)-m(x)\right)\right]\right)^2 dF_1(x)\\
&= \frac{1}{n_0 h^2}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[K\left(\frac{x-X_1^0}{h}\right)^2\left(\sigma^2(X_1^0)+\left(m(X_1^0)-m(x)\right)^2\right)\right]dF_1(x)\\
&+\frac{n_0-1}{n_0 h^2}\int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K\left(\frac{x-X_1^0}{h}\right)\left(m(X_1^0)-m(x)\right)\right]\right)^2 dF_1(x)\\
&= \frac{1}{n_0 h^2}\int \frac{1}{f^0(x)^2}\int K\left(\frac{y-x}{h}\right)^2\left(\sigma^2(y)+(m(y)-m(x))^2\right)f^0(y)\,dy\,dF_1(x)\\
&+\frac{n_0-1}{n_0 h^2}\int \frac{1}{f^0(x)^2}\left(\int K\left(\frac{y-x}{h}\right)(m(y)-m(x))\,f^0(y)\,dy\right)^2 dF_1(x)\\
&= \frac{1}{n_0 h}\int \frac{1}{f^0(x)^2}\int K(u)^2\left(\sigma^2(x+hu)+(m(x+hu)-m(x))^2\right)\\
&f^0(x+hu)\,du\,dF_1(x)+\frac{n_0-1}{n_0}\int \frac{1}{f^0(x)^2}\\
&\left(\int K(u)\,(m(x+hu)-m(x))\,f^0(x+hu)\,du\right)^2 dF_1(x).
\end{aligned}
\tag{29}
$$

Let us denote $\psi(y)=(m(y)-m(x))\,f^0(y)$. Then, $\psi(x)=0$; $\psi'(x)=m'(y)f(y)+(m(y)-m(x))\left(f^0\right)'(y)$; $\psi''(y)=m''(y)f^0(y)+2m'(y)\left(f^0\right)'(y)+(m(y)-m(x))\left(f^0\right)''(y)$. Using Taylor expansion:

$$
\begin{aligned}
\int K(u)\psi(x+hu)\,du &= \int K(u)\psi(x)\,du + h\int u K(u)\,\psi'(x)\,du\\
&+\frac{h^2}{2}\int u^2 K(u)\psi''(x)\,du+\frac{h^3}{3!}\int u^3 K(u)\psi'''(x)\,du+\mathscr{O}(h^4)\\
&=\frac{h^2}{2}\psi''(x)\mu_2(K)+\mathscr{O}(h^4),
\end{aligned}
$$

which leads to:

$$
\begin{aligned}
&\frac{n_0-1}{n_0}\int\frac{1}{f^0(x)^2}\left(\int K(u)\,\psi(x+hu)\,du\right)^2 dF_1(x)\\
&=\frac{n_0-1}{n_0}\int\frac{1}{f^0(x)^2}\frac{h^4}{4}\psi''(x)^2\mu_2(K)^2dF_1(x)+\mathscr{O}(h^6)\\
&=\frac{(n_0-1)h^4}{n_0}\frac{1}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.\\
&\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)dx+\mathscr{O}(h^6).
\end{aligned}\tag{30}
$$

On the other hand, denote $\phi(y)=\left[\sigma^2(y)+(m(y)-m(x))^2\right]f^0(y)$. Then, $\phi(x)=\sigma^2(x)f^0(x)$ and using Taylor expansion:

$$
\begin{aligned}
&\frac{1}{n_0 h}\int\frac{1}{f^0(x)^2}\int K(u)^2\phi(x+hu)\,du\,dF_1(x)\\
&=\frac{R(K)}{n_0 h}\int\frac{1}{f^0(x)^2}\sigma^2(x)f^0(x)f^1(x)\,dx+\mathscr{O}\left(\frac{h}{n_0}\right)\\
&=\frac{R(K)}{n_0 h}\int\frac{\sigma^2(x)\,f^1(x)}{f^0(x)}dx+\mathscr{O}\left(\frac{h}{n_0}\right).
\end{aligned}\tag{31}
$$

Assembling terms (30) and (31), and inserting them in expression (29), we have that Lemma 1 holds.

Lemma 2 (Technical result) *Consider a sequence of bandwidths $cn_0^{-1/5}, c>0$ and the function $MISE^a(h)$, given in (26) it turns out:*

$$
\begin{aligned}
MISE^a\left(cn_0^{-1/5}\right)&=R(K)n_0^{-4/5}c^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx\\
&+\frac{c^4\,n_0^{-4/5}}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.\\
&\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx+\mathscr{O}(n_0^{-6/5}),
\end{aligned}
$$

so that

$$
\begin{aligned}
\lim_{n_0\to\infty}n_0^{4/5}MISE^a\left(cn_0^{-1/5}\right)&=R(K)c^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx\\
&+\frac{c^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.\\
&\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx.
\end{aligned}\tag{32}
$$

Then,

$$
\lim_{n_0\to\infty}n_0^{1/5}\cdot h_{MISE^a}=c_0.\tag{33}
$$

Proof Given that h_{MISE^a} is the value which minimizes the function $MISE^a$, then

$$n_0^{4/5} MISE^a\left(c_0 \cdot n_0^{-1/5}\right) \geq n_0^{4/5} MISE^a\left(h_{MISE^a}\right), \forall n_0 \in \mathbb{N}, \tag{34}$$

which implies that

$$\begin{aligned}
&R(K) c_0^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \\
&+ \frac{c_0^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \\
&\geq \lim_{n_0 \to \infty} \sup n_0^{4/5} MISE^a(h_{MISE^a})
\end{aligned} \tag{35}$$

Bringing together expressions (26) and (35), we can conclude that:

$$\lim_{n_0 \to \infty} \sup n_0^{1/5} h_{MISE^a} < \infty.$$

Indeed,

$$\begin{aligned}
&\lim_{n_0 \to \infty} \sup n_0^{4/5} MISE^a(h_{MISE^a}) \\
&\geq \lim_{n_0 \to \infty} \sup \left[\frac{\left(n_0^{1/5} h_{MISE^a}\right)^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right.\right. \\
&\left.\left. + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \right],
\end{aligned}$$

and together with expression (35) lead to conclude that

$$\begin{aligned}
k \geq \lim_{n_0 \to \infty} \sup &\left[\frac{\left(n_0^{1/5} h_{MISE^a}\right)^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right.\right. \\
&\left.\left. + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \right],
\end{aligned}$$

being k a real positive number, which implies that:

$$\lim_{n_0 \longrightarrow \infty} \sup \left(n_0^{1/5} h_{MISE^a}\right)^4 \leq k \Leftrightarrow \lim_{n_0 \longrightarrow \infty} \sup \left(n_0^{1/5} h_{MISE^a}\right) \leq k.$$

By means of similar calculations, we can prove that $\lim_{n_0 \to \infty} \inf n_0^{1/5} h_{MISE^a} > 0$. As a matter of fact,

$$\lim_{n_0 \to \infty} \sup n_0^{4/5} MISE^a(h_{MISE^a}) \geq \lim_{n_0 \to \infty} \sup \left[\frac{R(K)}{n_0^{1/5} h_{MISE^a}} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \right],$$

which implies, together with expression (35), that

$$\lim_{n_0\to\infty}\sup a_{n_0} = \lim_{n_0\to\infty}\sup\left[\frac{1}{n_0^{1/5}h_{MISE^a}}\right] = k, \forall k\in\mathbb{R}^+. \tag{36}$$

Given that a_{n_0} is a positive sequence and using expression (36), we can conclude that $\lim\limits_{n_0\to\infty}\inf n_0^{1/5}h_{MISE^a} > 0$.

From both limit conditions, we can state that there exist two numbers $L, U \in \mathbb{R}^+, L < U$, satisfying:

$$L \leq n_0^{1/5}h_{MISE^a} \leq U, \text{ for almost all } n_0\in\mathbb{N}. \tag{37}$$

Using expressions (26) and (37), we obtain:

$$\begin{aligned}
&n_0^{4/5}MISE^a(h_{MISE^a})\\
&= R(K)\left(n_0^{1/5}h_{MISE^a}\right)^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{\left(n_0^{1/5}h_{MISE^a}\right)^4}{4}\mu_2(K)^2\\
&\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx\\
&+\mathscr{O}\left(\left(n_0^{1/5}h_{MISE^a}\right)^4 h^2_{MISE^a}\right) + \mathscr{O}\left(n_0^{-1/5}h_{MISE^a}\right)\\
&= R(K)\left(n_0^{1/5}h_{MISE^a}\right)^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{\left(n_0^{1/5}h_{MISE^a}\right)^4}{4}\mu_2(K)^2\\
&\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx\\
&+\mathscr{O}\left(h^2_{MISE^a}\right) + \mathscr{O}\left(n_0^{-2/5}\right).
\end{aligned} \tag{38}$$

Consider a subsequence of

$$\left\{n_0^{1/5}h_{MISE^a}\right\}_{n_0\in\mathbb{N}}, \tag{39}$$

which converges to a real number l (which is positive as a consequence of expression (37)). Furthermore, expression (38) assures that the corresponding subsequence of

$$\left\{n_0^{4/5}MISE^a\left(h_{MISE^a}\right)\right\}_{n_0\in\mathbb{N}},$$

converges to

$$\begin{aligned}
&R(K)l^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{l^4}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.\\
&\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx + \mathscr{O}\left(h^2_{MISE^a}\right) + \mathscr{O}\left(n_0^{-2/5}\right).
\end{aligned} \tag{40}$$

Moreover, expression (35) guarantees that

$$R(K)c_0^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx \tag{41}$$
$$+\frac{c_0^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx$$
$$\geq R(K)l^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx+\frac{l^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.$$
$$\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx. \tag{42}$$

As a consequence of c_0 being a strict absolute minimum (in $c>0$) of the second addend of expression (32), the previous inequality can only be satisfied when $l=c_0$. This reasoning establishes that c_0 is the unique adherent point of the sequence given in (39). Accordingly, considering expression (37), the proof is concluded.

Up to now, we have obtained an asymptotic result concerning a first order bandwidth which minimizes $MISE^a$. From now on, we investigate the second order term further.

Lemma 3 (Technical result) *Consider the sequence of functions defined below and L,U satisfying expression (37).*

$$\Gamma_{n_0}(z) := MISE^a(zn_0^{-1/5}), z>0, n_0\in\mathbb{N},$$

and

$$\gamma_{n_0} := \arg\min_{L\leq z\leq U}\Gamma_{n_0}(z).$$

It turns out that the relation among these two functions and expression (26) is as follows:

$$h_{MISE^a} = \gamma_{n_0}n_0^{-1/5}, \textit{ for almost all } n_0\in\mathbb{N}. \tag{43}$$

Proof Expression (26) guarantees the existence of L,U verifying expression (37). In addition, the continuity of the kernel function K assures the continuity of $MISE^a(h)$. Therefore, it exists a point, namely γ_{n_0}, within the interval $[L,U]$, in which the function Γ_{n_0} attains its minimum, whether it is unique or not.

Nevertheless, given that inequalities in expression (37) are fulfilled, any minimizer of Γ_{n_0} is attained within the interval $[L,U]$. Indeed, it it were not the case, there would exist a minimizer z_0 outside $[L,U]$ verifying

$$\Gamma_{n_0}(z_0)\leq\Gamma_{n_0}(z), \forall z>0, \tag{44}$$

and therefore,

$$MISE^a(z_0 n_0^{-1/5})\leq MISE^a(h), \forall h>0, \tag{45}$$

so that $z_0 n_0^{-1/5}$ would be a minimizer of $MISE^a$, but it would not satisfy inequalities given in (37) for that particular $n_0\in\mathbb{N}$. Finally, the equivalence between expressions (44) and (45) provides (43), and the proof is concluded.

In particular, the function Γ_{n_0} satisfies

$$\begin{aligned}\Gamma_{n_0}(z) &= R(K)n_0^{-4/5}z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx \\ &+ \frac{z^4\, n_0^{-4/5}}{4}\mu_2(K)^2\int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right. \\ &\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx + \mathscr{O}(z^6\, n_0^{-6/5}) + \mathscr{O}\left(z n_0^{-6/5}\right), z>0\end{aligned}$$

If we define the function Λ_{n_0} such that

$$\Lambda_{n_0}(z) := n_0^{4/5}\Gamma_{n_0}(z), z>0,$$

then,

$$\begin{aligned}\Lambda_{n_0}(z) &= R(K)z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{z^4}{4}\mu_2(K)^2\int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right. \\ &\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx + \mathscr{O}\left(z n_0^{-2/5}\right), z>0.\end{aligned}$$

If we restrict to those z within the interval $[L,U]$, then

$$\begin{aligned}\Lambda_{n_0}(z) &= R(K)z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{z^4}{4}\mu_2(K)^2\int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right. \\ &\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx + \mathscr{O}\left(n_0^{-2/5}\right),\end{aligned}$$

uniformly in $z \in [L,U]$.

Define now the function

$$\begin{aligned}T(z) &:= R(K)z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{z^4}{4}\mu_2(K)^2\int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right. \\ &\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx, \end{aligned} \tag{46}$$

then we have,

$$\sup_{L\le z\le U}\left|\Lambda_{n_0}(z) - T(z)\right| = \mathscr{O}\left(n_0^{-2/5}\right). \tag{47}$$

Denote, for the sake of brevity,

$$\begin{aligned}a &:= \frac{1}{4}\mu_2(K)^2\int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)\,dx \\ &= \frac{1}{4}\mu_2(K)^2\int \left[m''(x)f^0(x) + 2\,m'(x)\left(f^0\right)'(x)\right]^2 f^1(x)\left(f^0(x)\right)^{-1}\,dx, \text{ and} \\ b &:= R(K)\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx.\end{aligned}$$

Then, $T(z) = bz^{-1} + z^4 a$. Due to the fact that the kernel K is a nonnegative function and $\int K(u)\,du = 1$, then $R(K) > 0$ and $\mu_2(K) > 0$. Moreover, assume that K is bounded and $\int u^2 K(u)\,du < \infty$, $R(K)$ and $\mu_2(K)$ are finite numbers. In addition, assume that the density function, f^0, is at least one time differentiable and its first derivative is bounded and continuous in point x. As for the regression function m, assume it is two times differentiable and its first and second derivatives are bounded and continuous in point x. Then, we can conclude that $a > 0$ and $b > 0$.

Lemma 4 (Technical result) *Consider T the function defined in (46), it turns out that T attains a strict relative minimum in*

$$z_0 := \left(\frac{b}{4a}\right)^{1/5}. \tag{48}$$

Furthermore, fix $\delta > 0$ a real number,

$$T(z_0+\delta) - T(z_0) > 6az_0^2\delta^2. \tag{49}$$

If $\delta < z_0$, then we have

$$T(z_0-\delta) - T(z_0) > 6az_0^2\delta^2 - 4az_0\delta^3. \tag{50}$$

Proof Function T is differentiable in $(0,+\infty)$ and its derivative is given by:

$$T'(z) = 4az^3 + bz^{-2}, \forall z > 0.$$

It is straightforward to see that z_0 is the unique point in $(0,+\infty)$ where T' is zero. Moreover, T' takes negative values within the interval $(0,z_0)$ and positive values within the interval $(z_0,+\infty)$. Then, T is strictly decreasing in $(0,z_0)$ and strictly increasing in $(z_0,+\infty)$, leading to (48).

Now, to prove expression (49), fix $\delta > 0$ and consider

$$\begin{aligned}
T(z_0+\delta) - T(z_0) &= b(z_0+\delta)^{-1} + (z_0+\delta)^4 a - bz_0^{-1} - az_0^4 \\
&= b(z_0+\delta)^{-1} + a(z_0^4 + 4\delta z_0^3 + 6\delta^2 z_0^2 + 4\delta^3 z_0 + \delta^4) - bz_0^{-1} - az_0^4 \\
&= b\left((z_0+\delta)^{-1} - z_0^{-1}\right) + a(4\delta z_0^3 + 6\delta^2 z_0^2 + 4\delta^3 z_0 + \delta^4) \\
&= a(4\delta z_0^3 + 6\delta^2 z_0^2 + 4\delta^3 z_0 + \delta^4) - b\delta\left((z_0+\delta)z_0\right)^{-1} \\
&= \delta(4az_0^3 - b(z_0^2+\delta z_0)^{-1}) + 6a\delta^2 z_0^2 + 4a\delta^3 z_0 + a\delta^4 \\
&> \delta(4az_0^3 - bz_0^{-2}) + 6a\delta^2 z_0^2 + 4a\delta^3 z_0 + a\delta^4 \\
&> 6a\delta^2 z_0^2.
\end{aligned}$$

Analogously, in order to prove expression (50), consider $\delta < z_0$ and,

$$\begin{aligned}
T(z_0-\delta) - T(z_0) &= b(z_0-\delta)^{-1} + (z_0-\delta)^4 a - bz_0^{-1} - az_0^4 \\
&= b(z_0-\delta)^{-1} + a(z_0^4 - 4\delta z_0^3 + 6\delta^2 z_0^2 - 4\delta^3 z_0 + \delta^4) - bz_0^{-1} - az_0^4 \\
&= b\left((z_0-\delta)^{-1} - z_0^{-1}\right) + a(-4\delta z_0^3 + 6\delta^2 z_0^2 - 4\delta^3 z_0 + \delta^4) \\
&= a(-4\delta z_0^3 + 6\delta^2 z_0^2 - 4\delta^3 z_0 + \delta^4) - b\delta\left((z_0-\delta)z_0\right)^{-1} \\
&= \delta(-4az_0^3 + b(z_0^2 - \delta z_0)^{-1}) + 6a\delta^2 z_0^2 - 4a\delta^3 z_0 + a\delta^4 \\
&> \delta(-4az_0^3 + bz_0^{-2}) + 6a\delta^2 z_0^2 - 4a\delta^3 z_0 + a\delta^4 \\
&> 6a\delta^2 z_0^2 - 4a\delta^3 z_0.
\end{aligned}$$

These two last inequalities prove expressions (49) and (50).

Remark 1 The minimizer of T, z_0 is precisely c_0. Indeed, T happens to be the dominant part of Λ_{n_0}, whose minimization is (under certain conditions) equivalent to the minimization of $MISE^a$.

Theorem 3 *Under regularity conditions (B1)-(B3) from the paper, the bandwidth selector which minimizes the function $MISE^a$ has the following asymptotic expression:*

$$h_{MISE^a} = \left(\frac{R(K)\int \sigma^2(x) f^1(x) \left(f^0(x)\right)^{-1} dx}{\mu_2(K)^2 \int \beta(x) f^1(x)\, dx} \right)^{1/5} n_0^{-1/5} + \mathscr{O}\left(n_0^{-2/5}\right). \tag{51}$$

Proof (Proof of Theorem 3) First, assume that $z_0 \in (L,U)$. Indeed, if $z_0 \notin (L,U)$, we would consider a wider interval which would fulfill that condition. Furthermore, expression (47) guarantees the existence of some constant $C > 0$ such that

$$\sup_{L \le z \le U} \left|\Lambda_{n_0}(z) - T(z)\right| \le C n_0^{-2/5}, \text{ for almost all } n_0 \in \mathbb{N}. \tag{52}$$

Consider the following sequence of real positive numbers,

$$\delta_{n_0} := \left(C(2a)^{-1}\right)^{1/2} z_0^{-1} n_0^{-1/5}, n_0 \in \mathbb{N}. \tag{53}$$

Using expression (49), we have

$$T(z_0 + \delta_{n_0}) - T(z_0) > 6 a z_0^2 \delta_{n_0}^2 = 3 C n_0^{-2/5}, \forall n_0 \in \mathbb{N}. \tag{54}$$

On the other hand, given that $\delta_{n_0} < z_0$, for almost all natural n_0,

$$\begin{aligned} T(z_0 - \delta_{n_0}) - T(z_0) &> 6 a z_0^2 \delta_{n_0}^2 - 4 a z_0 \delta_{n_0}^3 \\ &= 3 C n_0^{-2/5} - 2 C^{3/2} z_0^{-2} (2a)^{-1/2} n^{-3/5}. \end{aligned} \tag{55}$$

As a consequence,

$$T(z_0 - \delta_{n_0}) - T(z_0) > \qquad 2 C n_0^{-2/5}, \text{ for almost all } n_0 \in \mathbb{N}. \tag{56}$$

Providing that z_0 is an interior point of the interval $[L,U]$ and $\{\delta_{n_0}\}$ tends to zero as n_0 tends to ∞, the sets given below are well defined for almost all $n_0 \in \mathbb{N}$

$$A_{n_0} := \left[L, z_0 - \delta_{n_0}\right] \cup \left[z_0 + \delta_{n_0}, U\right].$$

Consider now $z \in [L,U]$ and expressions (48), (52), (53), (54). For almost $n_0 \in \mathbb{N}$, we have

$$\begin{aligned} z \in \left[L, z_0 - \delta_{n_0}\right] &\Rightarrow z \le z_0 - \delta_{n_0} \Rightarrow \Lambda_{n_0}(z) + C n_0^{-2/5} \ge T(z) \ge T(z_0 - \delta_{n_0}) \\ &> T(z_0) + 2 C n_0^{-2/5} \ge \Lambda_{n_0}(z_0) + C n_0^{-2/5} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0). \\ z \in \left[z_0 + \delta_{n_0}, U\right] &\Rightarrow z \ge z_0 + \delta_{n_0} \Rightarrow \Lambda_{n_0}(z) + C n_0^{-2/5} \ge T(z) \ge T(z_0 + \delta_{n_0}) \\ &> T(z_0) + 3 C n_0^{-2/5} \ge \Lambda_{n_0}(z_0) + 2 C n_0^{-2/5} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0). \end{aligned}$$

Consequently,

$$z \in A_{n_0} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0).$$

Therefore, for all $n_0 \in \mathbb{N}$ except for a finite number of them (at most), any minimizer γ_{n_0} of Λ_{n_0} (and Γ_{n_0}) verifies $\gamma_{n_0} \in [L,U] \setminus A_{n_0}$. In other words, $\left|\gamma_{n_0} - z_0\right| \leq \delta_{n_0}$.

According to expression (43), for almost all $n_0 \in \mathbb{N}$, every minimizer h_{MISE^a} of $MISE^a$ satisfies:

$$\left|h_{MISE^a} - c_0\, n_0^{-1/5}\right| \leq \delta_{n_0} n_0^{-1/5} = \left(C\,(2\,a)^{-1}\right)^{1/2} z_0^{-1}\, n_0^{-2/5},$$

for almost all $n_0 \in \mathbb{N}$, which implies expression (51).

Proposition 2 *Suppose conditions (C1), (C2) and (C3) from the paper are fulfilled. Consider a sequence of bandwidths h_{n_0} such that $\sum_{n_0} h_{n_0}^{\lambda} < \infty$ for some $\lambda > 0$ and that $n_0^{\eta} h_{n_0} \longrightarrow \infty$ for some $\eta < 1 - s^{-1}$. Assume $(n_0\, h)^{-1/2} \log\left(h^{-1}\right) \longrightarrow 0$ as $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\, h \longrightarrow \infty$. Then,*

$$ISE(h) = ISE^a(h) + \mathscr{O}_P\left(h^6\right) + \mathscr{O}_P\left(\frac{h}{n_0}\log\frac{1}{h}\right) + \mathscr{O}_P\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + \mathscr{O}_P\left(\frac{\log\frac{1}{h}}{n_0^{3/2} h^{3/2}}\right), \tag{57}$$

where $ISE(h) = \int\left(\hat{m}_h(x) - m(x)\right)^2 dF^1(x)$ and $ISE^a(h) = \int\left(\tilde{m}_h(x) - m(x)\right)^2 dF^1(x)$.

Proof Under conditions (C1), (C2) and assuming that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\, h \longrightarrow \infty$, we have:

$$\sup_x \left|\hat{f}_h^0(x) - f^0(x)\right| \longrightarrow 0 \text{ almost sure as } n_0 \longrightarrow \infty,$$

and

$$\sup_x \left|\hat{f}_h^0(x) - f^0(x)\right| = \mathscr{O}_P\left(h^2 + n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right). \tag{58}$$

Additionally, thanks to results of Mack and Silverman[1], we have:

$$\sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right| \longrightarrow 0 \text{ almost sure as } n_0 \longrightarrow \infty,$$

and

$$\sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right| = \mathscr{O}_P\left(h^2 + n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right). \tag{59}$$

Consider now $\{\xi_{n_0}\}$ a sequence of random variables defined in the probability space (Ω, A, P) such that $\xi_{n_0} \geq 0$ and $\xi_{n_0} = \mathscr{O}_P\left(a_{n_0} + b_{n_0}\right)$, where $\left(a_{n_0}\right), \left(b_{n_0}\right)$ are sequences of real positive numbers. We can show that

$$\xi_{n_0} = \mathscr{O}_P\left(a_{n_0} + b_{n_0}\right) \Leftrightarrow \xi_{n_0} = \mathscr{O}_P\left(\max\{a_{n_0}, b_{n_0}\}\right) \tag{60}$$

'$\Leftarrow$' $a_{n_0} + b_{n_0} \geq \max\{a_{n_0}, b_{n_0}\} \Rightarrow \dfrac{\xi_{n_0}}{a_{n_0} + b_{n_0}} \leq \dfrac{\xi_{n_0}}{\max\{a_{n_0}, b_{n_0}\}}$.

'$\Rightarrow$' $a_{n_0} + b_{n_0} \leq 2\max\{a_{n_0}, b_{n_0}\} \Rightarrow \dfrac{2\xi_{n_0}}{a_{n_0} + b_{n_0}} \geq \dfrac{\xi_{n_0}}{\max\{a_{n_0,b_{n_0}}\}}$.

[1] Mack, Y.P. and Silverman B.W. (1982) Weak and strong uniform consistency of kernel regression estimates *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 61, 405–415.

It can be shown then that

$$MISE(h) = MISE^a(h) + \mathbb{E}\left[\int 2A_1 A_2\, dF^1(x)\right] + \mathbb{E}\left[\int A_2^2\, dF^1(x)\right],$$

where

$$A_1 = \frac{\hat{\Psi}_h(x)}{f^0(x)} - \frac{m(x)\hat{f}_h^0(x)}{f^0(x)} = \frac{1}{n_0 f^0(x)} \sum_{i=1}^{n_0} K_h(x - X_i^0)(Y_i^0 - m(x)),$$
$$A_2 = \left(\hat{m}_h^{NW}(x) - m(x)\right) \frac{\left(\hat{f}_h^0(x) - f^0(x)\right)}{f^0(x)},$$

and $MISE^a(h)$ is given in (26).

On the one hand, using expression (60), it turns out:

$$\begin{aligned}
\int_J A_2^2\, dF^1(x) &= \left|\int_J A_2^2\, dF^1(x)\right| = \left|\int_J \left(\hat{m}_h^{NW}(x) - m(x)\right) \frac{\left(\hat{f}_h^0(x) - f^0(x)\right)}{f^0(x)} f^1(x)\, dx\right| \\
&\leq \sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right|^2 \sup_J \left|\hat{f}_h^0(x) - f^0(x)\right|^2 \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(\mathscr{O}_P\left(h^2 + n_0^{-1/2} h^{-1/2} \left(\log\frac{1}{h}\right)^{1/2}\right)\right)^4 \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(\mathscr{O}_P\left(\max\left\{h^2, n_0^{-1/2} h^{-1/2} \left(\log\frac{1}{h}\right)^{1/2}\right\}\right)\right)^4 \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(\mathscr{O}_P\left(\max\left\{h^2, n_0^{-1/2} h^{-1/2} \left(\log\frac{1}{h}\right)^{1/2}\right\}^4\right)\right) \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(\mathscr{O}_P\left(\max\left\{h^8, n_0^{-2} h^{-2} \left(\log\frac{1}{h}\right)^{2}\right\}\right)\right) \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(\mathscr{O}_P\left(h^8 + n_0^{-2} h^{-2} \left(\log\frac{1}{h}\right)^{2}\right)\right) \\
&= \mathscr{O}_P\left(h^8\right) + \mathscr{O}_P\left(n_0^{-2} h^{-2} \left(\log\frac{1}{h}\right)^{2}\right).
\end{aligned} \tag{61}$$

The first addend in expression (61) is negligible in comparison to the second term in expression (26). Analogously, the second addend in expression (61) becomes insignificant compared to the first term in expression (26). In particular,

$$\begin{aligned}
\left(\int_J A_2^2\, dF^1(x)\right)^{1/2} &= \left(\mathscr{O}_P\left(h^8\right) + \mathscr{O}_P\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2} \\
&= \left(\mathscr{O}_P\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2} \\
&= \left(\mathscr{O}_P\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}\right)\right)^{1/2} \\
&= \mathscr{O}_P\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}^{1/2}\right) \\
&= \mathscr{O}_P\left(\max\left\{h^4, n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right\}\right) \\
&= \mathscr{O}_P\left(h^4 + n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right) \\
&= \mathscr{O}_P\left(h^4\right) + \mathscr{O}_P\left(n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right). \qquad (62)
\end{aligned}$$

Moreover, thanks to expression (26), we know that

$$\mathbb{E}\left[\int_J A_1^2\, dF_1(x)\right] = \mathscr{O}\left(n_0^{-1}h^{-1} + h^4\right). \qquad (63)$$

Applying Markov's inequality to expression (63), it turns out:

$$\int_J A_1^2\, dF^1(x) = \mathscr{O}_P\left(n_0^{-1}h^{-1} + h^4\right). \qquad (64)$$

Thus,

$$\begin{aligned}
\left(\int_J A_1^2\, dF^1(x)\right)^{1/2} &= \left(\mathscr{O}_P\left(n_0^{-1}h^{-1} + h^4\right)\right)^{1/2} = \mathscr{O}_P\left(\max\left\{n_0^{-1}h^{-1}, h^4\right\}^{1/2}\right) \\
&= \mathscr{O}_P\left(\max\left\{n_0^{-1/2}h^{-1/2}, h^2\right\}\right) = \mathscr{O}_P\left(n_0^{-1/2}h^{-1/2} + h^2\right) \\
&= \mathscr{O}_P\left(n_0^{-1/2}h^{-1/2}\right) + \mathscr{O}_P\left(h^2\right). \qquad (65)
\end{aligned}$$

Bringing together expressions (65) and (62) and using Cauchy-Schwarz inequality, we compute:

$$\begin{aligned}
\left|\int_J A_1 A_2\, dF^1(x)\right| &\le 2\left(\int_J A_1^2\, dF^1(x)\right)^{1/2}\left(\int_J A_2^2\, dF^1(x)\right)^{1/2} \\
&= 2\left(\mathscr{O}_p\left(h^4\right) + \mathscr{O}_P\left(n_0^{-1}h^{-1}\log\frac{1}{h}\right)\right) \\
&\cdot\left(\mathscr{O}_p\left(h^2\right) + \mathscr{O}_P\left(n_0^{-1/2}h^{-1/2}\log\frac{1}{h}\right)\right) \\
&= \mathscr{O}_P\left(h^6\right) + \mathscr{O}_P\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + \mathscr{O}_P\left(\frac{h}{n_0}\log\frac{1}{h}\right) + \mathscr{O}_P\left(\frac{\log\frac{1}{h}}{(n_0 h)^{3/2}}\right). \qquad (66)
\end{aligned}$$

The first addend in expression (66) is negligible as compared to the second term in expression (26). Furthermore, the fourth addend in expression (66) becomes insignificant in comparison to the first term in expression (26) if $\left(\dfrac{\log\frac{1}{h}}{n_0^{1/2}h^{1/2}}\right) \longrightarrow 0$ as $h \longrightarrow 0$, $n_0 \longrightarrow \infty$ and $n_0 h \longrightarrow \infty$.

It remains to be seen what happens with the second and third terms in expression (66). We begin with the second one. Given that $\left(n_0^{-1}h^{-1}+h^4\right)\cdot n_0 h = 1+h^5 n_0$ and the bandwidth h is of the form $n^{-\alpha}, \alpha > 0$, then

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then $n_0^{-1}h^{-1} \sim n_0^{-4/5}$ and $h^4 \sim n_0^{-4/5}$, which implies that $h \sim n_0^{-1/5}$, and

$$\frac{h^{7/2}}{n_0^{1/2}} \sim \frac{\left(n_0^{-1/5}\right)^{7/2}}{n_0^{1/2}} = \frac{n_0^{-7/10}}{n_0^{5/10}} = n_0^{-6/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{\frac{1}{n_0 h}} \longrightarrow 0 \Leftrightarrow n_0^{1/2}h^{9/2} \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $n_0 h^5 \longrightarrow 0$.
- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{h^4} \longrightarrow 0 \Leftrightarrow n_0^{-1/2}h^{-1/2} \longrightarrow 0 \Leftrightarrow n_0 h \longrightarrow \infty,$$

  which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\dfrac{h^{7/2}}{n_0^{1/2}} = o\left(\dfrac{1}{n_0 h} + h^4\right)$.

Finally, as for the third addend in (66),

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then

$$\frac{h}{n_0}\log\frac{1}{h} \sim n_0^{-6/5}\log n_0^{1/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h}{n_0}\log\frac{1}{h}}{\frac{1}{n_0 h}} = h^2\log\frac{1}{h} \longrightarrow 0 \Leftrightarrow h^2 \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $h \longrightarrow 0$.

– If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h}{n_0}\log\frac{1}{h}}{h^4} = \frac{\log\frac{1}{h}}{n_0 h^3} \longrightarrow 0 \Leftrightarrow n_0 h^3 \longrightarrow \infty,$$

which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\frac{h}{n_0}\log\frac{1}{h} = o\left(\frac{1}{n_0\, h} + h^4\right)$.

Considering this last argument and collecting terms (65), (62) and (66), expression (57) is proven.

1.2.2 Asymptotic expressions for the bootstrap criterion functions

Lemma 5 *Under regularity conditions (B1)-(B4), the function $MISE^{a*}$ is*

$$\begin{aligned} MISE^{a*}(h) &= \frac{R(K)}{n_0\, h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2\hat{B}_g + \mathscr{O}_P\left(h^6\, n_1^{-1}\, g^{-7}\left(g^{-2} + g^{-1} + 1\right)\right) \\ &+ \mathscr{O}_P(h^8\, n_1^{-1} g^{-9}) + \mathscr{O}_P\left(h^{-1}\, g^2\, n_1^{-1}\right) + \mathscr{O}_P\left(h n_1^{-1}\left(1 + g^{-1} + g^{-2}\right)\right). \end{aligned} \tag{67}$$

Thus, the dominant part of expression (67), namely $AMISE^{a*}(h)$, is given by:

$$AMISE^{a*}(h) = \frac{R(K)}{n_0\, h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2\hat{B}_g. \tag{68}$$

Proof Consider the smoothed bootstrap version of $MISE^a$. We start by computing $MISE^{a*}(h) := \mathbb{E}^*\left[\int\left(\tilde{m}_h^{NW}(x) - \hat{m}_g(x)\right)\right]$, which is the theoretical analogous to $MASE^*_{\tilde{m}_h^{NW}, X^1}$ given in (14).

Using a Taylor expansion and a change of variable, we obtain:

$$
\begin{aligned}
&\mathbb{E}^*\left[\int \left(\tilde{m}_h^{NW}(x)-\hat{m}_g(x)\right)^2 d\hat{F}_g^1(x)\right]\\
&=\mathbb{E}^*\left[\int \frac{1}{n_0\hat{f}_g^0(x)}\sum_{i=1}^{n_0} K_h\left(x-X_i^{0*}\right)\left(Y_i^{0*}-\hat{m}_g(x)\right)\right]^2 d\hat{F}_g^1(x)\\
&=\frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[\sum_{i=1}^{n_0} K_h\left(x-X_i^{0*}\right)\left(Y_i^{0*}-\hat{m}_g(x)\right)\right]^2 d\hat{F}_g^1(x)\\
&=\frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[Var^*\left[\sum_{i=1}^{n_0} K_h\left(x-X_i^{0*}\right)\left(Y_i^{0*}-\hat{m}_g(x)\right)\right]\right.\\
&\left.+\left(\mathbb{E}^*\left[\sum_{i=1}^{n_0} K_h\left(x-X_i^{0*}\right)\left(Y_i^{0*}-\hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x)\\
&=\frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[\sum_{i=1}^{n_0} Var^*\left[K_h\left(x-X_i^{0*}\right)\left(Y_i^{0*}-\hat{m}_g(x)\right)\right]\right.\\
&\left.+\left(\sum_{i=1}^{n_0}\mathbb{E}^*\left[K_h\left(x-X_i^{0*}\right)\left(Y_i^{0*}-\hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x)\\
&=\frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[n_0 Var^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right)\right]\right.\\
&\left.+n_0^2\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x)\\
&=\frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2} Var^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right)\right] d\hat{F}_g^1(x)\\
&+\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
&=\frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left[\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)^2\left(Y_1^{0*}-\hat{m}_g(x)\right)^2\right]\right.\\
&\left.-\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x)\\
&+\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
&=\frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)^2\left(Y_1^{0*}-\hat{m}_g(x)\right)^2\right] d\hat{F}_g^1(x)\\
&+\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)^2\left(Y_1^{0*}-\hat{m}_g(x)\right)^2 |X_1^{0*}\right]\right] d\hat{F}_g^1(x)\\
&+ \frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(Y_1^{0*}-\hat{m}_g(x)\right) |X_1^{0*}\right]\right]\right)^2 d\hat{F}_g^1(x)\\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)^2\mathbb{E}^*\left[\left(Y_1^{0*}-\hat{m}_g(x)\right)^2 |X_1^{0*}\right]\right] d\hat{F}_g^1(x)\\
&+ \frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(\mathbb{E}^*\left[Y_1^{0*} |X_1^{0*}\right]-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)^2\left(Var^*\left(Y_1^{0*}-\hat{m}_g(x)\,|X_1^{0*}\right)\right.\right.\\
&\left.\left.+\left(\mathbb{E}^*\left[Y_1^{0*}-\hat{m}_g(x)\,|X_1^{0*}\right]\right)^2\right)\right] d\hat{F}_g^1(x)\\
&+ \frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)^2\left(Var^*\left(Y_1^{0*} |X_1^{0*}\right)\right.\right.\\
&\left.\left.+\left(\mathbb{E}^*\left[Y_1^{0*} |X_1^{0*}\right]-\hat{m}_g(x)\right)^2\right)\right] d\hat{F}_g^1(x)\\
&+ \frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)^2\left(\hat{\sigma}_g^2(X_1^{0*})+g^2\mu_2(K)\right.\right.\\
&\left.\left.+\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)^2\right)\right] d\hat{F}_g^1(x)\\
&+ \frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x-X_1^{0*}\right)\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
&= \frac{1}{n_0 h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K\left(\frac{x-X_1^{0*}}{h}\right)^2\left(\hat{\sigma}_g^2(X_1^{0*})+g^2\mu_2(K)\right.\right.\\
&\left.\left.+\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)^2\right)\right] d\hat{F}_g^1(x)\\
&+ \frac{n_0-1}{n_0 h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K\left(\frac{x-X_1^{0*}}{h}\right)\left(\hat{m}_g(X_1^{0*})-\hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x)\\
&= \frac{1}{n_0 h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\int K\left(\frac{y-x}{h}\right)^2\\
&\left(\hat{\sigma}_g^2(y)+g^2\mu_2(K)+(\hat{m}_g(y)-\hat{m}_g(x))^2\right)\hat{f}_g^0(y)\,dy\,d\hat{F}_g^1(x)\\
&+ \frac{n_0-1}{n_0 h^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\int K\left(\frac{y-x}{h}\right)(\hat{m}_g(y)-\hat{m}_g(x))\,\hat{f}_g^0(y)\,dy\right)^2 d\hat{F}_g^1(x)\\
&= \frac{1}{n_0 h}\int \frac{1}{\hat{f}_g^0(x)^2}\int K(u)^2\left(\hat{\sigma}_g^2(x+hu)+g^2\mu_2(K)+(\hat{m}_g(x+hu)-\hat{m}_g(x))^2\right)\\
&\hat{f}_g^0(x+hu)\,du\,d\hat{F}_g^1(x)+\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\\
&\left(\int K(u)\,(\hat{m}_g(x+hu)-\hat{m}_g(x))\,\hat{f}_g^0(x+hu)\,du\right)^2 d\hat{F}_g^1(x). \qquad (69)
\end{aligned}
$$

Let us denote $\hat{\psi}_{g,x}(y) = (\hat{m}_g(y) - \hat{m}_g(x))\,\hat{f}_g^0(y)$. For the sake of simplicity, we will denote $\hat{\psi}_g(y) = \hat{\psi}_{g,x}(y)$ in the following. Then,

$$
\begin{aligned}
\hat{\psi}_g(x) &= 0,\\
\hat{\psi}_g'(y) &= \hat{m}_g'(y)\hat{f}_g^0(y) + (\hat{m}_g(y) - \hat{m}_g(x))\left(\hat{f}_g^0\right)'(y),\\
\hat{\psi}_g''(y) &= \hat{m}_g''(y)\hat{f}_g^0(y) + 2\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y) + (\hat{m}_g(y) - \hat{m}_g(x))\left(\hat{f}_g^0\right)''(y),\\
\hat{\psi}_g'''(y) &= \hat{m}_g'''(y)\hat{f}_g^0(y) + 3\hat{m}_g''(y)\left(\hat{f}_g^0\right)'(y) + 3\hat{m}_g'(y)\left(\hat{f}_g^0\right)''(y),\\
&\quad + (\hat{m}_g(y) - \hat{m}_g(x))\left(\hat{f}_g^0\right)'''(y),\\
\hat{\psi}_g^{(4)}(y) &= \hat{m}_g^{(4)}(y)\hat{f}_g^0(y) + 4\hat{m}_g'''(y)\left(\hat{f}_g^0\right)'(y) + 6\hat{m}_g''(y)\left(\hat{f}_g^0\right)''(y),\\
&\quad + 4\hat{m}_g'(y)\left(\hat{f}_g^0\right)'''(y)\,(\hat{m}_g(y) - \hat{m}_g(x))\left(\hat{f}_g^0\right)^{(4)}(y),\\
\hat{\psi}_g''(x) &= \hat{m}_g''(x)\hat{f}_g^0(x) + 2\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x),\\
\hat{\psi}_g^{(4)}(x) &= \hat{m}_g^{(4)}(x)\hat{f}_g^0(x) + 4\hat{m}_g'''(x)\left(\hat{f}_g^0\right)'(x), \text{ and}\\
&\quad + 6\hat{m}_g''(x)\left(\hat{f}_g^0\right)''(x) + 4\hat{m}_g'(x)\left(\hat{f}_g^0\right)'''(x).
\end{aligned}
$$

Using Taylor expansion:

$$
\begin{aligned}
&\int K(u)\,\hat{\psi}_g(x+hu)\,du\\
&= \int K(u)\,\hat{\psi}_g(x)\,du + h\int u\,K(u)\,\hat{\psi}_g'(x)\,du + \frac{h^2}{2}\int u^2 K(u)\,\hat{\psi}_g''(x)\,du\\
&+ \frac{h^3}{3!}\int u^3 K(u)\,\hat{\psi}_g'''(x)\,du + \frac{h^4}{4!}\int u^4 K(u)\,\hat{\psi}_g^{(4)}(x)\,du\\
&+ \frac{h^5}{5!}\int u^5 K(u)\,\hat{\psi}_g^{(5)}(x)\,du + \mathscr{O}_P(h^6)\\
&= \frac{h^2}{2}\,\hat{\psi}_g''(x)\mu_2(K) + \frac{h^4}{24}\,\hat{\psi}_g^{(4)}(x)\mu_4(K) + \mathscr{O}_P(h^6), \text{ and}\\
&\left(\int K(u)\,\hat{\psi}_g(x+hu)\,du\right)^2\\
&= \frac{h^4}{4}\,\hat{\psi}_g''(x)^2\mu_2(K)^2 + \frac{h^6}{24}\,\hat{\psi}_g''(x)\,\hat{\psi}_g^{(4)}(x)\mu_2(K)\mu_4(K)\\
&+ \frac{h^8}{576}\,\hat{\psi}_g^{(4)}(x)^2\mu_4(K) + \mathscr{O}_P(h^{12})\\
&= \frac{h^4}{4}\,\hat{\psi}_g''(x)^2\mu_2(K)^2 + \mathscr{O}_P(h^6 n_0^{-2} g^{-8}\left(g^{-2} + g^{-1} + 1\right)) + \mathscr{O}_P(h^8 n_0^{-2} g^{-10}),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\psi}_g''(x)^2 &= \hat{m}_g''(x)^2 \hat{f}_g^0(x)^2 \\
&\quad + 4\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 + 4\hat{m}_g''(x)\hat{f}_g^0(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x), \\
\hat{\psi}_g^{(4)}(x)^2 &= \hat{m}_g^{(4)}(x)^2 \hat{f}_g^0(x)^2 + 8\hat{m}_g^{(4)}(x)\hat{m}_g'''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x) \\
&\quad + 12\hat{m}_g^{(4)}(x)\hat{m}_g''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)''(x) \\
&\quad + 8\hat{m}_g^{(4)}(x)\hat{m}_g'(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'''(x) + 16\hat{m}_g'''(x)^2\left(\hat{f}_g^0\right)'(x)^2 \\
&\quad + 48\hat{m}_g'''(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)''(x) \\
&\quad + 32\hat{m}_g'''(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'''(x)\left(\hat{f}_g^0\right)'(x) + 36\hat{m}_g''(x)^2\left(\hat{f}_g^0\right)''(x)^2 \\
&\quad + 48\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)''(x)\left(\hat{f}_g^0\right)'''(x) + 16\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'''(x)^2, \\
\hat{\psi}_g''(x)\hat{\psi}_g^{(4)}(x) &= \hat{m}_g''(x)^2\hat{m}_g^{(4)}(x)\hat{f}_g^0(x)^2 + 4\hat{m}_g'''(x)\hat{m}_g''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x) \\
&\quad + 6\hat{m}_g''(x)^2\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'''(x) + 2\hat{m}_g'(x)\hat{m}_g^{(4)}(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x) \\
&\quad + 8\hat{m}_g'(x)\hat{m}_g'''(x)\left(\hat{f}_g^0\right)'(x)^2 + 12\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)''(x) \\
&\quad + 8\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)'''(x), \\
\left(\hat{f}_g^0\right)^{(r)}(x) &= n_0^{-1} g^{-r-1} \sum_{i=1}^{n_0} K^{(r)}\left(\frac{x - X_i^0}{g}\right), \\
\hat{m}_g'(x) &= g^{-1}\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^{-2} \cdot \left(\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) Y_i^0\right. \\
&\quad \left.\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) - \sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) Y_i^0\right),
\end{aligned}
\tag{70}
$$

$$\hat{m}_g''(x) = \left[\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)} + \frac{2\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \right.$$

$$- \frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}$$

$$\left. - \frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \right] \cdot g^{-2},$$

$$\hat{m}_g'''(x) = g^{-3} \left[\frac{\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)} - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \right.$$

$$+ \frac{4\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$- \frac{3\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}$$

$$- \frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}$$

$$+ \frac{6\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2 \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$\left. + \frac{6\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^3 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^4} \right], \text{ and}$$

$$\hat{m}_g^{(4)}(x) = g^{-4} \left[\frac{\sum_{i=1}^{n_0} K^{(4)}\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)} \right.$$

$$
\begin{aligned}
&+\frac{22\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\\
&+\frac{4\left(\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\\
&+\frac{12\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\\
&+\frac{4\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\\
&-\frac{3\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\\
&-\frac{6\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\\
&-\frac{4\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\\
&-\frac{12\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^3\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^4}\\
&-\frac{24\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^4\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^5}\\
&\left.+\frac{6\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^4}\right].
\end{aligned}
$$

Carrying on with calculations leads to:

$$
\begin{aligned}
&\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\int K(u)\,\hat{\psi}_g(x+hu)\,du\right)^2 d\hat{F}_g^1(x)\\
&=\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\frac{h^4}{4}\,\hat{\psi}_g''(x)^2\mu_2(K)^2 d\hat{F}_g^1(x)\\
&+\mathcal{O}_P(h^6\,n_1^{-1}\,g^{-7}\left(g^{-2}+g^{-1}+1\right))+\mathcal{O}_P(h^8\,n_1^{-1}g^{-9})\\
&=\frac{n_0-1}{n_0}\frac{h^4}{4}\mu_2(K)^2\\
&\int\left[\hat{m}_g''(x)^2+\frac{4\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)}{\hat{f}_g^0(x)}+\frac{4\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2}{\hat{f}_g^0(x)^2}\right]\hat{f}_g^1(x)dx\\
&+\mathcal{O}_P(h^6\,n_1^{-1}\,g^{-7}\left(g^{-2}+g^{-1}+1\right))+\mathcal{O}_P(h^8\,n_1^{-1}g^{-9}).
\end{aligned}
\tag{71}
$$

On the other hand, denote $\hat{\phi}_{g,x}(y)=\hat{\phi}_g(y)=\left[\hat{\sigma}_g^2(y)+g^2\,\mu_2(K)+(\hat{m}_g(y)-\hat{m}_g(x))^2\right]\hat{f}_g^0(y)$. Then,

$$
\begin{aligned}
\hat{\phi}_g(x)&=\hat{\sigma}_g^2(x)\hat{f}_g^0(x)+g^2\,\mu_2(K)\,\hat{f}_g^0(x),\\
\hat{\phi}_g'(y)&=\left(\hat{\sigma}_g^2\right)'(y)\hat{f}_g^0(y)+\hat{\sigma}_g^2(y)\left(\hat{f}_g^0\right)'(y)+g^2\mu_2(K)\left(\hat{f}_g^0\right)'(y),\\
&+2\hat{m}_g(y)\hat{m}_g'(y)\hat{f}_g^0(y)+\hat{m}_g(y)^2\left(\hat{f}_g^0\right)'(y)+\hat{m}_g(x)^2\left(\hat{f}_g^0\right)'(y)\\
&-2\hat{m}_g'(y)\hat{m}_g(x)\hat{f}_g^0(y)-2\hat{m}_g(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)'(y),\\
\hat{\phi}_g''(y)&=\left(\hat{\sigma}_g^2\right)''(y)\hat{f}_g^0(y)+2\left(\hat{\sigma}_g^2\right)'(y)\left(\hat{f}_g^0\right)'(y)+\hat{\sigma}_g^2(y)\left(\hat{f}_g^0\right)''(y)\\
&+g^2\mu_2(K)\left(\hat{f}_g^0\right)''(y)+2\hat{m}_g'(y)^2\hat{f}_g^0(y)+2\hat{m}_g(y)\hat{m}_g'(y)\hat{f}_g^0(y)\\
&+2\hat{m}_g(y)^2\left(\hat{f}_g^0\right)'(y)+2\hat{m}_g(y)\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y)\\
&+\hat{m}_g(x)^2(y)\left(\hat{f}_g^0\right)''(y)+\hat{m}_g^2(x)\left(\hat{f}_g^0\right)''(y)-2\hat{m}_g''(y)\hat{f}_g^0(y)\hat{m}_g(x)\\
&-2\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y)\hat{m}_g(x)-2\hat{m}_g'(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)'(y)\\
&-2\hat{m}_g(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)''(y),\text{ and}\\
\hat{\phi}_g''(x)&=\left(\hat{\sigma}_g^2\right)''(x)\hat{f}_g^0(x)+2\left(\hat{\sigma}_g^2\right)'(x)\left(\hat{f}_g^0\right)'(x)+\hat{\sigma}_g^2(x)\left(\hat{f}_g^0\right)''(x)\\
&+g^2\mu_2(K)\left(\hat{f}_g^0\right)''(x)+2\hat{m}_g'(x)^2\hat{f}_g^0(x)+2\hat{m}_g(x)\hat{m}_g'(x)\hat{f}_g^0(x)\\
&+2\hat{m}_g(x)^2\left(\hat{f}_g^0\right)'(x)+2\hat{m}_g(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x)\\
&-2\hat{m}_g''(x)\hat{f}_g^0(x)\hat{m}_g(x)-4\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x)\hat{m}_g(x),
\end{aligned}
$$

where

$$\begin{aligned}
\hat{\sigma}_g^2(x) &= \frac{1}{n_0 \hat{f}_g^0(x)} \sum_{i=1}^{n_0} K_g(x - X_i^0) \left(Y_i^0\right)^2 - \left[\frac{1}{n_0 \hat{f}_g^0(x)} \sum_{i=1}^{n_0} K_g(x - X_i^0) Y_i^0\right]^2 \\
&= \left[\frac{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)}\right] - \left[\frac{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)}\right]^2, \\
\left(\hat{\sigma}_g^2\right)'(x) &= g^{-1}\left(\left[\frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right.\right. \\
&\left. - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right] \\
&-2\left[\frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right. \\
&\left.\left. - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\right]\right), \text{ and} \\
\left(\hat{\sigma}_g^2\right)''(x) &= g^{-2}\left(\frac{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right. \\
&+ \frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\
&- \frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\
&+ \frac{2\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}
\end{aligned}$$

$$
\begin{aligned}
&-2\left[\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right.\\
&+\frac{\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\\
&-\frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\\
&-\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\\
&+\frac{3\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^5}\\
&\left.\left.-\frac{2\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0 \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0 \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\right]\right).
\end{aligned}
$$

Applying a Taylor expansion:

$$
\begin{aligned}
\int K(u)^2\hat{\phi}_g(x+hu)\,du &= R(K)\hat{\phi}_g(x)+\frac{h^2}{2}\hat{\phi}_g''(x)\int u^2K(u)^2\,du+\mathscr{O}_P\left(h^4\right)\\
&= R(K)\hat{\phi}_g(x)+\mathscr{O}_P\left(h^2 n_0^{-1}g^{-1}\left(1+g^{-1}+g^{-2}\right)\right).
\end{aligned}
$$

Carrying on with calculations leads to:

$$
\begin{aligned}
&\frac{1}{n_0 h}\int \frac{1}{\hat{f}_g^0(x)^2}\int K(u)^2\hat{\phi}_g(x+hu)\,du\,d\hat{F}_g^1(x)\\
&=\frac{R(K)}{n_0 h}\int \frac{\left(\hat{\sigma}_g^2(x)+g^2\mu_2(K)\right)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx+\mathscr{O}_P\left(h n_1^{-1}\left(1+g^{-1}+g^{-2}\right)\right)\\
&=\frac{R(K)}{n_0 h}\int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx+\frac{R(K)\,g^2\mu_2(K)}{n_0 h}\int \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx\\
&+\mathscr{O}_P\left(h n_1^{-1}\left(1+g^{-1}+g^{-2}\right)\right)\\
&=\frac{R(K)}{n_0 h}\int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx\\
&+\mathscr{O}_P\left(h n_1^{-1}\left(1+g^{-1}+g^{-2}\right)\right)+\mathscr{O}_P\left(h^{-1}g^2 n_1^{-1}\right).
\end{aligned}
\tag{72}
$$

Assembling terms (71) and (72), and inserting them in expression (69), Lemma 5 holds.

Proposition 3 *Assume conditions (C1), (C2), (C3). Suppose, additionally, that* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0 h \longrightarrow \infty$. *Consider* h_{n_0} *a sequence of bandwidths such that* $\sum_{n_0} h_{n_0}^{\lambda} < \infty$ *for some* $\lambda > 0$ *and that* $n_0^{\eta} h_{n_0} \longrightarrow \infty$ *for some* $\eta < 1 - s^{-1}$. *Assume further that* $(n_0 h)^{-1/2}\log\left(h^{-1}\right) \longrightarrow 0$ *as* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0 h \longrightarrow \infty$. *Then,*

$$
ISE^*(h)=ISE^{a*}(h)+\mathscr{O}_{P^*}\left(h^6\right)+\mathscr{O}_{P^*}\left(\frac{h}{n_0}\log\frac{1}{h}\right)+\mathscr{O}_{P^*}\left(\frac{h^{7/2}}{n_0^{1/2}}\right)+\mathscr{O}_{P^*}\left(\frac{\log\frac{1}{h}}{n_0^{3/2}h^{3/2}}\right),
\tag{73}
$$

almost sure with respect to P, where

$$
ISE^*(h)=\int\left(\hat{m}_h^{NW*}(x)-\hat{m}_g(x)\right)^2 d\hat{F}_g^1(x), \text{ and } ISE^{a*}(h)=\int\left(\tilde{m}_h^{NW*}(x)-\hat{m}_g(x)\right)^2 d\hat{F}_g^1(x).
$$

Proof Given that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0 h \longrightarrow \infty$, we have:

$$
\sup_x\left|\hat{f}_h^{0*}(x)-\hat{f}_g^0(x)\right| \longrightarrow 0 \text{ in bootstrap probability as } n_0 \longrightarrow \infty,
\tag{74}
$$

and

$$
\sup_x\left|\hat{f}_h^{0*}(x)-\hat{f}_g^0(x)\right|=\mathscr{O}_{P^*}\left(h^2+n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right).
\tag{75}
$$

Moreover,

$$
\sup_J\left|\hat{m}_h^{NW*}(x)-\hat{m}_g(x)\right| \longrightarrow 0 \text{ in bootstrap probability as } n_0 \longrightarrow \infty,
\tag{76}
$$

and

$$
\sup_J\left|\hat{m}_h^{NW*}(x)-\hat{m}_g(x)\right|=\mathscr{O}_{P^*}\left(h^2+n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right).
\tag{77}
$$

Focusing now on $MISE^*(h)$, it is straightforward that

$$
MISE^*(h)=MISE^{a*}(h)+\mathbb{E}^*\left[\int 2A_1^*A_2^*\,d\hat{F}_g^1(x)\right]+\mathbb{E}^*\left[\int\left(A_2^*\right)^2 d\hat{F}_g^1(x)\right],
$$

where

$$
\begin{aligned}
A_1^* &= \frac{\hat{\Psi}_h^*(x)}{\hat{f}_g^0(x)} - \frac{\hat{m}_g(x)\hat{f}_h^{0*}(x)}{\hat{f}_g^0(x)} = \frac{1}{n_0\,\hat{f}_g^0(x)}\sum_{i=1}^{n_0} K_h(x-X_i^{0*})(Y_i^{0*}-\hat{m}_g(x)),\\
A_2^* &= \left(\hat{m}_h^{NW*}(x)-\hat{m}_g(x)\right)\frac{\left(\hat{f}_h^{0*}(x)-\hat{f}_g^0(x)\right)}{\hat{f}_g^0(x)},
\end{aligned}
$$

and $MISE^{a*}(h)$ is given in (67). On the one hand, using expression (60) as well as (C1)-(C3), it turns out:

$$
\begin{aligned}
\int_J (A_2^*)^2\, d\hat{F}_g^1(x) &= \left|\int_J (A_2^*)^2\, d\hat{F}_g^1(x)\right|\\
&= \left|\int_J \left(\hat{m}_h^{NW*}(x)-\hat{m}_g(x)\right)\frac{\left(\hat{f}_h^0(x)-\hat{f}_g^0(x)\right)}{\hat{f}_g^0(x)}\hat{f}_g^1(x)\,dx\right|\\
&\le \sup_J\left|\hat{m}_h^{NW*}(x)-\hat{m}_g(x)\right|^2 \sup_J\left|\hat{f}_h^{0*}(x)-\hat{f}_g^0(x)\right|^2\int_{x\in J}\frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx\\
&= \int_{x\in J}\frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx\left(\mathcal{O}_{P^*}\left(h^2+n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right)\right)^4\\
&= \int_{x\in J}\frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx\left(\mathcal{O}_{P^*}\left(\max\left\{h^2,n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}\right)\right)^4\\
&= \int_{x\in J}\frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx\left(\mathcal{O}_{P^*}\left(\max\left\{h^2,n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}^4\right)\right)\\
&= \int_{x\in J}\frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx\left(\mathcal{O}_{P^*}\left(\max\left\{h^8,n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right\}\right)\right)\\
&= \int_{x\in J}\frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\,dx\left(\mathcal{O}_{P^*}\left(h^8+n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right)\right)\\
&= \mathcal{O}_{P^*}\left(h^8\right)+\mathcal{O}_{P^*}\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^{2}\right). \qquad (78)
\end{aligned}
$$

The first addend in expression (78) is negligible in comparison to the second term in expression (67). Analogously, the second addend in expression (78) becomes insignificant compared to the first term in expression (67). In particular,

$$\begin{aligned}\left(\int_J (A_2^*)^2\, d\hat{F}_g^1(x)\right)^{1/2} &= \left(\mathscr{O}_{P^*}\left(h^8\right) + \mathscr{O}_{P^*}\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2}\\ &= \left(\mathscr{O}_{P^*}\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2}\\ &= \left(\mathscr{O}_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}\right)\right)^{1/2}\\ &= \mathscr{O}_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}^{1/2}\right)\end{aligned}$$

$$\begin{aligned}&= \mathscr{O}_{P^*}\left(\max\left\{h^4, n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right\}\right)\\ &= \mathscr{O}_{P^*}\left(h^4 + n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right)\\ &= \mathscr{O}_{P^*}\left(h^4\right) + \mathscr{O}_{P^*}\left(n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right). \qquad (79)\end{aligned}$$

Moreover, thanks to expression (67), we have

$$\mathbb{E}\left[\int_J (A_1^*)^2\, dF_1(x)\right] = \mathscr{O}\left(n_0^{-1}h^{-1} + h^4\right). \qquad (80)$$

Applying Markov's inequality to expression (80), it turns out:

$$\int_J (A_1^*)^2\, d\hat{F}_g^1(x) = \mathscr{O}_{P^*}\left(n_0^{-1}h^{-1} + h^4\right). \qquad (81)$$

Thus,

$$\begin{aligned}\left(\int_J (A_1^*)^2\, d\hat{F}_g^1(x)\right)^{1/2} &= \left(\mathscr{O}_{P^*}\left(n_0^{-1}h^{-1} + h^4\right)\right)^{1/2} = \mathscr{O}_{P^*}\left(\max\left\{n_0^{-1}h^{-1}, h^4\right\}^{1/2}\right)\\ &= \mathscr{O}_{P^*}\left(\max\left\{n_0^{-1/2}h^{-1/2}, h^2\right\}\right) = \mathscr{O}_{P^*}\left(n_0^{-1/2}h^{-1/2} + h^2\right)\\ &= \mathscr{O}_{P^*}\left(n_0^{-1/2}h^{-1/2}\right) + \mathscr{O}_{P^*}\left(h^2\right). \qquad (82)\end{aligned}$$

Bringing together expressions (82) and (79) and using Cauchy-Schwarz inequality, we compute:

$$\left|\int_J A_1 A_2\, d\hat{F}_g^1(x)\right| \leq 2\left(\int_J (A_1^*)^2\, d\hat{F}_g^1(x)\right)^{1/2}\left(\int_J (A_2^*)^2\, d\hat{F}_g^1(x)\right)^{1/2} \tag{83}$$

$$= 2\left(\mathcal{O}_p\left(h^4\right) + \mathcal{O}_{P^*}\left(n_0^{-1} h^{-1}\log\frac{1}{h}\right)\right)$$

$$\cdot\left(\mathcal{O}_p\left(h^2\right) + \mathcal{O}_{P^*}\left(n_0^{-1/2} h^{-1/2}\log\frac{1}{h}\right)\right)$$

$$= \mathcal{O}_{P^*}\left(h^6\right) + \mathcal{O}_{P^*}\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + \mathcal{O}_{P^*}\left(\frac{h}{n_0}\log\frac{1}{h}\right)$$

$$+\,\mathcal{O}_{P^*}\left(\frac{\log\frac{1}{h}}{(n_0 h)^{3/2}}\right). \tag{84}$$

The first addend in expression (83) is negligible as compared to the second term in expression (67). Furthermore, the fourth addend in expression (83) becomes insignificant in comparison to the first term in expression (67) if

$$\left(\frac{\log\frac{1}{h}}{n_0^{1/2} h^{1/2}}\right) \longrightarrow 0,$$

as $h \longrightarrow 0$, $n_0 \longrightarrow \infty$ and $n_0 h \longrightarrow \infty$. It remains to be seen what happens with the second and third addends in expression (83). We begin with the second one. Given that $\left(n_0^{-1} h^{-1} + h^4\right)\cdot n_0 h = 1 + h^5 n_0$ and the bandwidth h is of the form $n^{-\alpha}, \alpha > 0$, then

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then $n_0^{-1} h^{-1} \sim n_0^{-4/5}$ and $h^4 \sim n_0^{-4/5}$, which implies that $h \sim n_0^{-1/5}$, and

$$\frac{h^{7/2}}{n_0^{1/2}} \sim \frac{\left(n_0^{-1/5}\right)^{7/2}}{n_0^{1/2}} = \frac{n_0^{-7/10}}{n_0^{5/10}} = n_0^{-6/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{\frac{1}{n_0 h}} \longrightarrow 0 \Leftrightarrow n_0^{1/2} h^{9/2} \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $n_0 h^5 \longrightarrow 0$.
- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{h^4} \longrightarrow 0 \Leftrightarrow n_0^{-1/2} h^{-1/2} \longrightarrow 0 \Leftrightarrow n_0 h \longrightarrow \infty,$$

  which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\dfrac{h^{7/2}}{n_0^{1/2}} = o\left(\dfrac{1}{n_0\, h} + h^4\right)$.

Finally, as for the third addend in (83),

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then

$$\frac{h}{n_0}\log\frac{1}{h} \sim n_0^{-6/5}\log n_0^{1/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\dfrac{h}{n_0}\log\dfrac{1}{h}}{\dfrac{1}{n_0\, h}} = h^2\log\frac{1}{h} \longrightarrow 0 \Leftrightarrow h^2 \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

which is true providing that $h \longrightarrow 0$.

- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\dfrac{h}{n_0}\log\dfrac{1}{h}}{h^4} = \frac{\log\dfrac{1}{h}}{n_0 h^3} \longrightarrow 0 \Leftrightarrow n_0 h^3 \longrightarrow \infty,$$

which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\dfrac{h}{n_0}\log\dfrac{1}{h} = o\left(\dfrac{1}{n_0\, h} + h^4\right)$.

Considering this last reasoning and collecting terms (82), (79) and (83), expression (73) is proven.

Theorem 4 *Consider h_{MISE^a}, its bootstrap version, $h^*_{MISE^a}$, and h_{AMISE^a}, which are the minimizers of expressions (26), (67) and (68), respectively. Under regularity conditions (B1)-(B4) and assuming that g is of order $n_0^{-1/2}$, it holds that*

$$h^*_{MISE^a} - h_{MISE^a} = \mathscr{O}_P\left(n_0^{-7/10}\right), \text{ and } \frac{h^*_{MISE^a} - h_{MISE^a}}{h_{MISE^a}} = \mathscr{O}_P\left(n_0^{-1/2}\right). \tag{85}$$

Proof Consider h_{MISE^a}, its bootstrap version, $h^*_{MISE^a}$, and h_{AMISE^a}, which are the minimizers of expressions (26), (67) and (68), respectively. Hence, it is obvious that $AMISE^{a*'}\left(h^*_{AMISE^a}\right) = 0$ as well as $MISE^{a'}\left(h_{MISE^a}\right) = 0$. Moreover, applying Taylor expansion to $MISE^{a*'}\left(h^*_{MISE^a}\right)$ towards the point h_{MISE^a}, leads to:

$$\begin{aligned} MISE^{a*'}\left(h^*_{MISE^a}\right) = {} & MISE^{a*'}\left(h_{MISE^a}\right) - AMISE^{a'}\left(h_{AMISE^a}\right) \\ & + \left(h^*_{MISE^a} - h_{MISE^a}\right)\cdot MISE^{a*''}\left(h_{MISE^a}\right) + \\ & + \frac{1}{2}\left(h^*_{MISE^a} - h_{MISE^a}\right)^2 \\ & \cdot MISE^{a*'''}\left(\widetilde{h_{MISE^a}}\right), \end{aligned} \tag{86}$$

where $\widetilde{h_{MISE^a}}$ is an intermediate value between $h^*_{MISE^a}$ and h_{MISE^a}. As a consequence of expression (86), it follows that:

$$\begin{aligned} h^*_{MISE^a} - h_{MISE^a} &= -\frac{MISE^{a*'}(h_{MISE^a}) - AMISE^{a'}(h_{AMISE^a})}{MISE^{a*''}(h_{MISE^a})} \\ &\cdot (1+o_P(1)) \\ &= -\frac{MISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a})}{MISE^{a*''}(h_{AMISE^a})} \\ &\cdot (1+o_P(1)) \\ &= -\frac{AMISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a})}{AMISE^{a*''}(h_{AMISE^a})} \\ &\cdot (1+o_P(1)). \end{aligned} \tag{87}$$

Thanks to expressions (12) and (35) in the paper, we have:

$$\begin{aligned} AMISE^{a''}(h_{AMISE^a}) &= \frac{R(K)}{n_0\, h^3_{AMISE^a}} A + 3\, h^2_{AMISE^a}\, \mu_2(K)^2 B \\ &= n_0^{-2/5} \cdot \left[R(K)A + 3\,\mu_2(K)^2 B\right], \end{aligned} \tag{88}$$

and,

$$\begin{aligned} & AMISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a}) \\ &= -\frac{R(K)}{n_0\, h^2_{AMISE^a}} \left(\hat{A}_g - A\right) + h^3_{AMISE^a}\, \mu_2(K)^2 \left(\hat{B}_g - B\right) \\ &= -n_0^{-3/5} \cdot \left[R(K)\left(\hat{A}_g - A\right) + \mu_2(K)^2 \left(\hat{B}_g - B\right)\right] \\ &= -n_0^{-3/5} \cdot n_0^{-1/2} \cdot \left[R(K) + \mu_2(K)^2\right] \\ &= -n_0^{-11/10} \cdot \left[R(K) + \mu_2(K)^2\right]. \end{aligned} \tag{89}$$

Considering now expressions (35) in the paper and (88) above, it turns out:

$$\begin{aligned} AMISE^{a*''}(h_{AMISE^a}) &= AMISE^{a*''}(h_{AMISE^a}) - AMISE^{a''}(h_{AMISE^a}) \\ &+ AMISE^{a''}(h_{AMISE^a}) \\ &= AMISE^{a''}(h_{AMISE^a}) + \mathscr{O}_P\left(n_0^{-9/10}\right). \end{aligned} \tag{90}$$

Therefore, combining expressions (87) and (90),

$$\begin{aligned} h^*_{MISE^a} - h_{MISE^a} &= -\frac{AMISE^{a*'}(h_{AMISE^a}) - AMISE^{a'}(h_{AMISE^a})}{AMISE^{a''}(h_{AMISE^a})} \\ &\cdot (1+o_P(1)). \end{aligned}$$

Finally, collecting terms (89) and (90), and plugging them into (87) leads to proof the result in Theorem 4.

2 Proof of the results presented in the Appendix

Lemma 6 *Consider the expressions for $\hat{A}_g$ and A from the paper. Then,*

$$\hat{A}_g - A = \sum_{i=1}^{k_0} a_i C^{[s_i]}_{\nu_i,\ell_i,r_i} + A_1, \tag{91}$$

where $k_0 = 6$, $a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1, a_5 = -2, a_6 = -2$, $\nu_1(x) = \dfrac{\sigma^2(x)}{f^0(x)}$, $\nu_2(x) = \dfrac{\sigma^2(x) f^1(x)}{f^0(x)^2}$, $\nu_3(x) = \dfrac{f^1(x)}{f^0(x)^2}$, $\nu_4(x) = \dfrac{f^1(x)\Psi_2(x)}{f^0(x)^3}$, $\nu_5(x) = \dfrac{f^1(x)\Psi_1(x)}{f^0(x)^3}$,
$\nu_6(x) = \dfrac{f^1(x)\Psi_1^2(x)}{f^0(x)^4}$, $\ell_1 = 0$, $\ell_2 = 0$, $\ell_3 = 2$, $\ell_4 = 0$, $\ell_5 = 1$, $\ell_6 = 0$, $r_1 = 0$, $r_2 = 0$, $r_3 = 0$, $r_4 = 0$, $r_5 = 0$, $r_6 = 0$, $[s_1] = 1$, $[s_2] = 0$, $[s_3] = 0$, $[s_4] = 0$, $[s_5] = 0$, $[s_6] = 0$ *and* $A_1 = \mathscr{O}\left(r_{0,n_0}\right)$, *with*

$$\begin{aligned} r_{0,n_0} = &\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx \\ &+ \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x) f^1(x)\right) dx \\ &+ \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx \\ &+ \int \left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx. \end{aligned} \tag{92}$$

Proof Our aim is to obtain upper bounds for $\hat{A}_g - A$. On the one hand,

$$
\begin{aligned}
\hat{A}_g - A &= \int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}\,dx - \int \frac{\sigma^2(x)f^1(x)}{f^0(x)}\,dx \\
&= \int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)}{f^0(x)}\,dx - \int \frac{\sigma^2(x)f^1(x)\left(\hat{f}_g^0(x) - f^0(x)\right)}{f^0(x)^2}\,dx \\
&+ \mathscr{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right)\,dx\Bigg) \\
&= \int \frac{\left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right)\hat{f}_g^1(x)}{f^0(x)}\,dx + \int \frac{\sigma^2(x)}{f^0(x)}\left[\hat{f}_g^1(x) - f^1(x)\right]dx \\
&- \int \frac{\sigma^2(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right]dx \\
&+ \mathscr{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right)\,dx\Bigg) \\
&= \int \frac{\left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)}{f^0(x)}\,dx + \int \frac{f^1(x)}{f^0(x)}\left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right]dx \\
&+ \int \frac{\sigma^2(x)}{f^0(x)}\left[\hat{f}_g^1(x) - f^1(x)\right]dx - \int \frac{\sigma^2(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right]dx \\
&+ \mathscr{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right)\,dx\Bigg) \\
&= \int \frac{f^1(x)}{f^0(x)}\left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right]dx + \int \frac{\sigma^2(x)}{f^0(x)}\left[\hat{f}_g^1(x) - f^1(x)\right]dx \\
&- \int \frac{\sigma^2(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right]dx \\
&+ \mathscr{O}\Bigg(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right)\,dx\Bigg) \\
&+ \mathscr{O}\left(\int \left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right).
\end{aligned}
\tag{93}
$$

Furthermore,

$$
\begin{aligned}
&\int \frac{f^1(x)}{f^0(x)}\left[\hat{\sigma}_g^2(x)-\sigma^2(x)\right]dx \\
&=\int \frac{f^1(x)}{f^0(x)}\left[\frac{\hat{\Psi}_{2,g}(x)}{\hat{f}_g^0(x)}-\frac{\Psi_2(x)}{f^0(x)}\right]dx-\int \frac{f^1(x)}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^2(x)}{\hat{f}_g^0(x)^2}-\frac{\Psi_1^2(x)}{f^0(x)^2}\right]dx \\
&=\int \frac{f^1(x)}{f^0(x)}\left[\frac{\hat{\Psi}_{2,g}(x)-\Psi_2(x)}{f^0(x)}-\frac{\Psi_2(x)\left(\hat{f}_g^0(x)-f^0(x)\right)}{f^0(x)^2}\right]dx \\
&-\int \frac{f^1(x)}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)}{f^0(x)^2}-\frac{\Psi_1^2(x)\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)}{f^0(x)^4}\right]dx \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\right. \\
&\left.\cdot\left(\hat{\Psi}_{2,g}(x)-\Psi_2(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\left.\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)dx\right) \\
&=\int \frac{f^1(x)}{f^0(x)^2}\left[\hat{\Psi}_{2,g}(x)-\Psi_2(x)\right]dx-\int \frac{f^1(x)\Psi_2(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&-\int \frac{f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right]dx-\int \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)^2-f^0(x)^2\right]dx \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right. \\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{2,g}(x)-\Psi_2(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right. \\
&\left.+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)dx\right) \\
&=\int \frac{f^1(x)}{f^0(x)^2}\left[\hat{\Psi}_{2,g}(x)-\Psi_2(x)\right]dx-\int \frac{f^1(x)\Psi_2(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&-2\int \frac{f^1(x)\Psi_1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-2\int \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{2,g}(x)-\Psi_2(x)\right)dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2 dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2 dx\right.\\
&\left.+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)dx\right).
\end{aligned}
\tag{94}
$$

In the following remark (expression (95)) is collected the proof used in last equality of (94).

Remark 2 Let $\hat{\Psi}_2$ be the estimator of Ψ_2. Then:

$$
\begin{aligned}
\hat{\Psi}_2^2-\Psi_2^2 &= \left(\hat{\Psi}_2+\Psi_2\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)=\left(\hat{\Psi}_2-\Psi_2+\Psi_2+\Psi_2\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=2\Psi_2\left[\hat{\Psi}_2-\Psi_2\right]+\mathscr{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)^2\right).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Psi}_2^3-\Psi_2^3 &= \left(\hat{\Psi}_2^2+\hat{\Psi}_2\Psi_2+\Psi_2^2\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=\left(\hat{\Psi}_2^2-\Psi_2^2+\Psi_2^2+\hat{\Psi}_2\Psi_2-\Psi_2^2+\Psi_2^2+\Psi_2^2\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=\left(3\Psi_2^2+(\hat{\Psi}_2+\Psi_2)\cdot(\hat{\Psi}_2-\Psi_2)+\Psi_2(\hat{\Psi}_2-\Psi_2)\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=3\Psi_2^2\left[\hat{\Psi}_2-\Psi_2\right]+\mathscr{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)^2+\left(\hat{\Psi}_2+\Psi_2\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Psi}_2^4-\Psi_2^4 &= \left(\hat{\Psi}_2^3+\hat{\Psi}_2^2\Psi_2+\Psi_2^3+\Psi_2^2\hat{\Psi}_2\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=\left(\left(\hat{\Psi}_2^3-\Psi_2^3\right)+\Psi_2^3+\left(\hat{\Psi}_2^2\Psi_2-\Psi_2^3\right)+\Psi_2^3+\Psi_2^3\right.\\
&\left.+\left(\Psi_2^2\hat{\Psi}_2-\Psi_2^3\right)+\Psi_2^3\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=\left(4\Psi_2^3+\left(\hat{\Psi}_2^3-\Psi_2^3\right)+\Psi_2\left(\hat{\Psi}_2^2-\Psi_2^2\right)+\Psi_2^2\left(\hat{\Psi}_2-\Psi_2\right)\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=4\Psi_2^3\left[\hat{\Psi}_2-\Psi_2\right]+\mathscr{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)^2\right.\\
&\left.+\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^2-\Psi_2^2\right)+\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Psi}_2^5-\Psi_2^5 &= \left(\hat{\Psi}_2^4+\Psi_2^4+\hat{\Psi}_2\Psi_2^3+\Psi_2\hat{\Psi}_2^3\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=\left(\left(\hat{\Psi}_2^4-\Psi_2^4\right)+\Psi_2^4\right.\\
&\left.+\left(\hat{\Psi}_2\Psi_2^3-\Psi_2^4\right)+\Psi_2^4+\Psi_2^4+\left(\Psi_2\hat{\Psi}_2^3-\Psi_2^4\right)+\Psi_2^4\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=\left(\left(\hat{\Psi}_2^4-\Psi_2^4\right)+4\Psi_2^4+\Psi_2^3\left(\hat{\Psi}_2-\Psi_2\right)+\Psi_2\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right)\cdot\left(\hat{\Psi}_2-\Psi_2\right)\\
&=4\Psi_2^4\left[\hat{\Psi}_2-\Psi_2\right]+\mathscr{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^4-\Psi_2^4\right)+\left(\hat{\Psi}_2-\Psi_2\right)^2\right)\\
&+\mathscr{O}\left(\left(\hat{\Psi}_2-\Psi_2\right)\cdot\left(\hat{\Psi}_2^3-\Psi_2^3\right)\right).
\end{aligned}
$$

Then,

$$\begin{aligned}
\hat{\Psi}_2^2 - \Psi_2^2 &= 2\Psi_2 \left[\hat{\Psi}_2 - \Psi_2\right] + \mathscr{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2\right), &(95)\\
\hat{\Psi}_2^3 - \Psi_2^3 &= 3\Psi_2^2 \left[\hat{\Psi}_2 - \Psi_2\right] \\
&\quad + \mathscr{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2 + \left(\hat{\Psi}_2 + \Psi_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right)\right), &(96)\\
\hat{\Psi}_2^4 - \Psi_2^4 &= 4\Psi_2^3 \left[\hat{\Psi}_2 - \Psi_2\right] + \mathscr{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2 + \left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^2 - \Psi_2^2\right)\right) \\
&\quad + \mathscr{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^3 - \Psi_2^3\right)\right), \text{ and} &(97)\\
\hat{\Psi}_2^5 - \Psi_2^5 &= 4\Psi_2^4 \left[\hat{\Psi}_2 - \Psi_2\right] + \mathscr{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^4 - \Psi_2^4\right) + \left(\hat{\Psi}_2 - \Psi_2\right)^2\right) \\
&\quad + \mathscr{O}\left(\left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^3 - \Psi_2^3\right)\right). &(98)
\end{aligned}$$

Combining expressions (94) and (93), Lemma 6 is concluded.

Lemma 7 *Given the expressions for $\hat{B}_g$ and B (from the paper), then $\hat{B}_g - B$ consists of a sum of 60 terms similar to those in expression (91). Specifically,*

$$\hat{B}_g - B = \sum_{i=7}^{k_1} a_i C^{[s_i]}_{\nu_i,\ell_i,r_i} + B_1, \tag{99}$$

where $k_1 = 66$. The functions $\nu(x)$, and the values of r, ℓ, a and $[s]$ are collected in Tables 1-3. Additionally, term B_1 is of order $\mathscr{O}\left(r_{1,n_0}\right)$ with r_{1,n_0} being also given in the tables.

i	$\nu_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
7	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}$	0	1	1	8
8	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi_1'(x)}{f^0(x)^4}$	0	0	0	-8
9	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x)}{f^0(x)^4}$	0	1	0	-8
10	$\frac{m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2}$	1	0	0	4
11	$\frac{m'(x)^2 f^1(x) \left(f^0\right)'(x)}{f^0(x)^2}$	0	0	1	8
12	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x) \Psi_1(x)}{f^0(x)^4}$	0	0	0	-16
13	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi_1(x)}{f^0(x)^4}$	0	0	1	-8
14	$\frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}$	0	0	0	-8
15	$\frac{f^1(x) m''(x)}{f^0(x)}$	0	1	2	2
16	$\frac{f^1(x) m''(x) \Psi_1''(x)}{f^0(x)^2}$	0	0	0	-2
17	$\frac{f^1(x) m''(x) \Psi_1(x)}{f^0(x)^2}$	0	0	2	-2
18	$\frac{f^1(x) m''(x) \left(f^0\right)''(x)}{f^0(x)^2}$	0	1	0	-2
19	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)''(x)}{f^0(x)}$	0	0	0	8
20	$\frac{f^1(x) m''(x) \Psi_1'(x)}{f^0(x)^2}$	0	0	1	-4
21	$\frac{f^1(x) m''(x) \left(f^0\right)'(x)}{f^0(x)^2}$	0	1	1	-4
22	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)}{f^0(x)^3}$	0	0	0	8
23	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)}{f^0(x)^3}$	0	0	1	8
24	$\frac{f^1(x) m''(x) \left(f^0\right)'(x)^2}{f^0(x)^3}$	0	1	0	4
25	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)^2}{f^0(x)^4}$	0	0	0	-12
26	$m''(x)^2$	1	0	0	1
27	$\frac{\Psi_1'(x) \Psi_1''(x) \left(f^0\right)'(x)}{f^0(x)^3}$	1	0	0	4

Table 1: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (99), $i \in \{7, \ldots, 27\}$.

i	$\nu_i(x)$	$[s_i]$	ℓ_i	r_i	a_i
28	$\frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}$	0	0	1	4
29	$\frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}$	0	1	2	4
30	$\frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}$	0	1	1	4
31	$\frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}$	0	0	0	-4
32	$\frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	0	0	-8
33	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}$	1	0	0	8
34	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}$	0	1	0	8
35	$\frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}$	0	0	1	24
36	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8
37	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	0	0	-32
38	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	1	0	0	-4
39	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	0	1	-4
40	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	0	-4
41	$\frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	0	2	-4
42	$\frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	1	-4
43	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}$	0	0	0	-12
44	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}$	1	0	0	-8
45	$\frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	-16
46	$\frac{\left(f^0\right)'(x)^2f^1(x)\Psi_1'(x)}{f^0(x)^4}$	0	1	1	-16
47	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}$	0	0	0	24

Table 2: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s_i]$ in expression (99), $i \in \{28,\ldots,47\}$.

i	$\nu_i(x)$	$[s_i]$	ℓ_i	r_i	a_i
48	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}$	1	0	0	-4
49	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	1	2	-4
50	$\frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}$	0	1	0	-4
51	$\frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	-8
52	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}$	0	0	0	12
53	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}$	1	0	0	-8
54	$\frac{\left(f^0\right)'(x)^4f^1(x)\Psi_1(x)}{f^0(x)^6}$	0	1	0	-16
55	$\frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}$	0	0	1	-32
56	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}$	0	0	0	32
57	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}$	1	0	0	4
58	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2f^1(x)}{f^0(x)^5}$	0	0	2	4
59	$\frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}$	0	1	0	8
60	$\frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}$	0	0	1	8
61	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$	0	0	0	-16
62	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}$	1	0	0	8
63	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8
64	$\frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}$	0	1	0	8
65	$\frac{\Psi_1(x)\Psi_1'(x)f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}$	0	0	1	24
66	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}$	0	0	0	-32

Table 3: Values of $\nu_i(x)$, r_i, ℓ_i, a_i and $[s_i]$ in expression (99), $i \in \{48,\ldots,66\}$.

Proof Focusing now on expression $\hat{B}_g - B$,

$$\begin{aligned}
\hat{B}_g - B = &\int \left[\hat{m}_g''(x)^2 \hat{f}_g^1(x) - m''(x)^2 f^1(x)\right] dx \\
&+4\int \left[\frac{\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)} - \frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)}\right] dx \\
&+4\int \left[\frac{\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2}\right] dx,
\end{aligned} \tag{100}$$

where

$$\begin{aligned}
B_1 &:= \int \left[\hat{m}_g''(x)^2 \hat{f}_g^1(x) - m''(x)^2 f^1(x)\right] dx, \\
B_2 &:= 4\int \left[\frac{\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)} - \frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)}\right] dx, \text{ and} \\
B_3 &:= 4\int \left[\frac{\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2}\right] dx.
\end{aligned}$$

We will focus on term B_1 in the first place. Carrying on with calculations, considering expression (95), it leads to:

$$\begin{aligned}
B_1 :=& \int \left[\hat{m}_g''(x)^2 \hat{f}_g^1(x) - m''(x)^2 f^1(x)\right] dx \\
=& \int \hat{m}_g''(x)^2 \hat{f}_g^1(x)\, dx - \int m''(x)^2 f^1(x)\, dx \\
=& \int \hat{m}_g''(x)^2 \left(\hat{f}_g^1(x) - f^1(x) + f^1(x)\right) dx - \int m''(x)^2 f^1(x)\, dx \\
=& \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int \hat{m}_g''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
=& \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int m''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
+& \int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
=& \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int m''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
+& \mathcal{O}\left(\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
=& 2\int f^1(x)m''(x)\left[\hat{m}_g''(x) - m''(x)\right] dx + \int m''(x)^2\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
+& \mathcal{O}\left(\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
+& \mathcal{O}\left(\int \left(\hat{m}_g''(x) - m''(x)\right)^2 dx\right).
\end{aligned} \tag{101}$$

Furthermore, considering expressions (95), (96) and (97), it turns out:

$$
\begin{aligned}
&2\int f^1(x)m''(x)\left[\hat{m}_g''(x)-m''(x)\right]dx\\
&=2\int f^1(x)m''(x)\left[\frac{\hat{\Psi}_{1,g}''(x)}{\hat{f}_g^0(x)}-\frac{\Psi_1''(x)}{f^0(x)}\right]dx\\
&-2\int f^1(x)m''(x)\left[\frac{\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)}{\hat{f}_g^0(x)^2}-\frac{\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)^2}\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-4\int f^1(x)m''(x)\left[\frac{\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)}{\hat{f}^0_g(x)^2}-\frac{\Psi'_1(x)\left(f^0\right)'(x)}{f^0(x)^2}\right]dx\\
&+4\int f^1(x)m''(x)\left[\frac{\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^2}{\hat{f}^0_g(x)^3}-\frac{\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^3}\right]dx\\
&=2\int\frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right]dx\\
&-2\int\frac{f^1(x)m''(x)\Psi''_1(x)}{f^0(x)^2}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+2\int\frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\hat{f}^0_g(x)^4-f^0(x)^4\right]dx\\
&+4\int\frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}^0_g(x)^2-f^0(x)^2\right]dx\\
&-2\int\frac{f^1(x)m''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right]dx\\
&-4\int\frac{f^1(x)m''(x)}{f^0(x)^2}\left[\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)-\Psi'_1(x)\left(f^0\right)'(x)\right]dx\\
&+4\int\frac{f^1(x)m''(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right]dx\\
&-4\int\frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^6}\left[\hat{f}^0_g(x)^3-f^0(x)^3\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)^2\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)-\Psi'_1(x)\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right)dx\right)\\
&=2\int\frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right]dx\\
&-2\int\frac{f^1(x)m''(x)\Psi''_1(x)}{f^0(x)^2}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&-2\int\frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-2\int\frac{f^1(x)m''(x)}{f^0(x)^2}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^0_g\right)''(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&-4\int \frac{f^1(x)m''(x)\Psi_1'(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int \frac{f^1(x)m''(x)}{f^0(x)^2}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}^0_g\right)'(x)\right]dx\\
&+8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+4\int \frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^3}\left[\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx\\
&+4\int \frac{f^1(x)m''(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^0_g\right)'(x)^2\right]dx\\
&-12\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)dx\right)+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}^0_g\right)'(x)-\Psi_1'(x)\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right)dx\right)\\
&=2\int \frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&-2\int \frac{f^1(x)m''(x)\Psi_1''(x)}{f^0(x)^2}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&-2\int \frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-2\int \frac{f^1(x)m''(x)\left(f^0\right)''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&-4\int \frac{f^1(x)m''(x)\Psi_1'(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\hat{\Psi}'_{1,g}(x)-\Psi_1'(x)\right]dx\\
&+8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)^2}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-12\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int \hat{f}^0_g(x)^2-f^0(x)^2\,dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi_1''(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)^2-f^0(x)^2\right)^2dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)-\Psi_1'(x)\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}'_{1,g}(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{f}^0_g(x)-f^0(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx\right). \qquad (102)
\end{aligned}
$$

Plugging expression (102) in (101) leads to the following expression for B_1:

$$
\begin{aligned}
B_1 := & \, 2\int \frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx+\int m''(x)^2\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\Psi_1''(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\left(f^0\right)''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& -4\int \frac{f^1(x)m''(x)\Psi_1'(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& -4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& +4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)^2}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& -12\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)-\Psi_1'(x)\left(f^0\right)'(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right)dx\right) \\
& +\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{m}_g''(x)-m''(x)\right)^2dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{m}_g^{''}(x)^2-m^{''}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)dx\right). \qquad (103)
\end{aligned}
$$

We now move on to achieve a deeper insight of term B_2, considering expressions (95), (96), (97) and (98), it follows that:

$$
\begin{aligned}
B_2 &:= 4\int\left[\frac{\hat{m}_g^{'}(x)\hat{m}_g^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}-\frac{m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)}\right]dx\\
&=4\int\frac{1}{f^0(x)}\left[\hat{m}_g^{'}(x)\hat{m}_g^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)-m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)\right]dx\\
&-4\int\frac{m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\right.\\
&\left.\cdot\left(\hat{m}_g^{'}(x)\hat{m}_g^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)-m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)\right)dx\right). \qquad (104)
\end{aligned}
$$

Computing further calculations with the first term in expression (104) using (95), (96), (97) and (98), we have:

$$
\begin{aligned}
&4\int\frac{1}{f^0(x)}\left[\hat{m}_g^{'}(x)\hat{m}_g^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)-m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)\right]dx\\
&=4\int\frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^{'}(x)\hat{\Psi}_{1,g}^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}-\frac{\Psi_1^{'}(x)\Psi_1^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^2}\right]dx\\
&+8\int\frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{'}(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3\Psi_1(x)f^1(x)}{f^0(x)^4}\right]dx\\
&-4\int\frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{''}(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right.\\
&\left.-\frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}-\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\right]dx\\
&-4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}-\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\right]dx\\
&-8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^5}-\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^5}\right]dx\\
&+4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}\right.\\
&\left.-\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\right]dx\\
&+8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^4}\right]dx. \qquad (105)
\end{aligned}
$$

The first addend in expression (105), using (95), (96), (97) and (98), happens to be:

$$
\begin{aligned}
&4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}-\frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\right]dx\\
&=4\int \frac{1}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)^2-f^0(x)^2\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
&=4\int \frac{1}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right). \qquad (106)
\end{aligned}
$$

Working out calculations with the first term in expression (106) leads to:

$$
\begin{aligned}
&4\int \frac{1}{f^0(x)^3}\left[\hat{\Psi}^{'}_{1,g}(x)\hat{\Psi}^{''}_{1,g}(x)\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)-\Psi^{'}_1(x)\Psi^{''}_1(x)\left(f^0\right)^{'}(x)f^1(x)\right]dx\\
&=4\int \frac{\Psi^{'}_1(x)}{f^0(x)^3}\left[\hat{\Psi}^{''}_{1,g}(x)\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)-\Psi^{''}_1(x)\left(f^0\right)^{'}(x)f^1(x)\right]dx\\
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}^{'}_{1,g}(x)-\Psi^{'}_1(x)\right)\hat{\Psi}^{''}_{1,g}(x)\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)\right]dx\\
&=4\int \frac{\Psi^{'}_1(x)\Psi^{''}_1(x)}{f^0(x)^3}\left[\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)-\left(f^0\right)^{'}(x)f^1(x)\right]dx\\
&+4\int \frac{\Psi^{'}_1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}^{''}_{1,g}(x)-\Psi^{''}_1(x)\right)\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{\Psi^{''}_1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}^{'}_{1,g}(x)-\Psi^{'}_1(x)\right)\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}^{'}_{1,g}(x)-\Psi^{'}_1(x)\right)\cdot\left(\hat{\Psi}^{''}_{1,g}(x)-\Psi^{''}_1(x)\right)\left(\hat{f}^0_g\right)^{'}(x)\hat{f}^1_g(x)\right]dx\\
&=4\int \frac{\Psi^{'}_1(x)\Psi^{''}_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\hat{f}^1_g(x)-f^1(x)\right]dx\\
&+4\int \frac{\Psi^{'}_1(x)\Psi^{''}_1(x)}{f^0(x)^3}\left[\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{\Psi^{'}_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\left(\hat{\Psi}^{''}_{1,g}(x)-\Psi^{''}_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{\Psi^{''}_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\left(\hat{\Psi}^{'}_{1,g}(x)-\Psi^{'}_1(x)\right)\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{\Psi^{'}_1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}^{''}_{1,g}(x)-\Psi^{''}_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{\Psi^{''}_1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}^{'}_{1,g}(x)-\Psi^{'}_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\left(\hat{\Psi}^{'}_{1,g}(x)-\Psi^{'}_1(x)\right)\cdot\left(\hat{\Psi}^{''}_{1,g}(x)-\Psi^{''}_1(x)\right)\hat{f}^1_g(x)\right]dx\\
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}^{'}_{1,g}(x)-\Psi^{'}_1(x)\right)\cdot\left(\hat{\Psi}^{''}_{1,g}(x)-\Psi^{''}_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\hat{f}^1_g(x)\right]dx\\
&=4\int \frac{\Psi^{'}_1(x)\Psi^{''}_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\hat{f}^1_g(x)-f^1(x)\right]dx\\
&+4\int \frac{\Psi^{'}_1(x)\Psi^{''}_1(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx\\
&+4\int \frac{\Psi^{'}_1(x)\Psi^{''}_1(x)}{f^0(x)^3}\left[\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\Psi^{'}_1(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}^{''}_{1,g}(x)-\Psi^{''}_1(x)\right]dx
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
&+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x) f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right] dx\\
&+4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right] dx\\
&+4\int \frac{\Psi_1'(x) f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right] dx\\
&+4\int \frac{\Psi_1'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right] dx\\
&+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right] dx\\
&+4\int \frac{\Psi_1''(x) f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right] dx\\
&+4\int \frac{\Psi_1''(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right] dx\\
&+4\int \frac{\left(f^0\right)'(x) f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right] dx\\
&+4\int \frac{\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right] dx\\
&+4\int \frac{f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right] dx\\
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right] dx\\
&=4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^1(x)-f^1(x)\right] dx\\
&+4\int \frac{\Psi_1'(x)\Psi_1''(x) f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right] dx\\
&+4\int \frac{\Psi_1'(x)\left(f^0\right)'(x) f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right] dx\\
&+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x) f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right] dx\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right)
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right). \qquad (109)
\end{aligned}
$$

Plugging expression (109) in (106), the first addend in expression (105) happens to be:

$$
\begin{aligned}
& 4 \int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^1(x)-f^1(x)\right] dx \\
& +4 \int \frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right] dx \\
& +4 \int \frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right] dx \\
& +4 \int \frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right] dx \\
& -8 \int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right] dx \\
& +\mathscr{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2 dx\right)+\mathscr{O}\left(\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_{g}^{0}(x)-f^{0}(x)\right)^{2}dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_{1}^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_{1}^{''}(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_{g}^{0}\right)^{'}(x)-\left(f^{0}\right)^{'}(x)\right)\cdot\left(\hat{f}_{g}^{1}(x)-f^{1}(x)\right)dx\right).
\end{aligned}
\tag{110}
$$

The second term in expression (105) turns out to be:

$$
\begin{aligned}
&8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^4}\right]dx\\
&=8\int \frac{1}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^9}\left[\hat{f}_g^0(x)^4-f^0(x)^4\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right)dx\right)\\
&=8\int \frac{1}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx\\
&-32\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right).
\end{aligned}
\tag{111}
$$

Carrying on with calculations considering the first term in expression (111) leads to:

$$8\int \frac{1}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx$$
$$=8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx$$
$$+8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)\right]dx$$
$$=8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1(x)f^1(x)\right]dx$$
$$+8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$
$$\left.\cdot\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)\right]dx$$
$$=8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$=8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx$$
$$+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$
\begin{aligned}
&+8\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right]dx\\
&+8\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&+8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int \frac{\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right]dx\\
&+8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int \frac{f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&=8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3 f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+24\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right). \qquad (112)
\end{aligned}
$$

Plugging expression (112) in (111), the second addend of expression (105) turns out ot be:

$$
\begin{aligned}
& 8 \int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3 \Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right] dx \\
& +8 \int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3 f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right] dx \\
& +24 \int \frac{\Psi_1^{'}(x) \Psi_1(x) f^1(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right] dx \\
& +8 \int \frac{\left(f^0\right)^{'}(x)^3 \Psi_1(x) f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right] dx \\
& -32 \int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3 \Psi_1(x) f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right] dx \\
& +\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2 dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{'}(x)^3 \hat{\Psi}_{1,g}(x) \hat{f}_g^1(x)-\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3 \Psi_1(x) f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right) \cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)+\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx\right) \\
& +\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2 dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) \cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right) \cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right) \cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right) dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) \cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right) \cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right) \cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right) \cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) \cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right) \cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right) \cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right) \cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right) \cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right) \cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right) \cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) dx\right) \\
& +\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right) \cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right) \cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) .
\end{aligned} \tag{113}
$$

Carrying on with calculations with the third addend in expression (105), using (95), (96), (97) and (98), it leads to:

$$
\begin{aligned}
& -4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right.\\
& \left.-\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\right]dx\\
& = -4\int \frac{1}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\\
& \left.-\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
& -4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)^3-f^0(x)^3\right]dx\\
& +\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)\\
& +\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\right.\\
& \left.\left.-\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
& = -4\int \frac{1}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\\
& \left.-\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
& -12\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
& +\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
& \cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)\right.\\
& \left.\left.f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\right.\\
& \left.\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right).
\end{aligned}
\tag{114}
$$

Working out further calculation with the first term in expression (114), using (95), (96), (97) and (98), it turns out:

$$
\begin{aligned}
&-4\int \frac{1}{f^0(x)^4}\Big[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\\
&-\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\Big]\,dx\\
&=-4\int \frac{\Psi_1'(x)}{f^0(x)^4}\Big[\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\\
&-\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\Big]\,dx\\
&-4\int \frac{1}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\Big]\,dx\\
&=-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)}{f^0(x)^4}\Big[\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1(x)\left(f^0\right)'(x)f^1(x)\Big]\,dx\\
&-4\int \frac{\Psi_1'(x)}{f^0(x)^4}\Big[\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\cdot\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\Big]\,dx\\
&-4\int \frac{\left(f^0\right)''(x)}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\Big]\,dx\\
&-4\int \frac{1}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\\
&\cdot\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\Big]\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= -4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \left(f^0\right)'(x)f^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right. \\
&\left.\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right. \\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right. \\
&\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right. \\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right]dx \\
&= -4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right. \\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx \\
&-4\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right. \\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}_g^1(x)\right]dx - 4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^4}
\end{aligned}
$$

$$\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int\frac{\Psi_1'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int\frac{\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int\frac{\left(f^0\right)^{''}(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int\frac{\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int\frac{\left(f^0\right)^{''}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx-4\int\frac{\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}$$
$$\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int\frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$-4\int\frac{\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx-4\int\frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.$$
$$\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)$$
$$\left.\cdot\hat{f}_g^1(x)\right]dx$$
$$=-4\int\frac{\Psi_1'(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$-4\int\frac{\Psi_1'(x)\left(f^0\right)^{''}(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx$$
$$-4\int\frac{\Psi_1'(x)\left(f^0\right)^{''}(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$
$$-4\int\frac{\Psi_1'(x)\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx$$
$$-4\int\frac{\Psi_1'(x)\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$
\begin{aligned}
&-4\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right]dx\\
&-4\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1^{'}(x)\Psi_1(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right]dx\\
&-4\int \frac{\Psi_1^{'}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1^{'}(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right]dx-4\int \frac{\Psi_1^{'}(x)\Psi_1(x)}{f^0(x)^4}\\
&\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-4\int \frac{\Psi_1^{'}(x)f^1(x)}{f^0(x)^4}\\
&\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right]dx\\
&-4\int \frac{\Psi_1^{'}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right]dx\\
&-4\int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.
\end{aligned}
\tag{115}
$$

$$
\begin{aligned}
&\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\\
&\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\Big(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\Big)\Big]\,dx\\
&-4\int\frac{\left(f^0\right)''(x)}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\\
&\cdot\Big(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\Big)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\\
&\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\Big]\,dx\\
&-4\int\frac{\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\\
&\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\Psi_1(x)f^1(x)}{f^0(x)^4}\\
&\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\Big)\Big]\,dx\\
&-4\int\frac{\Psi_1(x)}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\\
&\cdot\Big(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\Big)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\\
&\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\Big]\,dx\\
&-4\int\frac{\left(f^0\right)'(x)}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\\
&\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{f^1(x)}{f^0(x)^4}\\
&\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\\
&\cdot\Big(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\Big)\Big]\,dx\\
&-4\int\frac{1}{f^0(x)^4}\Big[\Big(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\Big)\cdot\Big(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\Big)\\
&\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\Big(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\Big)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= -4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)dx\right).
\end{aligned}
\tag{116}
$$

Plugging expression (116) in (114), the third addend in expression (105) happens to be:

$$
\begin{aligned}
&-4\int\frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}^1_g(x)-f^1(x)\right]dx\\
&-4\int\frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int\frac{\Psi'_1(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-4\int\frac{\Psi'_1(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-4\int\frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]dx\\
&-12\int\frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}^0_g\right)'(x)\hat{f}^1_g(x)\right.\right.\\
&\left.\left.-\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right).
\end{aligned}
\tag{117}
$$

The fourth addend in expression (105), using (95), (96), (97) and (98), happens to be:

$$
\begin{aligned}
&-8\int\frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}-\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^3}\right]dx\\
&=-8\int\frac{1}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right]dx\\
&+8\int\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)^3-f^0(x)^3\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&=-8\int\frac{1}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right]dx\\
&+24\int\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.
\end{aligned}
$$

$$\cdot \left(\hat{\Psi}_{1,g}'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \Psi_1'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \right)$$
$$+ \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \right)$$
$$= -8 \int \frac{\Psi_1'(x)^2}{f^0(x)^4} \left[\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \left(f^0\right)'(x)^2 f^1(x) \right] dx$$
$$- 8 \int \frac{1}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2 \right) \cdot \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) \right] dx$$
$$+ 24 \int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x) \right] dx$$
$$+ \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 \right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \right.$$
$$\left. \cdot \left(\hat{\Psi}_{1,g}'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \Psi_1'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) \right)$$
$$+ \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 + \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \right)$$
$$= -8 \int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^4} \left[\hat{f}_g^1(x) - f^1(x) \right] dx$$
$$- 8 \int \frac{\Psi_1'(x)^2}{f^0(x)^4} \left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2 \right) \cdot \hat{f}_g^1(x) \right] dx$$
$$- 8 \int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2 \right) \cdot \hat{f}_g^1(x) \right] dx - 8 \int \frac{1}{f^0(x)^4}$$
$$\left[\left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2 \right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2 \right) \cdot \hat{f}_g^1(x) \right] dx$$
$$+ 24 \int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x) \right] dx$$
$$+ \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx \right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \right.$$
$$\left. \cdot \left(\hat{\Psi}_{1,g}'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \Psi_1'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \right)$$
$$+ \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \right)$$
$$= -8 \int \frac{\Psi_1'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^4} \left[\hat{f}_g^1(x) - f^1(x) \right] dx$$
$$- 8 \int \frac{\Psi_1'(x)^2 f^1(x)}{f^0(x)^4} \left[\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2 \right] dx$$
$$- 8 \int \frac{\Psi_1'(x)^2}{f^0(x)^4} \left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2 \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) \right] dx$$
$$- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2 \right] dx$$
$$- 8 \int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2 \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) \right] dx$$

$$
\begin{aligned}
&-8\int \frac{f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right]dx\\
&-8\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+24\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&=-8\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-16\int \frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^2f^1(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+24\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right). \qquad (118)
\end{aligned}
$$

The fifth addend in expression (105) turns out to be, considering (95), (96), (97) and (98):

$$
\begin{aligned}
&-4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right.\\
&\left.-\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\right]dx\\
&=-4\int\frac{1}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx\\
&+4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)^3-f^0(x)^3\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&=-4\int\frac{1}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx\\
&+12\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&=-4\int\frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx\\
&-4\int\frac{1}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right]dx\\
&+12\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&=-4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\Psi_1''(x)f^1(x)\right]dx\\
&-4\int\frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right]dx\\
&-4\int\frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right]dx-4\int\frac{1}{f^0(x)^4}\\
&\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right]dx
\end{aligned}
\tag{119}
$$

$$
\begin{aligned}
&+12\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{''}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&=-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1^{''}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-4\int \frac{\Psi_1(x)\Psi_1^{''}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-4\int \frac{\Psi_1^{''}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-4\int \frac{1}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&+12\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{''}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right)\\
&=-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)\Psi_1^{''}(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right]dx\\
&-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1^{''}(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-4\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx\\
&-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1''(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-4\int \frac{f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx\\
&-4\int \frac{1}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&=-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-8\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right)\\
&\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\Big)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right).
\end{aligned}
\tag{120}
$$

Working out calculations with the sixth addend in expression (105), using (95), (96), (97) and (98), it leads to:

$$
\begin{aligned}
&-8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^5}-\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^5}\right]dx\\
&=-8\int \frac{1}{f^0(x)^6}\left[\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^{11}}\left[\hat{f}_g^0(x)^5-f^0(x)^5\right]dx\\
&+\mathscr{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)\\
&=-8\int \frac{1}{f^0(x)^6}\left[\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right]dx\\
&+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&=-8\int \frac{\left(f^0\right)'(x)^4}{f^0(x)^6}\left[\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\Psi_1^2(x)f^1(x)\right]dx\\
&-8\int \frac{1}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)\right]dx\\
&+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&=-8\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-8\int \frac{\left(f^0\right)'(x)^4}{f^0(x)^6}\left[\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-8\int \frac{\Psi_1^2(x)}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\hat{f}_g^1(x)\right]dx\\
&-8\int \frac{1}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\hat{f}_g^1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2 dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&=-8\int\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-8\int\frac{\left(f^0\right)'(x)^4f^1(x)}{f^0(x)^6}\left[\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right]dx\\
&-8\int\frac{\Psi_1^2(x)f^1(x)}{f^0(x)^6}\left[\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right]dx\\
&+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int\frac{\left(f^0\right)'(x)^4}{f^0(x)^6}\left[\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-8\int\frac{\Psi_1^2(x)}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-8\int\frac{f^1(x)}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\right]dx\\
&-8\int\frac{1}{f^0(x)^6}\left[\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right)dx\\
&+\int\mathscr{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2dx\right)\\
&=-8\int\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-16\int\frac{\left(f^0\right)'(x)^4f^1(x)\Psi_1(x)}{f^0(x)^6}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-32\int\frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx
\end{aligned}
\qquad (121)
$$

$$
\begin{aligned}
&+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]dx+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2\right.\\
&+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)+\left(\hat{f}_g^0(x)-f^0(x)\right)\\
&\left.\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&+\mathscr{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2dx\right).
\end{aligned}
\tag{122}
$$

Carrying on with calculations with the seventh addend in expression (105) following similar steps as in the previous expressions, it turns out:

$$
\begin{aligned}
&4\int\frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}\right.\\
&\left.-\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\right]dx\\
&=4\int\frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right]dx\\
&-4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^9}\left[\hat{f}_g^0(x)^4-f^0(x)^4\right]dx\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)
\end{aligned}
$$

$$= 4\int \frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right.$$

$$\left.-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right]dx$$

$$-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$= 4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right]dx$$

$$+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx$$

$$-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$= 4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)''(x)f^1(x)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx$$

$$+4\int \frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx$$

$$+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$$

$$\left[\hat{f}_g^0(x)-f^0(x)\right]dx+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$
$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$
$$=4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$+4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+4\int\frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+4\int\frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+4\int\frac{\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+4\int\frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\hat{f}_g^1(x)\right]dx$$
$$+4\int\frac{\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$
$$\left.\cdot\hat{f}_g^1(x)\right]dx+4\int\frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx-16\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$$
$$\left[\hat{f}_g^0(x)-f^0(x)\right]dx+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$
$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$
$$=4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$+4\int\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx$$
$$+4\int\frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right]dx$$
$$+4\int\frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx$$

$$
\begin{aligned}
&+4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)^{'}(x)^2\left(f^0\right)^{''}(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\Psi_1(x)^2\left(f^0\right)^{''}(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\Psi_1(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right]dx\\
&+4\int \frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right]dx\\
&+4\int \frac{\left(f^0\right)^{''}(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+4\int \frac{f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right]dx\\
&+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-16\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&=4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int \frac{\left(f^0\right)^{'}(x)^2\Psi_1(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right]dx
\end{aligned}
$$

$$+8\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx$$
$$+8\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx$$
$$-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\Bigg). \tag{123}$$

Finally, following analogous steps as in previous expressions, the eigthth addend in expression (105) happens to be:

$$8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^4}\right]dx$$
$$=8\int \frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right]dx$$
$$-8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^9}\left[\hat{f}_g^0(x)^4-f^0(x)^4\right]dx$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\Bigg(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right)dx\Bigg)$$
$$=8\int \frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right]dx$$
$$-32\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\Bigg(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right)dx\Bigg)$$
$$+\mathscr{O}\Bigg(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx$$
$$+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\Bigg)$$
$$=8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)-\Psi_1(x)\Psi'_1(x)f^1(x)\right]dx$$
$$+8\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)\right]dx$$
$$-32\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\Bigg(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right)dx\Bigg)$$
$$+\mathscr{O}\Bigg(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx$$
$$+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\Bigg)$$

$$= 8\int \frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x) - \Psi_1^{'}(x)f^1(x)\right]dx$$
$$+ 8\int \frac{\left(f^0\right)^{'}(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x)\right]dx$$
$$+ 8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right)\cdot\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x)\right]dx + 8\int \frac{1}{f^0(x)^5}$$
$$\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x)\right]dx$$
$$- 32\int \frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1^{'}(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x) - f^0(x)\right]dx$$
$$+ \mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) + \mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1^{'}(x)f^1(x)\right)dx\right)$$
$$+ \mathscr{O}\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)dx\right.$$
$$\left.+ \int\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)dx\right)$$
$$= 8\int \frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1^{'}(x)}{f^0(x)^5}\left[\hat{f}_g^1(x) - f^1(x)\right]dx$$
$$+ 8\int \frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+ 8\int \frac{\left(f^0\right)^{'}(x)^3\Psi_1^{'}(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+ 8\int \frac{\left(f^0\right)^{'}(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+ 8\int \frac{\Psi_1(x)\Psi_1^{'}(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+ 8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+ 8\int \frac{\Psi_1^{'}(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\hat{f}_g^1(x)\right]dx$$
$$+ 8\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^3 - \left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}^{'}(x) - \Psi_1^{'}(x)\right)\cdot\hat{f}_g^1(x)\right]dx - 32\int \frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1^{'}(x)f^1(x)}{f^0(x)^6}$$
$$\left[\hat{f}_g^0(x) - f^0(x)\right]dx + \mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right)$$
$$+ \mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1^{'}(x)f^1(x)\right)dx\right)$$

$$
\begin{aligned}
&+\mathscr{O}\bigg(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\\
&+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\bigg)\\
&=8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+8\int\frac{\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3\Psi_1'(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int\frac{\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int\frac{\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right]dx\\
&+8\int\frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+8\int\frac{\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&+8\int\frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+8\int\frac{f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right]dx\\
&+8\int\frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-32\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) + \mathcal{O}\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right. \\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^3 \hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x) - \left(f^0\right)'(x)^3 \Psi_1(x)\Psi_1'(x) f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right. \\
&\left.+\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right) \\
&= 8\int \frac{\left(f^0\right)'(x)^3 \Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+ 8\int \frac{\left(f^0\right)'(x)^3 \Psi_1(x) f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right] dx \\
&+ 8\int \frac{\left(f^0\right)'(x)^3 \Psi_1'(x) f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&+ 24\int \frac{\Psi_1(x)\Psi_1'(x) f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&- 32\int \frac{\left(f^0\right)'(x)^3 \Psi_1(x)\Psi_1'(x) f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+\mathcal{O}\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 + \left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right).
\end{aligned}
\tag{124}
$$

Collecting terms (110)-(124) and plugging them into (105), we obtain an expression for term B_2 (104), given by:

$$
\begin{aligned}
B_2 := 4 &\int \frac{\Psi_1^{'}(x)\Psi_1^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^3}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
+4 &\int \frac{\Psi_1^{'}(x)\Psi_1^{''}(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx \\
+4 &\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right]dx \\
+4 &\int \frac{\Psi_1^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right]dx \\
-4 &\int \frac{m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
-8 &\int \frac{\Psi_1^{'}(x)\Psi_1^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
+8 &\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
+8 &\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
+24 &\int \frac{\Psi_1^{'}(x)\Psi_1(x)f^1(x)\left(f^0\right)^{'}(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx \\
+8 &\int \frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right]dx \\
-4 &\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
-32 &\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{'}(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
-4 &\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx \\
-4 &\int \frac{\Psi_1^{'}(x)\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx
\end{aligned}
$$

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\begin{aligned}
&-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-12\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-16\int \frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^2f^1(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+24\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-8\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^4f^1(x)\Psi_1(x)}{f^0(x)^6}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-32\int \frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+8\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
&+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
&+8\int \frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
&+24\int \frac{\Psi_1(x)\Psi_1'(x)f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx \\
&-32\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right. \\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right. \\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \Big(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x) - \Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\Big)\,dx\Big)\\
&+\mathscr{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\,dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2\,dx\right)\\
&+\mathscr{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\,dx\right)\\
&+\mathscr{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\,dx\right)\\
&+\mathscr{O}\Big(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\\
&\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\Big) + \mathscr{O}\Big(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\\
&\cdot\left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\,dx\Big)\\
&+\mathscr{O}\Big(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\\
&\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\Big) + \mathscr{O}\Big(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\cdot\Big(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\\
&\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\Big)\,dx\Big)\\
&+\mathscr{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\,dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\,dx\right)
\end{aligned}
$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$
$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \qquad (125)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)^2-\Psi_1^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)^2-\Psi_1^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)^2-\Psi_1^{'}(x)^2\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-\Psi_1^{'}(x)^2\left(f^0\right)^{'}(x)^2f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)dx\right)
\end{aligned}
$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)+\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)+\mathscr{O}\left(\left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2\right)$$

$$+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)$$

$$+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)^{''}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^2\Psi_1(x)^2\left(f^0\right)^{''}(x)f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2dx+\int\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+\mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x) \right) dx \Big) \\
&+ \mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \right) \\
&+ \mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) dx \right) \\
&+ \mathcal{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \right) \\
&+ \mathcal{O}\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \right) \\
&+ \mathcal{O}\Big(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \\
&\cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x) \right) dx \Big) \\
&+ \mathcal{O}\Big(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x) \right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \Big) + \mathcal{O}\Big(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4 \right) \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x)^3 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}'(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^3 \Psi_1(x) \Psi_1'(x) f^1(x) \right) dx \Big) \\
&+ \mathcal{O}\Big(\int \left(\hat{f}_g^0(x) - f^0(x) \right) \\
&\cdot \left(\hat{m}_g'(x) \hat{m}_g''(x) \left(\hat{f}_g^0\right)'(x) \hat{f}_g^1(x) - m'(x) m''(x) \left(f^0\right)'(x) f^1(x) \right) dx \Big). \qquad (126)
\end{aligned}
$$

Finally, it remains to work out further calculations with term B_3. Considering expression (95), it turns out:

$$
\begin{aligned}
B_3 :=& \, 4\int \left[\frac{\hat{m}_g^{'}(x)^2 \left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^2} \right] dx \\
=& \, 4\int \frac{1}{f^0(x)^2} \left[\hat{m}_g^{'}(x)^2 \left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x) - m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x) \right] dx \\
& - 4\int \frac{m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^4} \left[\hat{f}_g^0(x)^2 - f^0(x)^2 \right] dx \\
& + \mathscr{O}\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right)^2 dx \right) + \mathscr{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) \right. \\
& \left. \cdot \left(\hat{m}_g^{'}(x)^2 \left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x) - m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x) \right) dx \right) \\
=& \, 4\int \frac{1}{f^0(x)^2} \left[\hat{m}_g^{'}(x)^2 \left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x) - m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x) \right] dx \\
& - 8\int \frac{m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
& + \mathscr{O}\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right)^2 dx \right) + \mathscr{O}\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&=4\int\frac{1}{f^0(x)^2}\left[\hat{m}_g'(x)^2\left(\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)+\left(f^0\right)'(x)^2f^1(x)-\left(f^0\right)'(x)^2f^1(x)\right)\right.\\
&\left.-m'(x)^2\left(f^0\right)'(x)^2f^1(x)\right]dx-8\int\frac{m'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&=4\int\frac{\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^2}\left[\hat{m}_g'(x)^2-m'(x)^2\right]dx\\
&+4\int\frac{1}{f^0(x)^2}\left[\hat{m}_g'(x)^2\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\left(f^0\right)'(x)^2f^1(x)\right)\right]dx\\
&-8\int\frac{m'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&=8\int\frac{\left(f^0\right)'(x)^2f^1(x)m'(x)}{f^0(x)^2}\left[\hat{m}_g'(x)-m'(x)\right]dx\\
&+4\int\frac{m'(x)^2}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\left(f^0\right)'(x)^2f^1(x)\right]dx\\
&+4\int\frac{1}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\left(f^0\right)'(x)^2f^1(x)\right)\right]dx\\
&-8\int\frac{m'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{m}_g'(x)-m'(x)\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&=8\int\frac{\left(f^0\right)'(x)^2f^1(x)m'(x)}{f^0(x)^2}\left[\frac{\hat{\Psi}_{1,g}'(x)}{\hat{f}_g^0(x)}-\frac{\Psi_1'(x)}{f^0(x)}\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^2}\left[\frac{\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x)}{\hat{f}_g^0(x)^2}-\frac{\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^2}\right]dx\\
&+4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot \hat{f}_g^1(x)\right]dx\\
&+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+4\int \frac{1}{f^0(x)^2}\\
&\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot \hat{f}_g^1(x)\right]dx\\
&-8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{m}_g'(x)-m'(x)\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)dx\right)\\
&=8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)'(x)\Psi_1(x)\right]dx\\
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)^2-f^0(x)^2\right]dx\\
&+4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int \frac{m'(x)^2 f^1(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx\\
&+4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right]dx\\
&+4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathscr{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathscr{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int\left(\hat{m}_g^{'}(x)-m^{'}(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{m}_g^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-m^{'}(x)^2\left(f^0\right)^{'}(x)^2f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)^{'}(x)\Psi_1(x)\right)dx\right)\\
&=8\int\frac{\left(f^0\right)^{'}(x)^2f^1(x)m^{'}(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right]dx\\
&-8\int\frac{\left(f^0\right)^{'}(x)^2f^1(x)m^{'}(x)\Psi_1^{'}(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int\frac{\left(f^0\right)^{'}(x)^2f^1(x)m^{'}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-16\int\frac{\left(f^0\right)^{'}(x)^2f^1(x)m^{'}(x)\left(f^0\right)^{'}(x)\Psi_1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+4\int\frac{m^{'}(x)^2\left(f^0\right)^{'}(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int\frac{m^{'}(x)^2f^1(x)\left(f^0\right)^{'}(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx\\
&-8\int\frac{\left(f^0\right)^{'}(x)^2f^1(x)m^{'}(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{\Psi}_{1,g}(x)\right]dx\\
&+4\int\frac{m^{'}(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int\frac{\left(f^0\right)^{'}(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int\frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right]dx\\
&+4\int\frac{1}{f^0(x)^2}\left[\left(\hat{m}_g^{'}(x)^2-m^{'}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^2-\left(f^0\right)^{'}(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-8\int\frac{m^{'}(x)^2\left(f^0\right)^{'}(x)^2f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{m}_g^{'}(x)-m^{'}(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{m}_g^{'}(x)^2\left(\hat{f}_g^0\right)^{'}(x)^2\hat{f}_g^1(x)-m^{'}(x)^2\left(f^0\right)^{'}(x)^2f^1(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)dx\right)\\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)^2dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathscr{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x)\Psi_1(x)\right)dx\right)\\
&= 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right]dx\\
&- 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x) - f^0(x)\right]dx\\
&- 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right]dx\\
&- 8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x) - f^0(x)\right]dx\\
&+ 4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x) - f^1(x)\right]dx\\
&+ 8\int \frac{m'(x)^2 f^1(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right]dx\\
&- 16\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x) - f^0(x)\right]dx\\
&- 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right]dx\\
&- 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right]dx\\
&+ 4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\right]dx\\
&+ 4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2 - m'(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\right]dx\\
&+ 4\int \frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2 - m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right]dx\\
&+ 4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2 - m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\right]dx + \mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) + \mathscr{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{m}_g'(x) - m'(x)\right)^2 dx\right) + \mathscr{O}\left(\int\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x) - m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)dx\right)\\
&+\mathscr{O}\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)dx\right)\\
&+\mathscr{O}\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x)\Psi_1(x)\right)dx\right)\\
&= 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi_1'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right] dx \\
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right] dx \\
&-8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right] dx \\
&+4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right] dx \\
&+8\int \frac{m'(x)^2 f^1(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right] dx \\
&-16\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right] dx \\
&-8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right] dx \\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right) dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right) dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{m}_g'(x)-m'(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right. \\
&\left.\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right) dx\right) \\
&+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)'(x)\Psi_1(x)\right) dx\right) \\
&+\mathcal{O}\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\right)+\mathcal{O}\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx\right) \\
&+\mathcal{O}\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2 dx\right)+\mathcal{O}\left(\int\left(\hat{m}_g'(x)^2-m'(x)^2\right)\right. \\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right) dx\right). \qquad (127)
\end{aligned}
$$

Bringing together terms (103), (126) and (127), and afterwards plugging them into (100), Lemma 7 is proven.

Furthermore, term r_{1,n_0} in Lemma 7 is given by:

$$
\begin{aligned}
r_{1,n_0} = & \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx \\
& + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \\
& \cdot \left(\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right) dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
& + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \\
& \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) \\
& \cdot \left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x) - \Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) + \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx + \int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)dx\\
&+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\\
&\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx+\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\\
&\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\\
&\left.-\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)
\end{aligned}
$$

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\begin{aligned}
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \\
&\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}'(x)^2 - \Psi_1'(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx
\end{aligned}
$$

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\begin{aligned}
&+\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)^2 dx+\int \left(\hat{f}^0_g(x)^3-f^0(x)^3\right) \\
&\cdot\left(\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)\right) dx \\
&+\int \left(\hat{f}^0_g(x)-f^0(x)\right)^2 dx+\int \left(\hat{f}^0_g(x)^2-f^0(x)^2\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)^2 dx \\
&+\int \left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx+\int \left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)^2 dx \\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx \\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right) dx \\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right) \\
&\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx+\int \left(\hat{f}^0_g(x)^3-f^0(x)^3\right)+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2 dx \\
&\cdot\left(\left(\hat{f}^0_g\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi''_1(x)f^1(x)\right) dx \\
&+\int \left(\hat{f}^0_g(x)-f^0(x)\right)^2 dx+\int \left(\hat{f}^0_g(x)^2-f^0(x)^2\right) dx \\
&+\int \left(\hat{f}^0_g(x)-f^0(x)\right)^2 dx+\int \left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^2-f^0(x)^2\right) dx \\
&+\int \left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^3-f^0(x)^3\right) dx \\
&+\int \left(\hat{\Psi}^2_{1,g}(x)-\Psi^2_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}^0_g\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}^2_{1,g}(x)-\Psi^2_1(x)\right) dx
\end{aligned}
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\begin{aligned}
&+\int \left(\left(\hat{f}_g^0\right)'(x)^4-\left(f^0\right)'(x)^4\right)\cdot\left(\hat{\Psi}_{1,g}^2(x)-\Psi_1^2(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\hat{f}_g^0(x)^5-f^0(x)^5\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)+\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx+\int \left(\hat{f}_g^0(x)^5-f^0(x)^5\right)^2dx\\
&+\int \left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx+\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\\
&\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\\
&\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int \left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\\
&\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx+\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx
\end{aligned}
$$

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\begin{aligned}
&+\int \left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)-\Psi_1^{'}(x)\right)\\
&\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx+\int \left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\\
&\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}^{'}(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1^{'}(x)f^1(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx+\int \left(\hat{m}_g^{'}(x)-m^{'}(x)\right)^2\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\\
&\cdot\left(\hat{m}_g^{'}(x)\hat{m}_g^{''}(x)\left(\hat{f}_g^0\right)^{'}(x)\hat{f}_g^1(x)-m^{'}(x)m^{''}(x)\left(f^0\right)^{'}(x)f^1(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}^{''}(x)-\Psi_1^{''}(x)\right)\,dx+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\,dx\\
&+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{''}(x)-\Psi_1(x)\left(f^0\right)^{''}(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}^{'}(x)\left(\hat{f}_g^0\right)^{'}(x)-\Psi_1^{'}(x)\left(f^0\right)^{'}(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)^{'}(x)^2-\Psi_1(x)\left(f^0\right)^{'}(x)^2\right)\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\,dx
\end{aligned}
$$

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\begin{aligned}
&+\int \left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\,dx\\
&+\int \left(\hat{m}_g''(x)^2-m''(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx+\int \left(\hat{m}_g''(x)-m''(x)\right)^2\,dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\,dx\\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)'(x)\Psi_1(x)\right)\,dx\\
&+\int \left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\,dx\\
&+\int \left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\,dx+\int \left(\hat{f}_g^0(x)-f^0(x)\right)^2\,dx. \qquad (128)
\end{aligned}
$$

Remark 3 Consider two random variables η, ξ. Thanks to Cauchy-Schwartz inequality, we have:

$$\mathbb{E}\left[(\eta+\xi)^2\right]\le 2\left(\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]\right). \qquad (129)$$

Similarly, given an additional random variable ζ, it follows that:

$$\mathbb{E}\left[(\eta+\xi+\zeta)^2\right]\le 3\left(\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]+\mathbb{E}\left[\zeta^2\right]\right). \qquad (130)$$

Proof (Proof of Remark 3) Using that $2\mathbb{E}[\eta]\mathbb{E}[\xi]\le\mathbb{E}[\eta]^2+\mathbb{E}[\xi]^2$, since $0\le(\mathbb{E}[\eta]-\mathbb{E}[\xi])^2=\mathbb{E}[\eta]^2+\mathbb{E}[\xi]^2-2\mathbb{E}[\eta]\mathbb{E}[\xi]$. Then,

$$
\begin{aligned}
0\le \mathbb{E}\left[(\eta+\xi)^2\right]&=\left|\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]+2\mathbb{E}[\eta\xi]\right|\\
&\le \mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]+\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]\\
&=2\left(\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
0 \leq \mathbb{E}\left[(\eta+\xi+\zeta)^2\right] &= \mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]+\mathbb{E}\left[\zeta^2\right] \\
&+2\mathbb{E}[\eta\xi]+2\mathbb{E}[\eta\zeta]+2\mathbb{E}[\zeta\xi] \\
&\leq \mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]+\mathbb{E}\left[\zeta^2\right]+\left(\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]\right) \\
&+\left(\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\zeta^2\right]\right)+\left(\mathbb{E}\left[\xi^2\right]+\mathbb{E}\left[\zeta^2\right]\right) \\
&= 3\left(\mathbb{E}\left[\eta^2\right]+\mathbb{E}\left[\xi^2\right]\right).
\end{aligned}
$$

Corollary 2 *Consider a pilot bandwidth $g>0$ of exact order $n_0^{-1/2}$, then:*

$$
\begin{aligned}
MISE^{a*}(h) &= \frac{R(K)}{n_0 h}\hat{A}_g+\frac{h^4}{4}\mu_2(K)^2\hat{B}_g+\mathscr{O}_P\left(h^6 n_1^{-1}\left(n_0^{7/2}+n_0^4+n_0^{9/2}\right)\right) \\
&+\mathscr{O}_P\left(h^{-1}n_0^{-1}n_1^{-1}\right)+\mathscr{O}_P\left(h n_1^{-1}\left(1+n_0^{1/2}+n_0\right)\right).
\end{aligned} \tag{131}
$$

Proof From now on, $C^{[s]}_{\nu,\ell,r}$, $\Psi^{(r)}_{s,\ell}$ will be denoted just by C and $\Psi^{(r)}_{\ell}$, respectively, for the sake of brevity, assuming $s=0$. We begin by analyzing the expectation of expression (32) in the paper,

$$
\begin{aligned}
\mathbb{E}\left[\hat{\Psi}^{(r)}_{\ell}(x)\right] &= \frac{1}{n_0 g^{r+1}}\sum_{i=1}^{n_0}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_i^0}{g}\right)Y_i^{0\ell}\right] \\
&= \frac{1}{g^{r+1}}\mathbb{E}\left[\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)Y_1^{0\ell}\middle|X_1^0\right]\right] \\
&= \frac{1}{g^{r+1}}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)\mathbb{E}\left[Y_1^{0\ell}\middle|X_1^0\right]\right] \\
&= \frac{1}{g^{r+1}}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)m_\ell(X_1^0)\right] \\
&= \int K_g^{(r)}(x-y)\,m_\ell(y)f^0(y)\,dy = K_g^{(r)}*\phi_\ell(x),
\end{aligned} \tag{132}
$$

where $\phi_\ell(u)=m_\ell(u)f^0(u)$. Considering now expression (132), we can compute further calculations with expression (32) in the paper. We denote by $\hat{\mu}_{i,g}(x) := K_g^{(r)}\left(x-X_i^0\right)Y_i^{0\ell} - K_g^{(r)}*\phi_\ell(x)$ and $\xi_i := \int \nu(x)\hat{\mu}_{i,g}(x)$. Then,

$$
\begin{aligned}
C &= \int \nu(x)\left(\frac{1}{n_0 g^{r+1}}\sum_{i=1}^{n_0}K^{(r)}\left(\frac{x-X_i^0}{g}\right)Y_i^{0\ell}-K_g^{(r)}*\phi_\ell(x)+K_g^{(r)}*\phi_\ell(x)-\Psi_\ell^{(r)}\right)dx \\
&= \int \nu(x)\left(\frac{1}{n_0}\sum_{i=1}^{n_0}\hat{\mu}_{i,g}(x)+K_g^{(r)}*\phi_\ell(x)-\Psi_\ell^{(r)}\right)dx \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}\int \nu(x)\hat{\mu}_{i,g}(x)\,dx+\int \nu(x)\left[K_g^{(r)}*\phi_\ell(x)-\Psi_\ell^{(r)}\right]dx \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}\xi_i+\int \nu(x)\left[K_g^{(r)}*\phi_\ell(x)-\Psi_\ell^{(r)}\right]dx.
\end{aligned} \tag{133}
$$

It follows that the MSE of $\hat{\Psi}_\ell^{(r)}(x)$ turns out to be, considering expression (133) and taking into account the fact that $\mathbb{E}[\xi_1]=0$:

$$\begin{aligned}\mathbb{E}\left[C^2\right] &= Var[C]+\mathbb{E}[C]^2 = \frac{Var[\xi_1]}{n_0}+\mathbb{E}[C]^2 = \frac{\mathbb{E}\left[\xi_1^2\right]}{n_0}+\mathbb{E}[C]^2 \\ &= \frac{\mathbb{E}\left[\xi_1^2\right]}{n_0}+\left(\int \nu(x)\left(K_g^{(r)}*\phi_\ell(x)-\Psi_\ell^{(r)}(x)\right)dx\right)^2. \end{aligned} \tag{134}$$

From now on, assume that f^0 and its derivatives tend to zero as $x\to\infty$ and m_ℓ and its derivatives are bounded as $x\to\infty$, where x is the point of evaluation. Focusing now on the second term of expression (134) leads to:

$$\begin{aligned}\int \nu(x)K_g^{(r)}*\phi_\ell(x)\,dx &= \int \nu(x)\int K_g^{(r)}(x-y)\phi_\ell(y)\,dy\,dx \\ &\overset{(1)}{=} \int \nu(x)\int K_g(x-y)\phi_\ell^{(r)}(y)\,dy\,dx \\ &= \int \phi_\ell^{(r)}(y)\int \nu(x)\frac{1}{g}K\left(\frac{x-y}{g}\right)dy\,dx \\ &= \int \phi_\ell^{(r)}(y)\int \nu(y+gu)K(u)\,du\,dy \\ &= \int \nu(x)\phi_\ell^{(r)}(x)\,dx+\frac{g^2}{2}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx \\ &\quad +\mathscr{O}\left(g^4\right). \end{aligned} \tag{135}$$

It is straightforward to proof equality (1) just by applying integration by parts to:

$$\begin{aligned} I &:= \int K_g^{(r)}(x-y)\phi_\ell(y)\,dy \\ &= \left[-K_g^{(r-1)}(x-y)\phi_\ell(y)\right]_{y=-\infty}^{y=+\infty}+\int_{-\infty}^{\infty}K_g^{(r-1)}(x-y)\phi_\ell'(y)\,dy \\ &= \int_{-\infty}^{\infty}K_g^{(r-1)}(x-y)\phi_\ell'(y)\,dy = \ldots = \int_{-\infty}^{\infty}K_g(x-y)\phi_\ell^{(r)}(y)\,dy. \end{aligned}$$

Combining the second term in (134) and expression (26) in the paper leads to conclude that:

$$\begin{aligned}\left(\int \nu(x)\left(K_g^{(r)}*\phi_\ell(x)-\Psi_\ell^{(r)}(x)\right)dx\right)^2 &= \left(\frac{g^2}{2}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right. \\ &\quad \left.+\mathscr{O}\left(g^4\right)\right)^2 \\ &= \frac{g^4}{4}\mu_2(K)^2\left(\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\ &\quad +\mathscr{O}\left(g^6\right). \end{aligned} \tag{136}$$

It remains to carry on with further computations with the first term of expression (134):

$$
\begin{aligned}
\frac{\mathbb{E}\left[\xi_1^2\right]}{n_0} &= \frac{1}{n_0}\mathbb{E}\left[\int \nu(x)\hat{\mu}_{1,g}(x)\,dx \int \nu(y)\hat{\mu}_{1,g}(y)\,dy\right] \\
&= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[\hat{\mu}_{1,g}(x)\hat{\mu}_{1,g}(y)\right]\,dx\,dy \\
&= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[\left(K_g^{(r)}\left(x-X_1^0\right)Y_1^{0\ell} - K_g^{(r)} * \phi_\ell(x)\right)\right. \\
&\cdot\left.\left(K_g^{(r)}\left(y-X_1^0\right)Y_1^{0\ell} - K_g^{(r)} * \phi_\ell(y)\right)\right]\,dx\,dy \\
&= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\left(Y_1^{0\ell}\right)^2\right]\,dx\,dy \\
&- \frac{1}{n_0}\left(\int \nu(x)K_g^{(r)} * \phi_\ell(x)\,dx\right)^2. \qquad (137)
\end{aligned}
$$

The second term in (137) has already been analyzed in (135), turning out:

$$
\begin{aligned}
\frac{1}{n_0}\left(\int \nu(x)K_g^{(r)} * \phi_\ell(x)\,dx\right)^2 = \frac{1}{n_0}&\left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
&+ \frac{g^2}{n_0}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx \\
&\int \nu(x)\phi_\ell^{(r)}(x)\,dx + \mathscr{O}\left(\frac{g^4}{n_0}\right). \qquad (138)
\end{aligned}
$$

Finally, it remains to work out further computations with:

$$
\begin{aligned}
&\frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\left(Y_1^{0\ell}\right)^2\right]\,dx\,dy \\
&= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\left(Y_1^{0\ell}\right)^2\middle|X_1^0\right]\right]\,dx\,dy \\
&= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\mathbb{E}\left[\left(Y_1^{0\ell}\right)^2\middle|X_1^0\right]\right]\,dx\,dy \\
&= \frac{1}{n_0}\int\int \nu(x)\nu(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)m_{2\ell}(X_1^0)\right]\,dx\,dy \\
&= \frac{1}{n_0}\int\int \nu(x)\nu(y)\int K_g^{(r)}(x-z)\,K_g^{(r)}(y-z)\,m_{2\ell}(z)f^0(z)\,dz\,dx\,dy \\
&= \frac{1}{n_0\,g^{2r+2}}\int\int \nu(x)\nu(y)\int K^{(r)}\left(\frac{x-z}{g}\right)K^{(r)}\left(\frac{y-z}{g}\right)m_{2\ell}(z)f^0(z)\,dz\,dx\,dy \\
&= \frac{1}{n_0\,g^{2r}}\int m_{2\ell}(z)f^0(z)\int\int \nu(z+g\,u)\nu(z+g\,v)K^{(r)}(u)\,K^{(r)}(v)\,du\,dv\,dz \\
&= \frac{2\,g^2}{r!\,(r+2)!\,n_0}\int u^r K^{(r)}(u)\,du\int u^{r+2}K^{(r)}(u)\,du\int \nu^{(r)}(z)\nu^{(r+2)}(z)m_{2\ell}(z)f^0(z)\,dz \\
&+ \frac{1}{(r!)^2\,n_0}\left(\int u^r K^{(r)}(u)\,du\right)^2\int \nu^{(r)}(z)^2 m_{2\ell}(z)f^0(z)\,dz + \mathscr{O}\left(\frac{g^4}{n_0}\right). \qquad (139)
\end{aligned}
$$

Assume K is a symmetric function. Therefore, K' and K'' are antisymmetric and symmetric functions, respectively. Thus, it is easy to proof by integration by parts that:

– If $r=0$,

$$\int K(u)\,du = 1, \int uK(u)\,du = 0, \int u^2K(u)\,du = \mu_2(K).$$

– If $r=1$,

$$\int K^{'}(u)\,du = 0, \int uK^{'}(u)\,du = -1, \int u^2K^{'}(u)\,du = 0,$$
$$\int u^3K^{'}(u)\,du = -3\,\mu_2(K).$$

– If $r=2$,

$$\int K^{''}(u)\,du = 0, \int uK^{''}(u)\,du = 0, \int u^2K^{''}(u)\,du = 2,$$
$$\int u^3K^{''}(u)\,du = 0, \int u^4K^{''}(u)\,du = 12\,\mu_2(K).$$

In general, $\int u^r K^{(r)}(u)\,du = (-1)^r\, r!$ and $\int u^{r+2}K^{(r)}(u)\,du = \frac{1}{2!}\,(-1)^r\,(r+2)!\mu_2(K)$.

Hence, expression (139) results in:

$$\frac{1}{n_0}\int \nu^{(r)}(z)^2 m_{2\ell}(z) f^0(z)\,dz$$
$$+\frac{g^2}{n_0}\mu_2(K)\int \nu^{(r)}(z)\nu^{(r+2)}(z) m_{2\ell}(z) f^0(z)\,dz + \mathscr{O}\left(\frac{g^4}{n_0}\right). \tag{140}$$

Using expression (129), our aim is to work out further computations for:

$$\mathbb{E}\left\{\left[\left(\hat{A}_g - A\right) + \frac{A}{4B}\left(\hat{B}_g - B\right)\right]^2\right\}$$
$$= \mathbb{E}\left[\left(\hat{A}_g - A\right)^2\right] + \frac{A^2}{16B^2}\mathbb{E}\left[\left(\hat{B}_g - B\right)^2\right] + \frac{A}{2B}\mathbb{E}\left[\left(\hat{A}_g - A\right)\cdot\left(\hat{B}_g - B\right)\right]$$
$$\leq 2\mathbb{E}\left[\left(\hat{A}_g - A\right)^2\right] + \frac{A^2}{8B^2}\mathbb{E}\left[\left(\hat{B}_g - B\right)^2\right]. \tag{141}$$

Finally, considering expression (130), collecting terms (136), (138) and (139), and afterwards plugging them into (134), it turns out that, $\forall \ell \in \{0,1,2\}, \forall r \in \{0,1,2\}$:

$$\mathbb{E}\left[C^2\right] = \frac{g^4}{4}\mu_2(K)^2\left(\int \nu^{''}(x)\phi_\ell^{(r)}(x)\,dx\right)^2$$
$$-\frac{g^2}{n_0}\mu_2(K)\int \nu^{''}(x)\phi_\ell^{(r)}(x)\,dx\int \nu(x)\phi_\ell^{(r)}(x)\,dx - \frac{1}{n_0}\left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2$$
$$+\frac{2\,g^2}{r!\,(r+2)!\,n_0}\int u^r K^{(r)}(u)\,du\int u^{r+2}K^{(r)}(u)\,du$$
$$\int \nu^{(r)}(x)\nu^{(r+2)}(x) m_{2\ell}(x) f^0(x)\,dx$$
$$+\frac{1}{(r!)^2\,n_0}\left(\int u^r K^{(r)}(u)\,du\right)^2\int \nu^{(r)}(x)^2 m_{2\ell}(x) f^0(x)\,dx$$
$$+\mathscr{O}\left(g^6\right) + \mathscr{O}\left(\frac{g^4}{n_0}\right). \tag{142}$$

In other words, considering expression (140) instead of (139):

$$\begin{aligned}
\mathbb{E}\left[C^2\right] = &\frac{g^4}{4}\left(\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
&+\frac{1}{n_0}\left[\int \nu^{(r)}(x)^2 m_{2\ell}(x) f^0(x)\,dx - \left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2\right] \\
&+\frac{g^2}{n_0}\mu_2(K)\left[\int \nu^{(r)}(x)\nu^{(r+2)}(x) m_{2\ell}(x) f^0(x)\,dx\right. \\
&\left.-\int \nu''(x)\phi_\ell^{(r)}(x)\,dx \int \nu(x)\phi_\ell^{(r)}(x)\,dx\right] \\
&+\mathscr{O}\left(g^6\right)+\mathscr{O}\left(\frac{g^4}{n_0}\right), \forall \ell \in \{0,1,2\}, \forall r \in \{0,1,2\}. \qquad (143)
\end{aligned}$$

Similarly, $\forall \ell \in \{0,1,2\}, \forall r \in \{0,1,2\}$,

$$\mathbb{E}\left[C^2\right] = g^4 C_1 + \frac{g^2}{n_0}C_2 + \frac{1}{n_0}C_3 + \mathscr{O}\left(g^6\right)+\mathscr{O}\left(\frac{g^4}{n_0}\right), \qquad (144)$$

where

$$\begin{aligned}
C_1 &:= \left(\frac{\mu_2(K)}{2}\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2, \qquad (145)\\
C_2 &:= \mu_2(K)\left[\int \nu^{(r)}(x)\nu^{(r+2)}(x) m_{2\ell}(x) f^0(x)\,dx\right. \\
&\quad \left.-\int \nu''(x)\phi_\ell^{(r)}(x)\,dx \int \nu(x)\phi_\ell^{(r)}(x)\,dx\right], \qquad (146)\\
C_3 &:= \int \nu^{(r)}(x)^2 m_{2\ell}(x) f^0(x)\,dx - \left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2. \qquad (147)
\end{aligned}$$

It is clear that C_1 in (145) is a non negative quantity. Let's focus now on C_2 in (146). Assume that f^0 and its derivatives tend to zero as $x \to \infty$; m_ℓ and ν and their derivatives are bounded as $x \to \infty$, where x is the point of evaluation. Thus, by applying integration by parts to the second term in (146), it turns out:

$$\begin{aligned}
\int \nu''(x)\phi_\ell^{(r)}(x)\,dx &= \left[\nu''(x)\phi_\ell^{(r-1)}(x)\right]_{-\infty}^{+\infty} - \int \nu'''(x)\phi_\ell^{(r-1)}(x)\,dx \\
&= -\int \nu'''(x)\phi_\ell^{(r-1)}(x)\,dx \\
&= -\left[\nu'''(x)\phi_\ell^{(r-2)}(x)\right]_{-\infty}^{+\infty} + \int \nu^{(4)}(x)\phi_\ell^{(r-2)}(x)\,dx \\
&= \int \nu^{(4)}(x)\phi_\ell^{(r-2)}(x)\,dx \\
&= \cdots = (-1)^r \int \nu^{(r+2)}(x)\phi_\ell(x)\,dx. \qquad (148)
\end{aligned}$$

Similarly,

$$\int \nu(x)\phi_\ell^{(r)}(x)\,dx = (-1)^r \int \nu^{(r)}(x)\phi_\ell(x)\,dx. \qquad (149)$$

Hence, expression (146) can be rewritten as:

$$
\begin{aligned}
C_2 := \mu_2(K) & \left[\int \nu^{(r)}(x)\nu^{(r+2)}(x) m_{2\ell}(x) f^0(x)\, dx \right. \\
& \left. -(-1)^{2r} \int \nu^{(r+2)}(x)\phi_\ell(x)\, dx \int \nu^{(r)}(x)\phi_\ell(x)\, dx \right] \\
& = \mu_2(K) \left[\int \nu^{(r)}(x)\nu^{(r+2)}(x) m_{2\ell}(x) f^0(x)\, dx \right. \\
& \left. - \int \nu^{(r+2)}(x) m_\ell(x) f^0(x)\, dx \int \nu^{(r)}(x) m_\ell(x) f^0(x)\, dx \right] \\
& = Cov\left[\nu^{(r)}(X^0) Y^{0\ell}, \nu^{(r+2)}(X^0) Y^{0\ell} \right]. \qquad (150)
\end{aligned}
$$

Considering now expressions (145) and (148), it follows that:

$$
C_1 := \left(\frac{\mu_2(K)}{2} \int \nu^{(r+2)}(x) m_\ell(x) f^0(x)\, dx \right)^2 = \left(\frac{\mu_2(K)}{2} \mathbb{E}\left[\nu^{(r+2)}(X^0) Y^{0\ell} \right] \right)^2.
$$

Finally, combining expressions (147) and (149) leads to:

$$
\begin{aligned}
C_3 & := \int \nu^{(r)}(x)^2 m_{2\ell}(x) f^0(x)\, dx - \left(\int \nu^{(r)}(x) m_\ell(x) f^0(x)\, dx \right)^2 \\
& = Var\left[\nu^{(r)}\left(X^0\right) Y^{0\ell} \right], \qquad (151)
\end{aligned}
$$

turning out that C_3 (147) is, as well, greater than 0.

As a consequence of expressions (145), (150) and (151), the optimal g will be determined by the positive or negative sign accompanying term C_2 in (150). In particular, denoting by $\vartheta(g) := \mathbb{E}\left[C^2\right]$ and considering expressions (91), (99) and (130), expression (141) could be rewritten as:

$$
\begin{aligned}
& \sum_{i=1}^{k_0} a_{0,i}^2 \vartheta_i(g) + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 \vartheta_i(g) + \sum_{i=1}^{k_2} a_{2,i}^2 \vartheta_i(g) + \sum_{i=1}^{k_3} a_{3,i}^2 \vartheta_i(g) \right) \\
& = \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] \cdot \vartheta_i(g) \\
& = \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] \cdot \left(g^4 C_{1,i} + \frac{g^2}{n_0} C_{2,i} + \frac{1}{n_0} C_{3,i} \right) \\
& = g^4 C_1^0 + \frac{g^2}{n_0} C_2^0 + \frac{1}{n_0} C_3^0,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] C_{1,i} \geq 0 \\
C_3^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] C_{3,i} \geq 0, \text{ and} \\
C_2^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] C_{2,i} \\
&= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] \\
&\quad \cdot Cov \left[\nu_i^{(r)}(X^0) Y^{0\ell}, \nu_i^{(r+2)}(X^0) Y^{0\ell} \right].
\end{aligned}
$$

If $C_2^0 \geq 0$, then the optimal pilot bandwidth g of the upper bound for $\hat{A}_g - A$ and $\hat{B}_g - B$ (expressions (91) and (128), respectively) is as close as 0 as possible. On the other hand, if $C_2^0 < 0$, then the optimal pilot bandwidth g of the upper bound for $\hat{A}_g - A$ and $\hat{B}_g - B$ (expressions (91) and (128), respectively) happens to be

$$
\begin{aligned}
g_{OPT} &\approx \arg\min_{g>0} \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] \cdot \vartheta_i(g) \\
&= \left(-\frac{C_2^0}{2C_1^0} \right)^{1/2} \cdot n_0^{-1/2}. \qquad (152)
\end{aligned}
$$

Moreover, it is straightforward that:

$$
\left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] \vartheta_i(g_{OPT}) = n_0^{-1} C_3^0 - n_0^{-2} \frac{\left(C_2^0\right)^2}{4C_1^0}.
$$

Combining expressions (67) and (152), Corollary 2 holds.

3 Additional material for the application in Section 4

In the following, Table 4 collects the sample sizes n_0 and n_1 from the real data application, while Table 5 contains the bandwidths for each sector, $h_{BOOT,Z}$.

	Studies					
CNAE	1	2	3	4	5	6
B	$n_{0,z} = 474$ $n_{1,z} = 13$	$n_{0,z} = 415$ $n_{1,z} = 14$	$n_{0,z} = 321$ $n_{1,z} = 39$	$n_{0,z} = 92$ $n_{1,z} = 23$	$n_{0,z} = 67$ $n_{1,z} = 28$	$n_{0,z} = 202$ $n_{1,z} = 97$
C	$n_{0,z} = 6592$ $n_{1,z} = 1712$	$n_{0,z} = 9276$ $n_{1,z} = 2778$	$n_{0,z} = 7479$ $n_{1,z} = 2222$	$n_{0,z} = 5096$ $n_{1,z} = 1227$	$n_{0,z} = 2134$ $n_{1,z} = 882$	$n_{0,z} = 3504$ $n_{1,z} = 2213$
D	$n_{0,z} = 43$ $n_{1,z} = 5$	$n_{0,z} = 67$ $n_{1,z} = 14$	$n_{0,z} = 219$ $n_{1,z} = 48$	$n_{0,z} = 348$ $n_{1,z} = 45$	$n_{0,z} = 360$ $n_{1,z} = 73$	$n_{0,z} = 419$ $n_{1,z} = 157$
E	$n_{0,z} = 1491$ $n_{1,z} = 145$	$n_{0,z} = 1636$ $n_{1,z} = 174$	$n_{0,z} = 609$ $n_{1,z} = 165$	$n_{0,z} = 350$ $n_{1,z} = 109$	$n_{0,z} = 186$ $n_{1,z} = 91$	$n_{0,z} = 354$ $n_{1,z} = 230$
F	$n_{0,z} = 3413$ $n_{1,z} = 88$	$n_{0,z} = 3148$ $n_{1,z} = 115$	$n_{0,z} = 1672$ $n_{1,z} = 338$	$n_{0,z} = 815$ $n_{1,z} = 189$	$n_{0,z} = 677$ $n_{1,z} = 267$	$n_{0,z} = 858$ $n_{1,z} = 424$
G	$n_{0,z} = 1044$ $n_{1,z} = 618$	$n_{0,z} = 2402$ $n_{1,z} = 1907$	$n_{0,z} = 1996$ $n_{1,z} = 1623$	$n_{0,z} = 635$ $n_{1,z} = 357$	$n_{0,z} = 346$ $n_{1,z} = 327$	$n_{0,z} = 913$ $n_{1,z} = 716$
H	$n_{0,z} = 1156$ $n_{1,z} = 171$	$n_{0,z} = 2402$ $n_{1,z} = 913$	$n_{0,z} = 1754$ $n_{1,z} = 640$	$n_{0,z} = 629$ $n_{1,z} = 196$	$n_{0,z} = 414$ $n_{1,z} = 210$	$n_{0,z} = 597$ $n_{1,z} = 374$
I	$n_{0,z} = 472$ $n_{1,z} = 526$	$n_{0,z} = 653$ $n_{1,z} = 601$	$n_{0,z} = 455$ $n_{1,z} = 372$	$n_{0,z} = 76$ $n_{1,z} = 63$	$n_{0,z} = 115$ $n_{1,z} = 157$	$n_{0,z} = 106$ $n_{1,z} = 117$
J	$n_{0,z} = 107$ $n_{1,z} = 61$	$n_{0,z} = 306$ $n_{1,z} = 169$	$n_{0,z} = 1385$ $n_{1,z} = 850$	$n_{0,z} = 950$ $n_{1,z} = 321$	$n_{0,z} = 1269$ $n_{1,z} = 522$	$n_{0,z} = 2540$ $n_{1,z} = 1783$
K	$n_{0,z} = 78$ $n_{1,z} = 115$	$n_{0,z} = 278$ $n_{1,z} = 227$	$n_{0,z} = 948$ $n_{1,z} = 916$	$n_{0,z} = 227$ $n_{1,z} = 352$	$n_{0,z} = 708$ $n_{1,z} = 786$	$n_{0,z} = 1934$ $n_{1,z} = 1815$
L	$n_{0,z} = 85$ $n_{1,z} = 66$	$n_{0,z} = 85$ $n_{1,z} = 73$	$n_{0,z} = 259$ $n_{1,z} = 225$	$n_{0,z} = 36$ $n_{1,z} = 53$	$n_{0,z} = 65$ $n_{1,z} = 88$	$n_{0,z} = 181$ $n_{1,z} = 199$
M	$n_{0,z} = 243$ $n_{1,z} = 172$	$n_{0,z} = 403$ $n_{1,z} = 411$	$n_{0,z} = 1022$ $n_{1,z} = 1273$	$n_{0,z} = 791$ $n_{1,z} = 606$	$n_{0,z} = 1077$ $n_{1,z} = 913$	$n_{0,z} = 3123$ $n_{1,z} = 2971$
N	$n_{0,z} = 1278$ $n_{1,z} = 842$	$n_{0,z} = 2857$ $n_{1,z} = 1390$	$n_{0,z} = 1627$ $n_{1,z} = 1118$	$n_{0,z} = 412$ $n_{1,z} = 333$	$n_{0,z} = 349$ $n_{1,z} = 593$	$n_{0,z} = 614$ $n_{1,z} = 670$
O	$n_{0,z} = 384$ $n_{1,z} = 206$	$n_{0,z} = 1339$ $n_{1,z} = 882$	$n_{0,z} = 1223$ $n_{1,z} = 1030$	$n_{0,z} = 373$ $n_{1,z} = 310$	$n_{0,z} = 457$ $n_{1,z} = 812$	$n_{0,z} = 721$ $n_{1,z} = 1019$
P	$n_{0,z} = 41$ $n_{1,z} = 73$	$n_{0,z} = 84$ $n_{1,z} = 120$	$n_{0,z} = 175$ $n_{1,z} = 281$	$n_{0,z} = 86$ $n_{1,z} = 183$	$n_{0,z} = 322$ $n_{1,z} = 1001$	$n_{0,z} = 985$ $n_{1,z} = 1239$
Q	$n_{0,z} = 460$ $n_{1,z} = 631$	$n_{0,z} = 615$ $n_{1,z} = 1144$	$n_{0,z} = 709$ $n_{1,z} = 2927$	$n_{0,z} = 223$ $n_{1,z} = 739$	$n_{0,z} = 551$ $n_{1,z} = 2752$	$n_{0,z} = 1173$ $n_{1,z} = 1712$
R	$n_{0,z} = 344$ $n_{1,z} = 188$	$n_{0,z} = 569$ $n_{1,z} = 327$	$n_{0,z} = 698$ $n_{1,z} = 441$	$n_{0,z} = 208$ $n_{1,z} = 92$	$n_{0,z} = 163$ $n_{1,z} = 137$	$n_{0,z} = 395$ $n_{1,z} = 414$
S	$n_{0,z} = 300$ $n_{1,z} = 242$	$n_{0,z} = 428$ $n_{1,z} = 354$	$n_{0,z} = 597$ $n_{1,z} = 546$	$n_{0,z} = 280$ $n_{1,z} = 164$	$n_{0,z} = 173$ $n_{1,z} = 266$	$n_{0,z} = 279$ $n_{1,z} = 425$

Table 4: Source sample size ($n_{0,z}$) and target sample size ($n_{1,z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group of belonging (B-S).

	Studies					
CNAE	1	2	3	4	5	6
B	10.34427	0.05311245	0.06555036	0.2250906	10.32863	11.46244
C	0.7827234	0.1570725	0.2240537	0.379917	1.036544	1.006386
D	10.23072	8.540002	11.09998	10.90716	11.87939	11.15287
E	4.034317	1.54504	3.425392	6.90935	8.561086	9.706537
F	0.1979207	0.1042053	0.1779684	0.7073216	2.416007	2.031799
G	10.59052	0.7064822	2.42826	5.206829	5.918734	10.74743
H	1.049699	0.8470396	2.464503	6.063806	6.250508	4.231824
I	7.865014	0.7109818	1.295192	4.747013	3.477528	2.816318
J	6.05535	1.539357	1.665482	0.6700639	1.620418	0.6657766
K	11.36344	11.8833	12.25746	11.09947	11.41423	6.882228
L	3.848873	1.101402	2.196453	3.877995	7.955811	5.016808
M	2.164264	1.776538	1.112405	1.194964	1.119651	0.4577655
N	0.4322325	0.1414672	0.193088	0.2586702	0.814847	0.4869169
O	10.67815	2.554323	11.71623	11.92024	10.88268	11.16583
P	10.27285	3.853947	10.83386	9.703337	8.053916	1.865665
Q	10.0757	4.310891	1.978999	9.339508	7.023242	3.641688
R	11.75589	4.868713	10.85612	10.14804	8.112695	9.912113
S	3.342381	0.8564468	5.826307	2.773484	4.111988	5.707007

Table 5: Bandwidth selector ($h_{BOOT,Z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group of belonging (B-S).