Analysed models

Source	Origin and function of the model	Features
Dewey (1910)	Often considered to be the first (modern) model of problem-solving processes.	Contains a first phase (*suggestions*) that is not represented in other models.
Wallas (1926)	Built on Poincaré's (1908) introspective reflection of his own problem-solving experiences.	Includes subconscious activities, especially *incubation* and *illumination*.
Pólya (1945)	Built on his own experiences as a problem solving mathematician as well as observations; *normative*.	Four phases, linear progression.
Mason, Burton, & Stacey (1982)	Developed within the course of a book with the focus on developing the readers' problem-solving competencies; *normative*.	Three phases (combining Pólya's (1945) *planning* and *carrying out* phases; focus on control / metacognition ("internal monitor").
Schoenfeld (1979; 1985, ch. 4)	Developed for his teaching problem solving at a university; *normative*.	Introduces a phase for heuristic searches (*exploration*).
Garofalo & Lester (1985); Lester, Garofalo, & Kroll (1989)	Theoretical considerations (1985) and empirical study with two seventh-grade classes with a focus on self-regulation (1989).	Four phases built on Pólya's (1945) phases ("More broadly defined") with a focus on metacognitive activities.
Wilson, Fernandez, & Hadaway (1993); Fernandez et al. (1994)	Developed and used for university teaching with a focus on pedagogical, curricular, and psychological aspects of problem solving; neither normative nor descriptive.	A "dynamic and cyclic" re-interpretation of Pólya's (1945) four-phase model.
Artzt & Armour-Thomas (1992)	Empirical study with 27 seventh-graders solving problems in groups. A model with a focus on cognition and metacognition as well as group work; *descriptive*.	Similar to Schoenfeld's (1985) model with the addition of a *watch and listen* phase addressing group work.
Leuders (2003)	Handbook for (pre-service) teachers; theoretically combining ideas from several models from the literature; neither normative nor descriptive.	In addition to Pólya's (1945) phases, the model contains *incubation* and *illumination* phases à la Wallas (1926).
Carlson & Bloom (2005)	Model designed to describe the problem-solving processes of 12 mathematicians, *descriptive*.	Four phases like Pólya's with "Conjecture – Imagine – Evaluate"-cycles in Planning phases.
Collet (2009; ch. 11 & 12)	Model derived from the literature, especially Pólya (1945) and Schoenfeld (1985)	Combines Pólya's *understanding* and *planning* phases; adds phases from a model for self-regulation.
Yimer & Ellerton (2010)	Empirical study with 17 pre-service teachers from a course on problem solving; three task-based interviews over the course of a semester with two problems (all from number theory) each; *descriptive*.	Five-phase model, cyclic.
Jacinto & Carreira (2017)	Merging of Schoenfeld's (1985) model with Martin and Grudziecki's (2006) model of digital technology. Case study with one 13-year-old student; *descriptive*.	Ten phases that reach a fine level of detail with some phases focusing on technology.

Organisation of the lecture

The course on *Elementary Geometry* was designed for students aspiring to become teachers for lower secondary schools. In addition to the weekly lecture, the students attended one of eight tutorials as shown in Fig. A1.

tutorial U1 paper-and-pencil (~30 students)	**tutorial Ulap2** DGS (~30 students)	**tutorial U3** paper-and-pencil (~30 students)	**tutorial Ulap4** DGS (~30 students)	**tutorial 5** paper-and-pencil (~30 students)	**tutorial 6** paper-and-pencil (~30 students)	**tutorial 7** paper-and-pencil (~30 students)	**tutorial 8** paper-and-pencil (~30 students)
U1-TV U1-A U1-B U1-C	Ulap2-TV Ulap2-A Ulap2-B Ulap2-C	U3-TV U3-A U3-B U3-C	Ulap4-TV Ulap4-A Ulap4-B Ulap4-C	not involved in the study	not involved in the study	not involved in the study	not involved in the study
(further students, not involved)	(further students, not involved)	(further students, not involved)	(further students, not involved)				

Fig. A1: The organisation of the tutorials

Four tutorials were involved in the study: U1, Ulap2, U3, and Ulap4 – “lap” being an abbreviation of “laptop computers” as in those two tutorials, the students worked with DGS. More precisely, in each of those four tutorials, four groups (named TV, A, B, or C), each consisting of three or four students, participated in the study (see Fig. A1).

Regarding our study, it was the plan to film the TV teams every time, i.e. for all five problems (see Fig. A2); the A, B, and C teams were supposed to be filmed in a rotating order: teams A for problems 1 and 4, teams B for problems 2 and 5, and teams C for problem 3. Thus, for each problem, there should be two groups (TV and one of A, B, or C) from each of the four tutorials. Because of students being absent due to illness or data loss because of technical difficulties, there are not 40 videotaped problem-solving processes (= 4 tutorials × 2 groups × 5 problems), but only 33 (15 paper-and-pencil groups from U1 and U3 as well as 18 DGS groups from Ulap2 and Ulap4) that were analysed.

Problems used in the study

(1) The Shortest Detour Given is a straight line *g* and two points *A* and *B* not on this line. Let *C* be any point on *g*, i.e. *C* can freely be chosen or moved on *g*. We are looking for the shortest detour from *A* via *C* to *B*. At which point C_{min} on the straight line *g* is the path minimal? Describe how to construct the point C_{min} exactly! Why is the way via C_{min} minimal?
(2) Locus Lines Problem Let *g* be a straight line in a plane and *A* be a point outside *g*. The point *P* moves along the line *g*. Determine the set of all points *X* of the plane, which together with *A* and *P* form the vertices of an equilateral triangle. Justify your solution!
(3) Airport problem Imagine that an airport is planned which should be easily accessible from three cities. a) The planners decide to position the airport so that it has the same distance from all three cities. Where should the airport be built? Justify your solution! b) Even before any measures for the construction of the airport have been implemented, there is a change of political power. The new government party is primarily oriented towards cost-cutting measures. It demands that as little money as possible is spent on the necessary roads. The planners therefore decide that the sum of the distances to the three cities should be minimal. Where must the airport be positioned? Justify your answer!
(4) Three Beaches Problem A woman lives on an island that has the shape of an equilateral triangle. On all three beaches, there are wonderful opportunities to surf. a) Where should she build her house so that the distances from the house to each of the beaches is identical? Where should she build her house so that the sum of the distances to all three beaches is minimal? (She surfs equally often on each beach.)
(5) Enlarge and Reduce **5 a) Construction problem:** Given two circles, one circle of the two can be mapped onto the other by scaling. Construct the centre of this scaling! There is a second point that can be chosen as the scaling centre. Construct this point as well! **5 b) And why?** Explain why the two points are actually centres of a scaling so that the circles can be mapped onto each other.

Fig. A2: The five problems used in our study

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