Supplementary material

Table A

Number of participants showing limited and strong evidence of different aspects when attending to student thinking in the first and last sessions

				No. of participants			
				First session		Last session	
Orientation	Aspect	Description	Example	Limited evidence	Strong evidence	Limited evidence	Strong evidence
Deficit-based	Error/mistake	Focuses on incorrect answers, computational errors, or flawed procedures in student work.	"Taylor incorrectly identifies the fraction as 1/10, which does not match his calculation. … He is not applying the principle of determining the greatest common divisor correctly."	4	1	2	0
	Failure	Emphasises what the student was unable to accomplish or complete correctly in relation to the task.	"He [Taylor] does not justify why he concludes both 1/10 and 5/12 are correct. He does not reconcile the discrepancy between the visual representation and the written calculation."	5	3	2	0
	Lack/gap	Identifies specific knowledge or skills that appear to be missing from	"His [Taylor's] work lacks an explicit explanation of how 4 twelfths plus 6 twelfths transitions to 10 twelfths."	7	5	3	0

		the student's mathematical understanding.					
	Misconception	References fundamental conceptual misunderstandings that appear to underlie the student's approach.	"Taylor thinks that the sum of the shaded parts of the circle is the denominator, rather than understanding it as the total number of parts of the circle. This suggests a misconception about numerator and denominator."	5	1	0	0
Strength-based	Ability	Recognises specific mathematical capabilities demonstrated by the student.	"Taylor correctly adds 1/4 and 1/6, demonstrating the skill of adding fractions symbolically."	14	5	4	16
	Strength	Emphasises particular mathematical strengths evidenced in student work.	"The visual representation demonstrates an understanding of how to decompose fractions into parts by shading and summing the areas, which reflects a foundational number sense."	5	3	6	9
Uncom-mitted	Student doing	Provides descriptive observations of the student's mathematical activity without evaluative judgment.	"In the circle, he [Taylor] shaded 1/4 horizontally and 1/6 vertically, added the shaded parts and identified 1/10 as the result."	5	15	5	18

Note. The numbers indicate the number of participants who demonstrated each aspect in their written responses. Limited evidence (coded as 1) refers to aspects that were present but not elaborated upon in the response; strong evidence (coded as 2) refers to aspects clearly articulated with specific reference to student work. Multiple aspects could be evident in a single participant's response.

Table B

Number of participants showing limited and strong evidence of different stances when interpreting student thinking in the first and last sessions

				No. of participants			
				First session		Last session	
Orientation	Stance	Description	Example	Limited evidence	Strong evidence	Limited evidence	Strong evidence
Deficit-based	Evaluative-negative	Makes judgmental assessments that emphasise shortcomings in the student's understanding.	"Taylor does not understand the relationship between the shaded areas and their fraction equivalence. He had a limited understanding of how fractions represent parts of a whole."	8	17	2	1
	Expectation-conflict	Focuses on contradictions or inconsistencies in student work with the assumption that students should recognise these discrepancies.	"Interestingly, he [Taylor] was not confused by the mismatch between his visual finding and his numerical result."	7	2	3	1
	Normative	Compares student work directly to normative standards with primary	"His [Taylor's] approach does not align with the usual approach for solving this kind of problem."	9	0	5	1

		focus on deviation from those standards.					
Strength-based	Evaluative-positive	Provides assessments that highlight achievements or progress in the student's mathematical thinking.	"He [Taylor] is … adept at partitioning and combining fractional parts."	7	7	9	10
	Interpretive-asset-based	Makes sense of student thinking in terms of the mathematical resources and capabilities they bring to the task.	"Taylor's partitioning strategy and his ability to sum fractions provide a strong basis for discussing fraction equivalence."	8	1	6	13
	Interpretive-in their own right	Understands student thinking on its own terms, acknowledging the underlying rationale shaping their reasoning.	"His [Taylor] approach to summing fractional parts reflects a consistent internal logic, even if it deviates from standard conventions."	9	1	9	4
Uncom-mitted	Assumption/ inference	Makes conjectures about student thinking without explicit judgment of correctness or quality.	"He [Taylor] might believe that the denominator changes when new parts are shaded, which could explain his response."	15	5	8	7
	Interpretive-non-evaluative	Makes sense of student work by interpreting their mathematical approach without evaluation.	"The two approaches (visual and calculation) are independent for him [Taylor]. Thus, he does not need to reconcile the two different results."	8	5	6	6

Note. The numbers indicate the number of participants who demonstrated each stance in their written responses. Limited evidence (coded as 1) refers to stances that were present but not elaborated upon in the response; strong evidence (coded as 2) refers to stances clearly articulated with specific reference to student work. Multiple stances could be evident in a single participant's response.

Table C

Number of participants showing limited and strong evidence of different instructional moves when responding to student thinking in the first and last sessions

				No. of participants			
				First session		Last session	
Orientation	Instructional move	Description	Example	Limited evidence	Strong evidence	Limited evidence	Strong evidence
Deficit-based	Challenging misconceptions	Directly addresses perceived conceptual misunderstandings by correcting or confronting them.	"I would challenge Taylor's approach by providing a circle divided into eighths and asking him to shade 3/8 and 5/8. … This would challenge his misunderstanding of the relationship between numerator and denominator."	5	4	1	1
	Flagging/ correcting errors	Points out or corrects mistakes or incorrect approaches without building on the student's existing thinking.	"I would point out the incorrect approach of translating 10 twelfths into 1/10 and explain what the numerator and denominator mean."	8	1	1	0
	Preventing obstacles	Intervenes to help students avoid potential	"I would introduce a step-by-step procedure for translating fractions from the symbolic to	7	6	3	2

		obstacles in future work.	the iconic representation in order to avoid such problems."				
	Redirecting understanding	Replaces the student's approach with a more conventional or standard method.	"I would guide the student [Taylor] by explaining what the denominator and numerator mean and how they can be visualised. Then, I would ask him to apply my explanation to the given task."	4	5	3	1
Strength-based	Accessing understanding	Designs questions or tasks specifically intended to elicit more information about the student's understanding.	"I would ask him [Taylor] to describe how he determined 4 twelfths, 6 twelfths and 10 twelfths. Then, I would ask, 'How are they represented in the circle?' This will help to determine if he understands the roles of the numerator and denominator in representing parts of a whole."	6	6	8	11
	Extending/ building upon understanding	Builds on the student's existing knowledge and approach to develop deeper understanding.	"I might say to Taylor, 'You have shown how 1/4 translates to 4 twelfths in the circle. What if we divided the circle differently, like into four parts instead of twelve? How might we describe the shaded portion then?' This encourages Taylor	9	5	5	16

			to consider how fractions relate to different partitions of the whole and builds on his existing understanding."				
	Positive reinforcement	Acknowledges and affirms productive aspects of student thinking, validating their mathematical efforts.	"I would … say [to Taylor] that he did a great job in explaining his visual approach."	3	2	8	6
Uncom-mitted	Clarifying student work	Seeks additional information about student thinking before proceeding with instruction.	"I might ask him [Taylor], 'Can you walk me through why you think 1/10 and 10 twelfths are the same?'"	4	3	4	3
	Giving general response	Offers generic instructional feedback or guidance that could apply to many students rather than addressing the particular student's mathematical reasoning.	"I would say, 'Check your final answer again. Let's review your steps together."	6	2	4	3

Note. The numbers indicate the number of participants who demonstrated each instructional move in their written responses. Limited evidence (coded as 1) refers to instructional moves that were present but not elaborated upon in the response; strong evidence (coded as 2) refers to instructional moves clearly articulated with specific reference to student work. Multiple instructional moves could be evident in a single participant's response.

Table D

Written noticing responses of Ari

	First session	Last session
Attending	"Taylor does not draw the correct number of twelfths when representing the fractions as parts of the circle. It is noticeable that he misinterprets the denominator of the quarter or the sixth (and the tenth) as the number of twelfths. When adding the fractions, he has no problems extending them correctly and expressing them as twelfths. When the teacher points out the difference between the two solutions, it becomes clear that Taylor does not recognise that both problems represent the same task or that he assumes that one task can have two different solutions."	"What is remarkable about Taylor's mathematical thinking is that he shaded the two fractions as parts of a circle, assigning a different shading style to each fraction. It is also clever that he added a legend to explain the shading styles. When looking at the circle, Taylor noticed that it is divided into twelve parts. In explaining how he solved the problem, Taylor interpreted the denominator of 1/4 and 1/6 as the number of twelfths and, based on this assumption, logically derived the quantities. He coherently noted the result as 1/10. It is also noticeable that Taylor's systematic approach to fraction addition reflects an understanding of how denominators relate to partitioning a whole into equivalent parts. Interestingly, when asked which of the two solutions is correct, Taylor viewed both answers as valid."
Interpreting	"Since Taylor interprets the denominators 4, 6, and 10 as the number of twelfths, he does not grasp the roles of the numerator and denominator, which suggests he has not developed a conceptual understanding of fractions. Specifically, he does not understand that the denominator describes the type of fraction, while the numerator describes the quantity. This also suggests that Taylor has not understood the addition of fractions but has instead memorised the algorithm. Taylor's addition	"Taylor's alignment of denominators reflects an understanding of equivalence, even if his interpretation of visual representations is incomplete. His logical approach to aligning denominators and adding fractions shows procedural fluency and emerging understanding, even if his interpretation of denominators is incomplete. Taylor's use of a legend shows his awareness that mathematical solutions should be communicated in a clear and understandable manner for others. His

	of fractions is correct, but it seems like he is just following an algorithm without understanding. He does not understand that the two tasks represent the same problem.”	coherent addition of fractions as twelfths demonstrates his basic logical skills. Taylor’s view of the tasks as distinct may stem from his focus on their differing representations rather than a lack of understanding of equivalence.”
Responding	“If I were Taylor’s teacher, I would explain again that the denominator shows how many parts the whole is divided into and use visual aids to illustrate this concept. Specifically, I would show him this using a circle as an example with sixths and then give him the task of dividing a circle into four equal parts, and then make the transition to the twelfths. I would explain that dividing a whole into one set of parts can also be expressed with other denominators, provided these are multiples of the first division. Then, I would ask Taylor to redraw the fractions correctly and check his work step-by-step. Finally, I would explain why his solutions differ and demonstrate how to correctly align the fractions using visual aids.”	“First, I would highlight how Taylor’s legend effectively clarifies his reasoning and use this as a starting point to discuss how visual representations connect to calculations. Then, I would ask Taylor what the 12 in his calculation represents and guide him to connect this to his visual representation. He then may realise that the number of parts is noted in the numerator and the type of fraction is named in the denominator. I could also point out to Taylor that he already made this association correctly in his written calculation. Using the circle, I would ask him to verify whether a quarter of a circle truly equals three twelfths of a circle and make a connection to the calculation. I would encourage Taylor to see the visual and numerical solutions as different representations of the same problem, using consistent notation.”

Table E

Written noticing responses of Sam

	First session	Last session
Attending	"Taylor was unable to differentiate between numerators and denominators in his drawings and did not make the fractions equal before drawing them. This shows that he struggles with understanding the basic principles of fractions, particularly finding a common denominator. For the total fraction, he added the shaded parts but again did not differentiate between numerators and denominators, leading to an incorrect visual solution. In the written solution to the task, he managed to carry out the correct steps and arrived at the right answer. However, in conversation with the teacher, he did not realise that he arrived at two different solutions and could not distinguish between the correct and incorrect answers. This confirms that Taylor has developed an automated approach to calculation without critically reflecting on the results."	"Taylor identified 1/4 as four parts of the whole and drew this according to his understanding. He did the same with 1/6. He translated 1/4 into four twelfths. Similarly, he translated 1/6 into six twelfths. He then added the shaded parts and arrived at ten twelfths, which he consistently expressed as 1 over 10 or 1/10. By shading the two fractions differently, he demonstrated structured thinking. His written calculation of fractions was effortless. He recognised the different denominators of the fractions given and made them equal before adding them. Taylor's recognition of 1/4 and 1/6 as parts of a whole demonstrates progress in understanding fraction equivalence. In the conversation with the teacher, he experienced no issue, as he has first counted and then calculated according to his conception."
Interpreting	"Taylor seems to have understood the steps for adding fractions and can perform them. However, based on his visual solution, he does not have a conceptual understanding of fractions and instead relies on an automated approach to calculation. His reliance on automated rules prevents him from adapting to unfamiliar problems. Taylor applies	"Taylor's logical process in representing 1/4 and 1/6 suggests that he is developing an understanding of how fractions relate to parts of a whole. His reasoning and execution of the task are highly consistent. He can correctly extend and add two fractions with different denominators. This clearly shows that he understands the basic principles of adding fractions

	automated steps without understanding the underlying concepts, as shown by his inability to distinguish between correct and incorrect solutions. This is evident in how he fails to notice that his first solution is incorrect and inconsistent with his second solution. Taylor cannot explain why his two solutions differ. This confirms my suspicion that Taylor lacks the ability to critically reflect on and evaluate his results. This inability to self-monitor and identify errors indicates that he has not internalised the mathematical concepts.”	and can apply them in his written work. Taylor uses the concepts available to him to develop a precise visual solution. Taylor’s consistent approach to shading parts and aligning fractions demonstrates his progress in connecting visual and numeric representations. Taylor has the potential to further develop his skills and deepen his understanding of mathematical principles. His structured thinking and consistent execution of visual and written solutions show a foundation for further conceptual growth.”
Responding	“I would have Taylor redraw the circles to correctly represent 1/4 and 1/6, ensuring he understands how to represent the fractions visually. Taylor should use a visual aid (such as a pie chart or a similar representation) to check his solutions and ensure they match the written calculations. These visual tools can make abstract concepts more tangible. I would show Taylor how to divide a circle into equivalent parts and explain why the denominators need to match. The numerator indicates how many parts are taken, while the denominator shows the total number of equal parts into which the whole is divided. Once this visual understanding is achieved, I would ask Taylor to compare his two solutions using this newly acquired knowledge. Through this comparison, he could realise that the two solutions are not the same and cannot both be correct.”	“I would praise Taylor for his efforts to articulate his ideas and ask him to verbally explain his approach to the visual solution again. Specifically, I would ask: What does the shading in your diagram tell us about the fractions you added? Can you explain how the numbers relate to the parts you drew? We could then discuss these questions together, particularly in dialogue with other students in the group. I would then guide Taylor to articulate how shading relates to his written solution, helping him see the connections between his representations. Taylor’s ability to align fractions shows his progress; I would encourage him to explore how this relates to equivalence.”