1 On line Appendix to "Agency theory meets matching theory," by Inés Macho-Stadler and David Pérez-Castrillo

Proof of Proposition 6. a) Substituting the efforts in the program that maximizes the principal, and rewriting $(PCLT)$, we obtain

$$\max_{(B,F_H,s_H,F_L,s_L)} \left\{ q\left(1-s_H\right)s_H \frac{R_i^2\left(p_H+\Theta\right)^2}{v} + (1-q)(1-s_L)s_L \frac{R_i^2\left(p_L+\Theta\right)^2}{v} - \frac{1}{\delta}B - qF_H - (1-q)F_L \right\}$$

$$\text{s.t.} \quad -\frac{1}{\delta}B - qF_H - (1-q)F_L =$$

$$q\frac{s_H^2 R_i^2\left(p_H+\Theta\right)^2}{2v} - q\frac{s_H^2 r\sigma^2}{2} + (1-q)\frac{s_L^2 R_i^2\left(p_L+\Theta\right)^2}{2v} - (1-q)\frac{s_L^2 r\sigma^2}{2} - \frac{1}{\delta}\left(1+\delta\right)U^o.$$

We rewrite the principal's program as

$$\max_{(s_H,s_L)} \left\{ q\frac{\left(2-s_H\right)s_H R_i^2\left(p_H+\Theta\right)^2}{2v} + (1-q)\frac{(2-s_L)s_L R_i^2\left(p_L+\Theta\right)^2}{2v} \right.$$
$$\left. -q\frac{s_H^2 r\sigma^2}{2} - (1-q)\frac{s_L^2 r\sigma^2}{2} - \frac{1+\delta}{\delta}U^o \right\}$$

whose FOCs are

$$\begin{aligned}
\frac{\partial}{\partial s_H} &= q\left(1-s_H\right)\frac{R_i^2\left(p_H+\Theta\right)^2}{v} - qs_H r\sigma^2 = 0 \Leftrightarrow s_H = \frac{R_i^2\left(p_H+\Theta\right)^2}{R_i^2\left(p_H+\Theta\right)^2 + rv\sigma^2} \\
\frac{\partial}{\partial s_L} &= 0 \Leftrightarrow s_L = \frac{R_i^2\left(p_L+\Theta\right)^2}{R_i^2\left(p_L+\Theta\right)^2 + rv\sigma^2}.
\end{aligned}$$

Moreover, any vector of fixed-payments (B, F_H, F_L) that satisfy $(PCLT)$ (and which ensures that the senior agent obtains at least U^o in both states of the world, which is always possible) is equivalent for both the principal and the agent.

b) The expressions for e_H and e_L follow easily.

c) Finally, $E\widetilde{\pi}^{LT}\left(R_i\right) = q\frac{(2-s_H)s_H(R_i(p_H+\Theta))^2}{2v} + (1-q)\frac{(2-s_L)s_L(R_i(p_L+\Theta))^2}{2v} - q\frac{s_H^2 r\sigma^2}{2} - (1-q)\frac{s_L^2 r\sigma^2}{2} - \frac{1+\delta}{\delta}U^o$. Substituting s_H and s_L by their expression at the optimum, and after some calculations, we obtain $E\widetilde{\pi}^{LT}\left(R_i\right)$ as stated in the proposition.

Proof of Proposition 7. a) Substituting the ICC in the program, we obtain

$$\max_{(F_I,s_I)}\left\{(1-s_I)\,s_I\frac{R_i^2p_I^{\sharp 2}}{v}-F_I\right\}$$
$$\text{s.t. } F_I+\frac{1}{2v}\left(s_IR_ip_I^{\sharp}\right)^2-\frac{1}{2}rs_I^2\sigma^2\geq U_I$$

Note also that it is straightforward that the PC is binding, so we can rewrite the principal's profits as

$$\max_{s_I}\left\{(1-s_I)\,s_I\frac{R_i^2p_I^{\sharp 2}}{v}+\frac{1}{2v}s_I^2R_i^2p_I^{\sharp 2}-\frac{1}{2}rs_I^2\sigma^2-U_I\right\}$$

and from the FOC of this program we obtain:

$$s_I^{\sharp}=\frac{R_i^2p_I^{\sharp 2}}{R_i^2p_I^{\sharp 2}+rv\sigma^2}.$$

The expression for $F_I^{\sharp}$ easily follows from the binding PC and the optimal s_I.

b) The expression for the effort follows easily.

c) Consider a principal that signs ST contracts and specializes in seniors of type H. Then, with probability q she will keep the junior that she hired in the previous period (because this agent is high ability) but with probability $(1-q)$ she has to hire a senior who worked for another principal when junior (and hence he does not have principal-specific training). If she hires a senior with productivity $p_H^{\sharp}$ (where either $p_H^{\sharp}=p_H+\Theta$ or $p_H^{\sharp}=p_H$), her profits are:

$$\left(1-s_H^{\sharp}\right)s_H^{\sharp}\frac{R_i^2p_H^{\sharp 2}}{v}+\frac{1}{2v}s_H^{\sharp 2}R_i^2p_H^{\sharp 2}-\frac{1}{2}rs_H^{\sharp 2}\sigma^2-U_H=$$
$$\frac{1}{2v}R_i^4p_{\;\;H}^{\sharp 4}{}_{H}^{\sharp 2}\frac{1}{R_i^2p_H^{\sharp 2}+rv\sigma^2}\left(2-\frac{2R_i^2p_H^{\sharp 2}}{R_i^2p_H^{\sharp 2}+rv\sigma^2}+\frac{R_i^2p_H^{\sharp 2}}{R_i^2p_H^{\sharp 2}+rv\sigma^2}-\frac{1}{R_i^2p_H^{\sharp 2}+rv\sigma^2}rv\sigma^2\right)-U_H=$$
$$\frac{1}{2v}\frac{R_i^4p_H^{\sharp 4}}{R_i^2p_H^{\prime 2}+rv\sigma^2}-U_H.$$

Given that the probability that $p_H^{\sharp}=p_H+\Theta$ is q, and also taking into account the (discounted) cost of the junior agent, her one-period profit is:

$$E\widetilde{\pi}_H^{ST}(R_i,B,U_H)=q\frac{1}{2v}\frac{R_i^4(p_H+\Theta)^4}{R_i^2(p_H+\Theta)^2+rv\sigma^2}+(1-q)\frac{1}{2v}\frac{R_i^4p_H^4}{R_i^2p_H^2+rv\sigma^2}-U_H-\frac{1}{\delta}B=$$
$$\frac{1}{2v}R_i^4\left(q\frac{(p_H+\Theta)^4}{R_i^2(p_H+\Theta)^2+rv\sigma^2}+(1-q)\frac{p_H^4}{R_i^2p_H^2+rv\sigma^2}\right)-U_H-\frac{1}{\delta}B.$$

Similar calculations lead to an expression of the expected one-period profit for a principal that hires agents with a low industry-specific ability, taking into account that the probability that the senior agent has a principal-specific ability is now $(1-q)$.

Proof of Proposition 8. Before we prove the proposition, we prove in Claim 1 the properties on the derivatives of the differences between LT and ST profits.

Claim 1 *i) The function* $E\widetilde{\pi}^{LT}(R_i) - E\widetilde{\pi}_L^{ST}(R_i, B = U^o - q\delta(U_H - U^o), U_L = U^o)$ *is increasing in* R_i.
ii) The function $E\widetilde{\pi}_H^{ST}(R_i, B = U^o - q\delta(U_H - U^o), U_H) - E\widetilde{\pi}^{LT}(R_i)$ *is increasing in* R_i.

Proof of Claim 1. i) Denoting $k \equiv rv\sigma^2$ and $x \equiv R_i^2$, we can write the difference in profits as:

$$\frac{1}{2v}x^2\left(\frac{q(p_H+\Theta)^4}{x(p_H+\Theta)^2+k} - \frac{qp_L^4}{xp_L^2+k}\right) - q(U_H - U^o). \tag{A1}$$

The derivative of the function depicted in $(A1)$ with respect to x is positive if:

$$\frac{x^2(p_H+\Theta)^6 + 2x(p_H+\Theta)^4 k}{\left(x(p_H+\Theta)^2+k\right)^2} - \frac{x^2p_L^6 + 2xp_L^4 k}{xp_L^2+k} > 0. \tag{A2}$$

Equation $(A2)$ is satisfied given that $p_H + \Theta > p_L$ and

$$\frac{\partial}{\partial d}\left(\frac{x^2d^6 + 2xd^4k}{(xd^2+k)^2}\right) = \frac{2x^3d^7 + 6x^2d^5k + 8xd^3k^2}{(xd^2+k)^3} > 0. \tag{A3}$$

ii) Proceeding as in the proof of i), the difference in profits is:

$$\frac{1}{2v}x^2\left(\frac{p_H^4}{xp_H+k} - \frac{(p_L+\Theta)^4}{x(p_L+\Theta)^2+k}\right). \tag{A4}$$

The function in $(A4)$ is increasing in x because $p_H > p_L + \Theta$ and then the derivative of $(A4)$ with respect to x is the similar to the derivative of $(A1)$ with respect to x. ■

We now prove Proposition 8.

a) If there is an equilibrium with only ST contracts then there is a principal i such that principals with $R < R_i$ hire low-ability seniors and principals with $R > R_i$ hire high-ability seniors. The principal with the threshold level R_i should be indifferent between the two types of contract, that is, $E\widetilde{\pi}_H^{ST}(R_i, B = U^o - q\delta(U_H - U^o), U_H) = E\widetilde{\pi}_L^{ST}(R_i, B = U^o - q\delta(U_H - U^o), U_L = U^o)$. Then, easy calculations lead to

$$U_H = \frac{1}{2v}R_i^4\left(\frac{q(p_H+\Theta)^4}{R_i^2(p_H+\Theta)^2 + rv\sigma^2} + \frac{(1-q)p_H^4}{R_i^2p_H + rv\sigma^2} - \frac{qp_L^4}{R_i^2p_L^2 + rv\sigma^2} - \frac{(1-q)(p_L+\Theta)^4}{R_i^2(p_L+\Theta)^2 + rv\sigma^2}\right) + U^o$$

and the profit under any of the two types of ST contract is

$$\frac{1}{2v}R_i^4\left(\frac{q(1-q)p_L^4}{R_i^2p_L^2+rv\sigma^2}+\frac{(1-q)^2(p_L+\Theta)^4}{R_i^2(p_L+\Theta)^2+rv\sigma^2}+\frac{q^2(p_H+\Theta)^4}{R_i^2(p_H+\Theta)^2+rv\sigma^2}+\frac{q(1-q)p_H^4}{R_i^2p_H+rv\sigma^2}\right)-\frac{(1+\delta)}{\delta}U^o.$$

Given this expression for the profits, the profit under LT contracts is larger than the profit under ST contracts for R_i if and only if

$$\frac{q(1-q)(p_H+\Theta)^4}{R_i^2(p_H+\Theta)^2+rv\sigma^2}+\frac{q(1-q)(p_L+\Theta)^4}{R_i^2(p_L+\Theta)^2+rv\sigma^2}>\frac{q(1-q)p_H^4}{R_i^2p_H+rv\sigma^2}+\frac{q(1-q)p_L^4}{R_i^2p_L^2+rv\sigma^2},$$

which holds for any $\Theta>0$.

b) We rewrite equation (6) as

$$\frac{1}{2v}R^{L4}\left(\frac{qp_L^4}{R^{L2}p_L^2+rv\sigma^2}+\frac{(1-q)(p_L+\Theta)^4}{R^{L2}(p_L+\Theta)^2+rv\sigma^2}\right)-U^o-\frac{1}{\delta}\left(U^o-q\delta\left(U_H-U^o\right)\right)=$$
$$\frac{1}{2v}R^{L4}\left(\frac{q(p_H+\Theta)^4}{R^{L2}(p_H+\Theta)^2+rv\sigma^2}+\frac{(1-q)(p_L+\Theta)^4}{R^{L2}(p_L+\Theta)^2+rv\sigma^2}\right)-\frac{1}{\delta}(1+\delta)U^o,$$

that is,

$$U_H-U^o=\frac{1}{2v}R^{L4}\left(\frac{(p_H+\Theta)^4}{R^{L2}(p_H+\Theta)^2+rv\sigma^2}-\frac{p_L^4}{R^{L2}p_L^2+rv\sigma^2}\right). \tag{A5}$$

Similarly, equation (7) is

$$\frac{1}{2v}R^{H4}\left(\frac{q(p_H+\Theta)^4}{R^{H2}(p_H+\Theta)^2+rv\sigma^2}+\frac{(1-q)p_H^4}{R^{H2}p_H^2+rv\sigma^2}\right)-U_H-\frac{1}{\delta}\left(U^o-q\delta\left(U_H-U^o\right)\right)=$$
$$\frac{1}{2v}R^{H4}\left(\frac{q(p_H+\Theta)^4}{R^{H2}(p_H+\Theta)^2+rv\sigma^2}+\frac{(1-q)(p_L+\Theta)^4}{R^{H2}(p_L+\Theta)^2+rv\sigma^2}\right)-\frac{1}{\delta}(1+\delta)U^o$$

that is,

$$U_H-U^o=\frac{1}{2v}R^{H4}\left(\frac{p_H^4}{R^{H2}p_H^2+rv\sigma^2}-\frac{(p_L+\Theta)^4}{R^{H2}(p_L+\Theta)^2+rv\sigma^2}\right). \tag{A6}$$

Therefore, an equilibrium with the coexistence of ST and LT contracts exists if we can find R^L and R^H with $\underline{R}<R^L<R^H<\overline{R}$, and U_H that satisfy (5), $(A5)$, and $(A6)$.

Equations $(A5)$ and $(A6)$ imply

$$R^{H4}\left(\frac{p_H^4}{R^{H2}p_H^2+rv\sigma^2}-\frac{(p_L+\Theta)^4}{R^{H2}(p_L+\Theta)^2+rv\sigma^2}\right)=R^{L4}\left(\frac{(p_H+\Theta)^4}{R^{L2}(p_H+\Theta)^2+rv\sigma^2}-\frac{p_L^4}{R^{L2}p_L^2+rv\sigma^2}\right). \tag{A7}$$

Equation $(A7)$ implicitly defines a function $R^H(R^L)$. We prove several properties of this function through the following claims:

Claim 2 *$R^H(R^L) > R^L$ for any R^L and any $\Theta > 0$.*

Proof of Claim 2. Denote $k \equiv rv\sigma^2$, $x \equiv R^{L2}$, and $y \equiv R^{H2}$. Then, rewrite $(A7)$ as

$$y^2\left(\frac{p_H^4}{yp_H^2+k}-\frac{(p_L+\Theta)^4}{y\,(p_L+\Theta)^2+k}\right)=x^2\left(\frac{(p_H+\Theta)^4}{x\,(p_H+\Theta)^2+k}-\frac{p_L^4}{xp_L^2+k}\right) \tag{A8}$$

and equation $(A8)$ implicitly defines the function $y(x)$. Proving $R^H(R^L) > R^L$ is equivalent to proving $y(x) > x$. Define

$$h(z,t) \equiv z^2\left(\frac{(p_H+t)^4}{z\,(p_H+t)^2+k}-\frac{(p_L+\Theta-t)^4}{z\,(p_L+\Theta-t)^2+k}\right).$$

Then, we can write $(A8)$ as

$$h(z=y, t=0) = h(z=x, t=\Theta). \tag{A9}$$

Notice:

$$\frac{\partial h(z,t)}{\partial t}=z^2\frac{2\,(p_H+t)^5\,z+4\,(p_H+t)^3\,k}{\left(z\,(p_H+t)^2+k\right)^2}+\frac{2\,(p_L+\Theta-t)^5\,z+4\,(p_L+\Theta-t)^3\,k}{\left(z\,(p_L+\Theta-t)^2+k\right)^2}>0$$

and the derivative of $h(z,t)$ with respect to z is similar to the derivative of the function in $(A1)$ with respect to x; hence $\frac{\partial h(z,t)}{\partial z} > 0$ for any z and any $t \in [0,\Theta]$.

Therefore, $\frac{\partial h(z,t)}{\partial t} > 0$ and $\frac{\partial h(z,t)}{\partial z} > 0$ which, together with $h(z = y(x), t = 0) = h(z = x, t = \Theta)$, imply that $y(x) > x$ whenever $\Theta > 0$. ■

Claim 3 *The function $R^H(R^L)$ is increasing.*

Proof of Claim 3. Given that $R^H(R^L) = \sqrt{y\,(R^{L2})}$, $R^{H\prime}\left(R^L\right) = \frac{1}{2}\frac{2R^L y'\left(R^{L2}\right)}{\sqrt{y(R^{L2})}}$. Therefore, $R^H(R^L)$ is increasing if $y(x)$ is increasing. To prove that $y(x)$ is increasing, note that as stated above, we can implicitly define the function $y(x)$ as $h(y,0) - h(x,\Theta) = 0$. Therefore,

$$\frac{\partial h(z,t)}{\partial z}\Big|_{(y,0)}\,dy-\frac{\partial h(z,t)}{\partial z}\Big|_{(x,\Theta)}\,dx=0.$$

We have shown that $\frac{\partial h(z,t)}{\partial z} > 0$ for any z and any $t \in [0,\Theta]$, hence $\frac{dy}{dx} > 0$. ■

Claim 4 *i) If $\Theta = 0$, then $R^H(R^L) = R^L$ for any R^L. In particular, $R^H\left(\underline{R}\right) = \underline{R}$.*
ii) $R^H\left(\underline{R}\right)$ increases with Θ.

Proof of Claim 4. Part i) trivially follows from equation $(A8)$.

For part ii), and following $(A7)$, define $R^H(\underline{R})$ as a function of Θ as $m\left(R^H(\underline{R}),\Theta\right) = 0$, where

$$m\left(R^H,\Theta\right) \equiv R^{H4}\left(\frac{p_H^4}{R^{H2}p_H^2+k} - \frac{(p_L+\Theta)^4}{R^{H2}(p_L+\Theta)^2+k}\right) - \underline{R}^4\left(\frac{(p_H+\Theta)^4}{\underline{R}^2(p_H+\Theta)^2+k} - \frac{p_L^4}{\underline{R}^2p_L^2+k}\right)$$

and where we have denoted $k = rv\sigma^2$. Then,

$$\frac{\partial m\left(R^H,\Theta\right)}{\partial\Theta} = -R^{H4}\frac{2R^{H2}(p_L+\Theta)^5+4(p_L+\Theta)^3k}{\left(R^{H2}(p_L+\Theta)^2+k\right)^2} - \underline{R}^4\frac{2\underline{R}^2(p_H+\Theta)^5+4(p_H+\Theta)^3k}{\left(\underline{R}^2(p_H+\Theta)^2+k\right)^2} < 0$$

and

$$\frac{\partial m\left(R^H,\Theta\right)}{\partial R^H} > 0$$

where the last inequality follows the same logic as $\frac{\partial h(z,t)}{\partial z} > 0$ above. Therefore, $\frac{dR^H(\underline{R})}{d\Theta} > 0$, given that

$$\frac{\partial m\left(R^H,\Theta\right)}{\partial R^H}\Big|_{(R^H(\underline{R}),\Theta)} dR^H(\underline{R}) + \frac{\partial m\left(R^H,\Theta\right)}{\partial\Theta}\Big|_{(R^H(\underline{R}),\Theta)} d\Theta = 0.$$

■

Claim 5 *If $\Theta > 0$, then there is an equilibrium with LT and ST contracts simultaneously iff $R^H(\underline{R}) < \overline{R}$.*

Proof of Claim 5. Any such equilibrium exists if and only if we can find R^L and R^H with $\underline{R} < R^L < R^H < \overline{R}$ satisfying $(A7)$ and (5).

Consider the function $R^o\left(R^L\right)$ implicitly defined by $G\left(R^L\right) = 1 - G(R^o)$. There is an equilibrium with LT and ST contracts simultaneously iff $R^H\left(R^L\right)$ and $R^o\left(R^L\right)$ intersect in a point $\left(R^L,R^H\right)$ with $\underline{R} < R^L < R^H < \overline{R}$. Claim 2 ensures that any intersect satisfies $R^L < R^H$ if $\Theta > 0$. The facts that $R^H\left(R^L\right)$ is increasing (Claim 3), $R^o\left(R^L\right)$ is decreasing whenever $R^L < R^o\left(R^L\right)$, and $R^o(\underline{R}) = \overline{R}$ imply that the functions intersect at one and only one interior point if and only if $R^H(\underline{R}) < \overline{R}$. ■

Therefore, Claim 4 implies that there is an equilibrium with LT and ST contracts simultaneously if and only if $\Theta \leq \Theta^o$, where Θ^o is implicitly defined by equation (8). Note that we can write equation (8) as $m(R^H = \overline{R}, \Theta = \Theta^o) = 0$. There is only one Θ^o that satisfies the equation because $\frac{\partial m(R^H=\overline{R},\Theta)}{\partial\Theta} < 0$, $m(R^H = \overline{R}, \Theta = 0) > 0$ and $m(R^H = \overline{R}, \Theta = p_H - p_L) < 0$.

c) Given that there is no equilibrium involving ST contracts if $\Theta > \Theta^o$, the only equilibrium is one where every principal signs LT contracts.