Supplementary Material
Enhancing Pairwise Comparisons in Randomized Block Designs through Covariate Post-Stratification

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Proof of Lemma 1: We first show the unbiasedness of the estimator. We define the vector $\boldsymbol{I} = (I_{i11}, \ldots, I_{inn}, i = 1, \ldots, b)$ which represents the membership of rows to set type 1 and 2. By conditioning on $\boldsymbol{I}$, we can express the following

$$E(\tilde{\Delta}_{1(t,t')}) \quad = \quad E\{E(T_1|\boldsymbol{I})\} + E\{E(T_2|\boldsymbol{I})\}.$$

We now consider

$$\begin{aligned} E\{E(T_1|\boldsymbol{I})\} &= E\left[\sum_{i=1}^{b}\sum_{j=1}^{n} E\left\{\frac{I_{ijj}d_{ijj}}{m_{t,t'}D_{t,t'}}|I_{ijj}=1\right\}\right] \\ &= E\left[\left\{\frac{I_1}{D_{t,t'}}\right\}(\mu_1-\mu_2+\sigma(\phi_{[1]}-\phi_{[2]})\right] = \frac{1}{2}(\mu_1-\mu_2+\sigma\delta_1), \end{aligned}$$

where $I_1 = 1$ if the set type 1 is not empty and zero otherwise.

In a similar fashion, we can show that

$$E\{E(T_2|\boldsymbol{I})\} = \frac{\mu_1 - \mu_2 + \sigma\delta_2}{2}.$$

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Putting two conditional expectations together, we have

$$E(\tilde{\Delta}_{1(t,t')}) = E\{E(T_1|\boldsymbol{I})\} + E\{E(T_2|\boldsymbol{I}) = \mu_1 - \mu_2 = \Delta$$

which shows that the estimator $\tilde{\Delta}_{1(t,t')}$ is unbiased for $\Delta_{t,t'}$.

To prove the variance, we use total variance formula

$$Var(\tilde{\Delta}_{1(t,t')}) = E\{Var(\tilde{\Delta}_{1(t,t')}|\boldsymbol{I})\} + Var\{E(\tilde{\Delta}_{1(t,t')}|\boldsymbol{I})\}.$$

We first consider

$$\begin{aligned} Var\{E(\tilde{\Delta}_{1(t,t')}|\boldsymbol{I})\} &= Var\{E(T_1|\boldsymbol{I})\} + Var\{E(T_2|\boldsymbol{I})\} \\ &\quad +2Cov\left\{E(T_1|\boldsymbol{I}), E(T_2|\boldsymbol{I})\right\}. \end{aligned} \tag{1}$$

The first term in the right side of the above equation equals

$$Var\{E(T_I|\boldsymbol{I})\} = Var(\frac{I_1}{D_{t,t'}})[\mu_1 - \mu_2 + \sigma\delta_1]^2$$

Using the part (iii) of Lemma 2 in Ozturk (2014), we obtain $Var(I_1/D_{t,t'}) = (1/2)^{nb+1}$ and write

$$Var\{E(T_I|\boldsymbol{I})\} = \frac{1}{2^{nb+1}}[\mu_1 - \mu_2 + \sigma\delta_1]^2. \tag{2}$$

In a similar fashion, we can show

$$Var\{E(T_2|\boldsymbol{I})\} = \frac{1}{2^{nb+1}}[\mu_1 - \mu_2 + \sigma\delta_2]^2. \tag{3}$$

We now consider the covariance between the conditional expected values of T_1 and T_2 given $\boldsymbol{I}$

$$\begin{aligned} Cov\left\{E(T_1|\boldsymbol{I}), E(T_2|\boldsymbol{I})\right\} &= Cov(\frac{I_1}{D_{t,t'}}, \frac{I_2}{D_{t,t'}})\left\{(\mu_1-\mu_2)+\sigma\delta_1)\right\}\left\{(\mu_1-\mu_2)-\sigma\delta_1)\right\} \\ &= Cov(\frac{I_1}{D_{t,t'}}, \frac{I_2}{D_{t,t'}})[(\mu_1-\mu_2)^2-\sigma^2\delta_1^2], \end{aligned}$$

where $I_2 = 1$ if the set type 2 is not empty and zero otherwise. The part (iv) of Lemma 2 in Ozturk (2014) gives that $Cov(\frac{I_1}{D_{t,t'}}, \frac{I_2}{D_{t,t'}}) = -(1/2)^{nb+1}$. Using this expression in the above equation we obtain

$$Cov\left\{E(T_1|\boldsymbol{I}), E(T_2|\boldsymbol{I})\right\} = -(1/2)^{nb+1}\{(\mu_1-\mu_2)^2-\sigma^2\delta_1^2\}. \tag{4}$$

Combining equations (2), (3) and (4) in equation (1), we have

$$Var\{E(\hat{\Delta}_{JPS}|\boldsymbol{I})\} = \frac{\sigma^2}{2^{nb-1}}\{\phi_{[1]}-\phi_{[2]}\}^2. \tag{5}$$

We now consider

$$E\{Var(\hat{\Delta}_{1(t,t')}|\boldsymbol{I})\} = E\{Var(T_1|\boldsymbol{I})\} + E\{Var(T_2|\boldsymbol{I})\}.$$

By conditioning on $\boldsymbol{I}$, we compute

$$\begin{aligned} E\{Var(T_I|\boldsymbol{I})\} &= E\{\frac{I_1^2}{m_{t,t'}D_{t,t'}^2}\}\{\sigma_{[1]}^2+\sigma_{[2]}^2-2\sigma_{[1,2]}\} \\ &= C_{nb}^2\{\sigma_{[1]}^2+\sigma_{[2]}^2-2\sigma_{[1,2]}\}. \end{aligned}$$

For a set of random variable $\{\sigma\epsilon_1, \sigma\epsilon_2\}$ with mean zero and variance σ^2, we have

$$2\sigma^2 = Var(\sigma\epsilon_1+\sigma\epsilon_2) = Var(\sigma\epsilon_{[1]}+\sigma\epsilon_{[2]}) = \sigma_{[1]}^2+\sigma_{[2]}^2+2\sigma_{[1,2]},$$

where $\epsilon_{[1]}$ and $\epsilon_{[1]}$ the first and second judgment order statistics in a sample of size 2. The solution of the above equation for 2 $\sigma_{[1,2]}$ yields

$$2\sigma_{[1,2]} = 2\sigma^2 - \sigma_{[1]}^2 - \sigma_{[2]}^2.$$

Using this expression, we write

$$E\{Var(T_I|\boldsymbol{I})\} = C_n^2 \left\{2\sigma_{[1]}^2 + 2\sigma_{[2]}^2 - 2\sigma^2\right\}.$$

The part v of Lemma 2 in Ozturk (2014) provides an analytic expression for C_n^2.

$$C_{nb}^2 = E\{\frac{I_1^2}{m_{t,t'}D_{t,t'}^2}\} = \frac{1}{2^{nb}}\left[\frac{1}{nb} + \sum_{n_1=1}^{nb-2+1}\frac{\binom{nb}{n_1}}{2^2 n_1}\right].$$

Using the equality $\sigma_{[1]}^2 + \sigma_{[2]}^2 = 2\sigma^2 - \sigma^2\sum_{j=1}^{2}\phi_{[j]}^2$, we write

$$E\{Var(T_I|\boldsymbol{I})\} = 2C_{nb}^2\sigma^2\{1 - \sum_{j=1}^{2}\phi_{[j]}\}.$$

In a similar fashion, we can show that

$$E\{Var(T_2|\boldsymbol{I})\} = 2C_{nb}^2\sigma^2\{1 - \sum_{j=1}^{2}\phi_{[j]}^2\}.$$

Combining these two terms, we have

$$E\{Var(\tilde{\Delta}_{1(t,t')}|\boldsymbol{I})\} = 4C_{nb}^2\sigma^2\{1 - \sum_{j=1}^{2}\phi_{[j]}^2\}.$$

The variance of $\tilde{\Delta}_{1(,t,t')}$ is then given by

$$Var(\tilde{\Delta}_{1(,t,t')}) = \frac{\sigma^2}{2^{nb-1}}(\phi_{[1]} - \phi_{[2]})^2 + 4C_{nb}^2\sigma^2\{1 - \sum_{j=1}^{2}\phi_{[j]}^2\}$$

which completes the proof.

Proof of Lemma 2: We first consider the expected values of U_i^2, i=1,2,3. The expected value of U_1^2 can be constructed by conditioning on $\boldsymbol{I}$

$$\begin{aligned}
E(U_1^2) &= E\{E(U_1^2|\boldsymbol{I})\} = E(\frac{I(m_{t,t'} > 1)m_{t,t'}(m_{t,t'}-1)}{m_{t,t'}(m_{t,t'}-1)})E\left\{(d_{111} - d_{122})^2|I_{111}, I_{122}\right\} \\
&= E\{I(m_{t,t'} > 1)\}E\left\{[(\mu_1 + \epsilon_{1[1]} - \mu_2 - \sigma\epsilon_{1[2]}) - (\mu_1 + \sigma\epsilon_{2[1]} - \mu_2 - \sigma\epsilon_{2[2]})]^2\right\} \\
&= E\{I(m_{t,t'} > 1)\}\sigma^2 E\left\{[(\epsilon_{1[1]} - \epsilon_{1[2]}) - (\epsilon_{2[1]} - \epsilon_{2[2]})]^2\right\} \\
&= E\{I(m_{t,t'} > 1)\}\sigma^2 \left\{E(\epsilon_{1[1]} - \epsilon_{1[2]})^2 + E(\epsilon_{2[1]} - \epsilon_{2[2]})^2\right. \\
&\quad \left. -2E(\epsilon_{1[1]} - \epsilon_{1[2]})(\epsilon_{2[1]} - \epsilon_{2[2]})\right\} \\
&= E\{I(m_{t,t'} > 1)\}\left\{2(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}) + 2\sigma^2(\phi_{[1]} - \phi_{[2]})^2 - 2\sigma^2(\phi_{[1]} - \phi_{[2]})^2\right\} \\
&= 2E\{I(m_{t,t'} > 1)\}(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}).
\end{aligned}$$

Using similar argument, we can show that

$$E(U_2^2) = E\{E(U_2^2|\boldsymbol{I})\} = 2E\{I(n_2 > 1)\}(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}),$$

where $n_2 = nb - m_{t,t'}$. We now consider the expected value of U_3^2

$$E(U_3^2) = E\left\{E(U_3^2|\boldsymbol{I})\right\}.$$

For the conditional expectation $E(U_3^2|\boldsymbol{I})$ we write

$$\begin{aligned}
E(U_3^2|\boldsymbol{I}) &= \frac{I(0 < m_{t,t'} < nb)m_{t,t'}n_2}{m_{t,t'}n_2} E\left\{(d_{111} - d_{122})^2 | I_{111} = 1, I_{122} = 0\right\} \\
&= I(0 < m_{t,t'} < nb)\left\{E(d_{111}^2|I_{111} = 1) + E(d_{122}^2|I_{122} = 0)\right. \\
&\quad \left. -2E(d_{111}d_{122}|I_{111} = 1, I_{122} = 0)\right\} \\
&= I(0 < m_{t,t'} < nb)\left\{2(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}) + 4\sigma^2(\phi_{[1]} - \phi_{[2]})^2\right\}.
\end{aligned}$$

Using the above conditional expectation we have

$$E(U_3^2) = E(0 < m_{t,t'} < nb)\left\{2(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}) + 4\sigma^2(\phi_{[1]} - \phi_{[2]})^2\right\}.$$

We now first consider the expected value of S_1^2

$$\begin{aligned}
E(S_1^2) &= E\left\{(1 < m_{t,t'} < nb)\left(U_3^2 - \frac{U_1^2 + U_2^2)}{2}\right) + I(m_{t,t'} = 1)\left(U_3^2 - U_2^2\right) + I(n_2 = 1)\left(U_3^2 - U_1^2\right)\right\} \\
&= E(I(1 < m_{t,t'} < nb))\left\{2(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}) + 4\sigma^2(\phi_{[1]} - \phi_{[2]})^2\right. \\
&\quad \left. -2(\sigma_{[1]} + \sigma_{[2]}^2 - 2\sigma_{[1,2]})\right\} \\
&+ E(I(m_{t,t'} = 1))\left\{2(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}) + 4\sigma^2(\phi_{[1]} - \phi_{[2]})^2\right\} \\
&+ E(I(n_2 = 1))\left\{2(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}) + 4\sigma^2(\phi_{[1]} - \phi_{[2]})^2\right\} \\
&= 4\sigma^2((\phi_{[1]} - \phi_{[2]})^2\left\{EI(1 < m_{t,t'} < nb) + E(I(m_{t,t'} = 1)) + E(I(n_2 = 1))\right\}.
\end{aligned}$$

Since $m_{t,t'}$ and n_2 have a binomial distribution with success probability 1/2 and sample size nb, expected value of S_1^2 reduces to following form

$$E(S_1^2) = 4\sigma^2(\phi_{[1]} - \phi_{[2]})^2\left\{1 - \frac{2}{2^{nb}} - \frac{2nb}{2^{nb}} + \frac{2nb}{2^{nb}}\right\} = 4\sigma^2(\phi_{[1]} - \phi_{[2]})^2\left\{1 - \frac{2}{2^{nb}}\right\}.$$

Since $K_{nb} = \frac{1}{4(2^{(nb-1)})(1-2^{-(nb-1)})}$, we have

$$E(K_{nb}S_1^2) = 4K_{nb}\sigma^2(\phi_{[1]} - \phi_{[2]})^2 \left\{1 - \frac{2}{2^{nb}}\right\} = \frac{\sigma^2}{2^{(nb-1)}}(\phi_{[1]} - \phi_{[2]})^2.$$

We now consider $E(S_2^2)$

$$\begin{aligned} E(S_2^2) &= E\left\{I(1 < m_{t,t'} < nb-1)\frac{[U_1^2 + U_2^2]}{2} + I(m_{t,t'} \leq 1)U_2^2 + I(m_{t,t'} \geq nb-1\ )U_1^2\right\} \\ &= 2\gamma^2\left(E\{I(1 < m_{t,t'} < nb-1)\} + E\{I(m_{t,t'} \leq 1)\} + E\{I(m_{t,t'} \geq nb-1)\}\right) \\ &= 2\gamma^2 = 2(\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}). \end{aligned}$$

It is now clear that

$$E(C_{nb}S_2^2) = 2C_{nb}\left\{\sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}\right\} = 4C_{nb}\sigma^2\left(1 - \sum_{j=1}^{2}\phi_{[j]}^2\right)$$

which completes the proof.

Proof of Theorem 1: The sample size $m_{t,t'}$ in type 1 set is random variable. This requires to consider joint distribution of sample $m_{t,t'}$ and difference of treatment and control group sample averages. Let

$$\boldsymbol{V}_{ij} = (I_{ijj}d_{ijj}, (1 - I_{ijj})d_{ijj}, I_{ijj}, 1 - I_{ijj})^\top, i = 1, \ldots, b, j = 1, \ldots, n$$

be a four dimentional random vector. Expected value and variance of the vector $\boldsymbol{V}_{ij}$ can be established

$$\begin{aligned} E(\boldsymbol{V}_{ij}) &= E\{E(\boldsymbol{V}_{ij}|I_{ijj})\} = (\delta_{1tt'}/2, \delta_{2tt'}/2, 1/2, 1/2)^\top = (\theta_1, \theta_2, \theta_3, \theta_4)^\top \\ Var(\boldsymbol{V}_{ij}) &= E\{Var(\boldsymbol{V}_{ij}|I_{ijj})\} + Var\{E(\boldsymbol{V}_{ij}|I_{ijj})\} = \boldsymbol{A} + \boldsymbol{B}, \end{aligned}$$

where

$$\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{B} = \frac{1}{4}\begin{bmatrix} \delta_{1tt'}^2 & -\delta_{1tt'}\delta_{tt'}2 & \delta_1 & -\delta_{1tt'} \\ -\delta_{1tt'}\delta_{2tt'} & \delta_2^2 & -\delta_{2tt'} & \delta_{2tt'} \\ \delta_{1tt'} & -\delta_{2tt'} & 1 & -1 \\ -\delta_{1tt'} & \delta_{2tt'} & -1 & 1 \end{bmatrix}$$

and $\gamma = \sigma_{[1]}^2 + \sigma_{[2]}^2 - 2\sigma_{[1,2]}$. Let $\bar{\boldsymbol{V}}_{nb} = \frac{1}{nb}\sum_{i=1}^{b}\sum_{j=1}^{n}\boldsymbol{V}_{ij}$. Since $\bar{\boldsymbol{V}}_{nb}$ is the average of independent identically distributed variables, $\sqrt{2nb}\left\{\bar{\boldsymbol{V}}_{nb} - E(\bar{\boldsymbol{V}}_{nb})\right\}$ converges to multivariate normal distribution with mean $\mathbf{0}$ and variance $2(\boldsymbol{A}+\boldsymbol{B})$. Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^\top$ be a four-dimensional vector. We define a function

$$H(\boldsymbol{x}) = \frac{1}{2}(\frac{x_1}{x_3} + \frac{x_2}{x_4}).$$

It is clear that

$$H(\bar{\boldsymbol{V}}_{nb}) = \frac{1}{2}\left(\frac{\sum_{i=1}^{b}\sum_{j=1}^{n} I_{ijj}d_{ijj}}{m_{t,t'}} + \frac{\sum_{i=1}^{b}\sum_{j=1}^{n}(1-I_{i,i})d_{ijj}}{nb - m_{t,t'}}\right)$$

and

$$H(\boldsymbol{\theta}) = \mu_t - \mu_{t'} = \Delta_{t,t'}.$$

Using first order Taylor expansion of function $H(.)$ around $\boldsymbol{\theta}$, we have

$$H(\bar{\boldsymbol{V}}_n) \approx H(\boldsymbol{\theta}) + H'(\boldsymbol{\theta})(\bar{\boldsymbol{V}}_n - \boldsymbol{\theta}),$$

where

$$H'(\boldsymbol{\theta}) = \frac{1}{2}(\frac{1}{\theta_3} + \frac{1}{\theta_4} - \frac{\theta_1}{\theta_3^2} - \frac{\theta_2}{\theta_4^2}) = (1, 1, -\delta_{1tt'}, -\delta_{2tt'})$$

For large sample sizes we then conclude that $\sqrt{2nb}\left\{H(\bar{\boldsymbol{V}}_n) - H(\boldsymbol{\theta})\right\}$ converges to a normal distribution with mean zero and variance $2H'(\boldsymbol{\theta})\left\{\boldsymbol{A}+\boldsymbol{B}\right\}(H'(\boldsymbol{\theta}))^\top$. It can be shown that

$$2H'(\boldsymbol{\theta})\left\{\boldsymbol{A}+\boldsymbol{B}\right\}(H'(\boldsymbol{\theta}))^\top = 2\gamma = 4\sigma^2\left\{1-\sum_{i=1}^{2}\phi_{[i]}^2\right\}$$

which completes the proof.

Proof of Theorem 2: Using the generalized U-statistics (Lee, 1990), we can write

$$\begin{aligned}\sum_{i=1}^{b}\sum_{j=1}^{n_t}E(\boldsymbol{W}|C_{ij}) &= \frac{1}{bn_t}\sum_{i=1}^{b}\sum_{j=1}^{n_t}\boldsymbol{Z}_{1,ij}\\ \sum_{i=1}^{b}\sum_{k=1}^{n_{t'}}E(\boldsymbol{W}|L_{ik}) &= \frac{1}{bn_{t'}}\sum_{i=1}^{b}\sum_{k=1}^{n_{t'}}\boldsymbol{Z}_{2,ik}.\end{aligned}$$

We note that $\boldsymbol{Z}_{1,ij}$, and $Z_{2,ik}$ are independent. It follows that

$$\begin{aligned}Var\left(\sqrt{N_{tt'}}\left\{\sum_{i=1}^{b}\sum_{j=1}^{n_t}E(\boldsymbol{W}|C_{ij})-\boldsymbol{\theta})\right\}\right) &= \frac{k_1^2}{\lambda_t}\boldsymbol{\Sigma}_1\\ Var\left(\sqrt{N_{tt'}}\left\{\sum_{i=1}^{b}\sum_{k=1}^{n_{t'}}E(\boldsymbol{W}|L_{ik})-\boldsymbol{\theta}\right\}\right) &= \frac{k_2^2}{\lambda_{t'}}\boldsymbol{\Sigma}_2,\end{aligned}$$

putting these two expression together we have that $\sqrt{N_{tt'}}(\boldsymbol{U}-\boldsymbol{\theta})$ converges in distribution to a multivariate normal distribution with mean $\mathbf{0}$ and variance

$$\boldsymbol{\Sigma} = \frac{k_1^2}{\lambda_t}\boldsymbol{\Sigma}_1 + \frac{k_2^2}{\lambda_{t'}}\boldsymbol{\Sigma}_2.$$

Using the function $G(\boldsymbol{x}) = \frac{1}{2}(\frac{x_1}{x_3}+\frac{x_2}{x_4})$ along with Delta method we have

$$\sqrt{N_{tt'}}\{G(\boldsymbol{W})-G(\boldsymbol{\theta})\} = \sqrt{N_{tt'}}(\tilde{\Delta}_{2,t,t'}-\Delta_{t,t'}) \xrightarrow{D} N(0, G'(\boldsymbol{\theta})\boldsymbol{\Sigma}G'^\top(\boldsymbol{\theta})),$$

where $G'(\boldsymbol{\theta}) = (1, 2, -\delta_{1tt'}, -\delta_{2tt'})$. Applying Slutsky's theorem we have

$$\frac{\sqrt{N_{tt'}}(\tilde{\Delta}_{2,t,t'} - \Delta_{t,t'})}{\sqrt{G'(\boldsymbol{\theta}^*)\boldsymbol{\Sigma} G'^{\top}(\boldsymbol{\theta}^*)}} \xrightarrow{D} N(0,1),$$

where

$$G'(\boldsymbol{\theta}^*) = \frac{bn_t n_{t'}}{2}(\frac{1}{m_{t,t'}}, \frac{1}{bn_t n_{t'} - m_{t,t'}}, \frac{-\delta_{1tt'}}{m_{t,t'}}, \frac{-\delta_{2tt'}}{bn_t n_{t'} - m_{t,t'}}).$$

We now derive explicit expressions for the entries of $\boldsymbol{\Sigma}_1$ and Σ_2

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} \sigma_{1,0}^{2(1,1)} & \sigma_{1,0}^{2(1,2)} & \sigma_{1,0}^{2(1,3)} & \sigma_{1,0}^{2(1,4)} \\ & \sigma_{1,0}^{2(2,2)} & \sigma_{1,0}^{2(2,3)} & \sigma_{1,0}^{2(2,4)} \\ & & \sigma_{1,0}^{2(3,3)} & \sigma_{1,0}^{2(3,4)} \\ & & & \sigma_{1,0}^{2(4,4)} \end{pmatrix}, \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} \sigma_{0,1}^{2(1,1)} & \sigma_{0,1}^{2(1,2)} & \sigma_{0,1}^{2(1,3)} & \sigma_{0,1}^{2(1,4)} \\ & \sigma_{0,1}^{2(2,2)} & \sigma_{0,1}^{2(2,3)} & \sigma_{0,1}^{2(2,4)} \\ & & \sigma_{0,1}^{2(3,3)} & \sigma_{0,1}^{2(3,4)} \\ & & & \sigma_{0,1}^{2(4,4)} \end{pmatrix},$$

where

$$\sigma_{1,0}^{2(r,r')} = Cov\{\psi^{(r)}(C_{ij}, L_{ik}), \psi^{(r')}(C_{ij}, L_{ik'})\}, \quad \sigma_{0,1}^{2(r,r')} = Cov\{\psi^{(r)}(C_{ij}, L_{ik}), \psi^{(r')}(C_{ij'}, L_{ik})\},$$

Using definition of $\psi^{(r)}$, C_{ij} and L_{ik}, we write

$$\begin{aligned} \sigma_{1,0}^{2(1,1)} &= E\left\{\psi^{(1)}(C_{ij}, L_{ik})\psi^{(1)}(C_{ij}, L_{ik'})\right\} - E\left\{\psi^{(1)}(C_{ij}, L_{ik})\right\} E\{\psi^{(1)}(C_{ij}, L_{ik'})\} \\ &= E\left\{(\mu_t + \sigma\epsilon_{tij} - \mu_{t'} - \sigma\epsilon_{t'ik})\left(\mu_t + \sigma\epsilon_{tij} - \mu_{t'} - \sigma\epsilon_{t'ik'}\right) I_{ijk} I_{ijk'}\right\} - \frac{1}{4}\delta_{1tt'}^2 \\ &= E\left\{(Y_{tij}^* - Y_{t'ik}^*)(Y_{tij}^* - Y_{t'ik'}^*) I_{ijk} I_{ijk'}\right\} - \frac{1}{4}\delta_{1tt'}^2 \\ &= D_{tt'} - \frac{1}{4}\delta_{1tt'}^2, \end{aligned}$$

where $Y_{tij}^* = \mu_t + \sigma\epsilon_{tij}$ is the response variable after the block effect is removed. We partition

$D_{tt'}$ as follows

$$D_{tt'} = E\{Y_{tij}^{*^2} I_{ijk} I_{ijk'}\} - 2E\{Y_{tij}^* Y_{t'ik'}^* I_{ijk} I_{ijk'}\} + E\{Y_{t'ik}^* Y_{t'ik'}^* I_{ijk} I_{ijk'}\}.$$

Using conditional expectation we have

$$E\{Y_{tij}^{*^2} I_{ijk} I_{ijk'}\} = E\left\{I_{ijk} I_{ijk'}\right\} E\left\{Y_{tij}^{*^2} | I_{ijk} I_{ijk'}\right\} = \frac{1}{3}\left\{\sigma_{t[1:3]}^{*2} + \mu_{t[1:3]}^{*2}\right\},$$

where $\mu_{t[1:3]}^*$ and $\sigma_{t[1:3]}^{*2}$ are the mean and variance of the smallest observation of Y_{tij}^* in a set of size 3. In a similar fashion, we obtain

$$\begin{aligned} E\{Y_{tij}^* Y_{t'ik'}^* I_{ijk} I_{ijk'}\} &= E\left\{I_{ijk} I_{ijk'}\right\} E\left\{Y_{tij}^* Y_{t'ik'}^* | I_{ijk} I_{ijk'}\right\} = \frac{1}{3}\left\{\sigma_{tt'[1,2:3]}^* + \mu_{t[1:3]}^* \mu_{t'[2:2]}^*\right\} \\ E\{Y_{t'ik}^* Y_{t'ik'}^* I_{ijk} I_{ijk'}\} &= E\left\{I_{ijk} I_{ijk'}\right\} E\{Y_{t'ik}^* Y_{t'ik'}^* | I_{ijk} I_{ijk'}\} = \frac{1}{3}(\mu_{t'[2:2]}^{*2}). \end{aligned}$$

Putting all these expression in $D_{tt'}$, we obtain

$$\sigma_{1,0}^{(1,1)} = \frac{1}{3}\left\{\sigma_{t[1:3]}^{*2} - 2\sigma_{tt'[1,2:3]}^* + (\mu_{t[1:3]}^* - \mu_{t'[2:2]}^*)^2\right\} - \frac{1}{4}\delta_{1tt'}^2.$$

We now consider

$$\begin{aligned} \sigma_{1,0}^{(1,2)} &= Cov\{\psi^{(1)}(C_{ij}, L_{ik}), \psi^{(2)}(C_{ij}, L_{ik'})\} \\ &= E\left\{(Y_{tij}^* - Y_{t'ik}^*)(Y_{tij}^* - Y_{t'ik'}^*) I_{ijk} I_{ijk'}^*\right\} - \frac{1}{4}\delta_{1tt'}\delta_{2tt'} \\ &= E\{Y_{tij}^{*^2} I_{ijk} I_{ijk'}^*\} - E\{Y_{tij}^* Y_{t'ik'}^* I_{ijk} I_{ijk'}^*\} \\ &\quad - E\{Y_{t'ik}^* Y_{tij}^* I_{ijk} I_{ijk'}^*\} + E\{Y_{t'ik}^* Y_{t'ik'}^* I_{ijk} I_{ijk'}^*\} - \frac{1}{4}\delta_{1tt'}\delta_{2tt'}. \end{aligned}$$

We note that $I_{ijk} I_{ijk'}^* = I(\epsilon_{tij} < \epsilon_{tik}) I(\epsilon_{tik'} < \epsilon_{tij})$ implies that $\epsilon_{tik'} < \epsilon_{tij} < \epsilon_{tik}$. Conditioning on this ordering, we obtain

$$
\begin{aligned}
E\{Y_{tij}^{*^2} I_{ijk} I_{ijk'}^*\} &= \frac{1}{6}\{\sigma_{t[2:3]}^{*2} + \mu_{t[2:3]}^{*2}\} \\
E\{Y_{tij}^{*} Y_{t'ik'}^{*} I_{ijk} I_{ijk'}^*\} &= \frac{1}{6}\{\sigma_{tt'[1,2:3]}^{*} + \mu_{t[2:3]}^{*}\mu_{t'[1:3]}^{*}\} \\
E\{Y_{t'ik}^{*} Y_{tij}^{*} I_{ijk} I_{ijk'}^*\} &= \frac{1}{6}\{\sigma_{t't[2,3:3]}^{*} + \mu_{t'[3:3]}^{*}\mu_{t[2:3]}^{*}\} \\
E\{Y_{t'ik}^{*} Y_{t'ik'}^{*} I_{ijk} I_{ijk'}^*\} &= \frac{1}{6}\{\sigma_{t't'[1,3:3]}^{*} + \mu_{t'[1:3]}^{*}\mu_{t'[3:3]}^{*}\}.
\end{aligned}
$$

combining these expressions in $\sigma_{1,0}^{(1,2)}$ we obtain

$$
\begin{aligned}
\sigma_{1,0}^{(1,2)} &= \frac{1}{6}\left\{\sigma_{t[2:3]}^{*2} - \sigma_{tt'[1,2:3]}^{*} - \sigma_{t't[2,3:3]}^{*} + \sigma_{t't'[1,3:3]}^{*} + \mu_{t[2:3]}^{*2} - \mu_{t[2:3]}^{*}\mu_{t'[1:3]}^{*} - \mu_{t'[3:3]}^{*}\mu_{t[2:3]}^{*} + \mu_{t'[1:3]}^{*}\mu_{t'[3:3]}^{*}\right\} \\
&\quad -\frac{1}{4}\delta_{1tt'}\delta_{2tt'}.
\end{aligned}
$$

For the derivation $\sigma_{1,0}^{(1,3)}$, we have

$$
\begin{aligned}
\sigma_{1,0}^{(1,3)} &= Cov\{\psi^{(1)}(C_{ij}, L_{ik}), \psi^{(3)}(C_{ij}, L_{ik'})\} \\
&= E\left\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk} I_{ijk'}\right\} - E\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk}\} E\{I_{ijk'}\} \\
&= \frac{1}{3}\left\{\mu_{t[1:3]}^* - \mu_{t'[2:2]}^*\right\} - \frac{1}{4}\delta_{1tt'}.
\end{aligned}
$$

The expression for $\sigma_{1,0}^{(1,3)}$ follows from

$$
\begin{aligned}
\sigma_{1,0}^{(1,4)} &= Cov\{\psi^{(1)}(C_{ij}, L_{ik}), \psi^{(4)}(C_{ij}, L_{ik'})\} \\
&= E\left\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk} I_{ijk'}^*\right\} - E\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk}\} E\{I_{ijk'}^*\} \\
&= \frac{1}{6}\left\{\mu_{t[2:3]}^* - \mu_{t'[3:3]}^*\right\} - \frac{1}{4}\delta_{1tt'}.
\end{aligned}
$$

The entry $\sigma_{1,0}^{(2,2)}$ is computed as follows

$$\begin{aligned}
\sigma_{1,0}^{(2,2)} &= E\left\{\psi^{(2)}(C_{ij}, L_{ik})\psi^{(2)}(C_{ij}, L_{ik'})\right\} - E\left\{\psi^{(2)}(C_{ij}, L_{ik})\right\} E\{\psi^{(2)}(C_{ij}, L_{ik'})\} \\
&= E\{Y_{tij}^{*^2} I_{ijk}^* I_{ijk'}^*\} - E\{Y_{tij}^* Y_{t'ik'}^* I_{ijk}^* I_{ijk'}^*\} \\
&\quad - E\{Y_{t'ik}^* Y_{tij}^* I_{ijk}^* I_{ijk'}^*\} + E\{Y_{t'ik}^* Y_{t'ik'}^* I_{ijk}^* I_{ijk'}^*\} - \frac{1}{4}\delta_{2tt'}^2 \\
&= \frac{1}{3}\left\{\sigma_{t[3:3]}^{*2} + \mu_{t[3:3]}^{*2} - 2(\sigma_{tt'[1,3:3]}^* + \mu_{t[3:3]}^* \mu_{t'[1:2]}^*) + \mu_{t'[1:2]}^{*2}\right\} - \frac{1}{4}\delta_{2tt'}^2 \\
&= \frac{1}{3}\left\{\sigma_{t[3:3]}^{*2} - 2\sigma_{tt'[1,3:3]}^* + (\mu_{t[3:3]}^* - \mu_{t'[1:2]}^*)^2\right\} - \frac{1}{4}\delta_{2tt'}^2.
\end{aligned}$$

For the $\sigma_{1,0}^{(2,3)}$, we evaluate

$$\begin{aligned}
\sigma_{1,0}^{(2,3)} &= Cov\{\psi^{(2)}(C_{ij}, L_{ik}), \psi^{(3)}(C_{ij}, L_{ik'})\} \\
&= E\left\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk}^* I_{ijk'}\right\} - E\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk}^*\} E\{I_{ijk'}\} \\
&= \frac{1}{6}\left\{\mu_{t[2:3]}^* - \mu_{t'[1:3]}^*\right\} - \frac{1}{4}\delta_{2tt'}.
\end{aligned}$$

The entry $\sigma_{1,0}^{(2,4)}$ is computed from

$$\begin{aligned}
\sigma_{1,0}^{(2,4)} &= Cov\{\psi^{(2)}(C_{ij}, L_{ik}), \psi^{(4)}(C_{ij}, L_{ik'})\} \\
&= E\left\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk}^* I_{ijk'}^*\right\} - E\{(Y_{tij}^* - Y_{t'ik}^*) I_{ijk}^*\} E\{I_{ijk'}^*\} \\
&= \frac{1}{3}\left\{\mu_{t[3:3]}^* - \mu_{t'[1:2]}^*\right\} - \frac{1}{4}\delta_{2tt'}.
\end{aligned}$$

We also have $\sigma_{1,0}^{(3,3)} = \sigma_{1,0}^{(4,4)} = 1/12$ and $\sigma_{1,0}^{(3,4)} = -1/12$.

In a similar fashion, we can compute the entries of $\boldsymbol{\Sigma}_2$

$$
\begin{aligned}
\sigma_{0,1}^{(1,1)} &= Cov\{\psi^{(1)}(C_{ij}, L_{ik}), \psi^{(1)}(C_{ij'}, L_{ik})\} \\
&= E\{Y^*_{tij}Y^*_{tij'}I_{ijk}I_{ij'k}\} - 2E\{Y^*_{tij}Y^*_{t'ik}I_{ijk}I_{ij'k}\} + E\{Y^*_{t'ik}Y^*_{t'ik}I_{ijk}I_{ij'k}\} - \frac{1}{4}\delta^2_{1tt'} \\
&= \frac{1}{6}\{\mu^{*2}_{t[1:2]} - 2(\sigma^*_{tt'[1,3:3]} + \mu^*_{t[1:2]}\mu^*_{t'[3:3]}) + \sigma^{*2}_{t'[3:3]} + \mu^*_{t'[3:3]}\} - \frac{1}{4}\delta^2_{1tt'} \\
&= \frac{1}{6}\{\sigma^{*2}_{t'[3:3]} - 2\sigma^*_{tt'[1,3:3]} + (\mu^*_{t[1:2]} - \mu^*_{t'[3:3]})^2\} - \frac{1}{4}\delta^2_{1tt'}, \\
\sigma_{0,1}^{(1,2)} &= Cov\{\psi^{(1)}(C_{ij}, L_{ik}), \psi^{(2}(C_{ij'}, L_{ik})\} \\
&= E\left(\{Y^*_{tij}Y^*_{tij'} - Y^*_{tij}Y^*_{t'ik} - Y^*_{t'ik}Y^*_{tij'} + Y^{*2}_{t'ik}\}I_{ijk}I^*_{ij'k}\right) - \frac{1}{4}\delta_{1tt'}\delta_{2tt'} \\
&= \frac{1}{6}\left\{\sigma^*_{tt[1,3:3]} + \mu^*_{t[1:3]}\mu^*_{t[3:3]} - \sigma^*_{tt'[1,2:3]} - \mu^*_{t[1:3]}\mu^*_{t'[2:3]}\right. \\
&\quad \left. -\sigma^*_{t't[2,3:3]} - \mu^*_{t'[2:3]}\mu^*_{t[3:3]} + \sigma^{*2}_{t'[2:3]} + \mu^{*2}_{t'[2:3]}\right\} - \frac{1}{4}\delta_{1tt'}\delta_{2tt'}, \\
\sigma_{0,1}^{(1,3)} &= Cov\{\psi^{(1)}(C_{ij}, L_{ik}), \psi^{(3}(C_{ij'}, L_{ik})\} \\
&= E(\left\{Y^*_{tij} - Y^*_{t'ik}\right\} I_{ijk}I_{ij'k}) - \frac{1}{4}\delta_{1tt'} \\
&= \frac{1}{3}\{\mu^*_{t[1:2]} - \mu^*_{t'[3:3]}\} - \frac{1}{4}\delta_{1tt'}, \\
\sigma_{0,1}^{(1,4)} &= Cov\{\psi^{(1)}(C_{ij}, L_{ik}), \psi^{(4}(C_{ij'}, L_{ik})\} \\
&= E(\left\{Y^*_{tij} - Y^*_{t'ik}\right\} I_{ijk}I^*_{ij'k}) - \frac{1}{4}\delta_{1tt'} \\
&= \frac{1}{6}\{\mu^*_{t[1:3]} - \mu^*_{t'[2:3]}\} - \frac{1}{4}\delta_{1t't}, \\
\sigma_{0,1}^{(2,2)} &= Cov\{\psi^{(2)}(C_{ij}, L_{ik}), \psi^{(2}(C_{ij'}, L_{ik})\} \\
&= \frac{1}{3}\left(\{Y^*_{tij}Y^*_{tij'}I^*_{ijk}I^*_{ij'k}\} - 2E\{Y^*_{tij}Y^*_{t'ik}I^*_{ijk}I^*_{ij'k}\} + E\{Y^*_{t'ik}Y^*_{t'ik}I^*_{ijk}I^*_{ij'k}\}\right) - \frac{1}{4}\delta_{2tt'} \\
&= \frac{1}{3}\left\{\mu^{*2}_{t[2:2]} - 2\sigma^*_{tt'[1,2:3]} - 2\mu^*_{t[2:2]}\mu^*_{t'[1:3]} + \sigma^{*2}_{t'[1:3]} + \mu^{*2}_{t'[1:3]}\right\} - \frac{1}{4}\delta_{2tt'} \\
&= \frac{1}{3}\left\{\sigma^{*2}_{t'[1:3]} - 2\sigma^*_{tt'[1,2:3]} + (\mu^*_{t[2:2]} - \mu^*_{t'[1:3]})^2\right\} - \frac{1}{4}\delta_{2tt'},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{0,1}^{(2,3)} &= Cov\{\psi^{(2)}(C_{ij}, L_{ik}), \psi^{(3}(C_{ij'}, L_{ik})\} \\
&= \frac{1}{6}\left\{\mu^*_{t[3:3]} - \mu_{t'[2:3]}\right\} - \frac{1}{4}\delta_{2tt'}, \\
\sigma_{0,1}^{(2,4)} &= Cov\{\psi^{(2)}(C_{ij}, L_{ik}), \psi^{(4}(C_{ij'}, L_{ik})\} \\
&= E\{(Y^*_{tij} - Y^*_{t'ik})I^*_{ijk}I^*_{ij'k}\} - \frac{1}{4}\delta_{2tt'} \\
&= \frac{1}{3}\{\mu^*_{t[2:2]} - \mu^*_{t'[1:3]}\} - \frac{1}{4}\delta_{2tt'}, \\
\sigma_{0,1}^{(3,3)} &= Cov\{\psi^{(3)}(C_{ij}, L_{ik}), \psi^{(3}(C_{ij'}, L_{ik})\} = 1/12, \\
\sigma_{0,1}^{(3,4)} &= Cov\{\psi^{(3)}(C_{ij}, L_{ik}), \psi^{(4}(C_{ij'}, L_{ik})\} = -1/12, \\
\sigma_{0,1}^{(4,4)} &= Cov\{\psi^{(4)}(C_{ij}, L_{ik}), \psi^{(4}(C_{ij'}, L_{ik})\} = 1/12.
\end{aligned}
$$

We have that $\boldsymbol{\Sigma_1} = Var(\boldsymbol{Z}_{1,ij})$ and $\boldsymbol{\Sigma_2} = Var(\boldsymbol{Z}_{2,ik})$. It then follows that $\sqrt{N_{tt'}}(\boldsymbol{U} - \boldsymbol{\theta})$ converges to a normal distribution with mean $\mathbf{0}$ and covariance matrix

$$
\boldsymbol{\Sigma} = \frac{k_1^2}{\lambda_t}\boldsymbol{\Sigma}_1 + \frac{k_2^2}{\lambda_{t'}}\boldsymbol{\Sigma}_2.
$$

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