Supplement to:
Propeller-Rotor Aerodynamic Interaction in Helicopter Air-to-Air Refueling: An Analytical Solution for Rotor Trim, Including a Sensitivity Study

Berend G. van der Wall
Senior Scientist
German Aerospace Center (DLR)
Braunschweig, Germany

1. Example aircraft, helicopter, operational, and rotor trim data

Table 1: Technical data of tanker aircraft and transport helicopter. (*) = estimated

Tanker aircraft Parameter, symbol	Value	Transport helicopter Parameter, symbol	Value
Max. take-off mass, -, t	141	Max. take-off mass, -, t	19.5
Min. speed, V_{min}, m/s	63.9 (*)	Max. speed, V_{max}, m/s	66.7
		Never exceed speed, V_{NE}, m/s	82.0
Number of propellers, N_p, -	4	Number of rotors, N_R, -	1
Number or propeller blades, N_{bp}, -	8	Number of rotor blades, N_b, -	6
Propeller radius, R_p, m	2.67	Rotor radius, R, m	11
Blade chord length, c_p, m	0.5(*)	Blade chord length, c, m	0.74
Propeller solidity, σ_p, -	0.477(*)	Rotor solidity, σ, -	0.128
Propeller tilt angle, $\Delta\alpha_p$, deg	-2	Blade twist, Θ_{tw}, deg/R	-6.0

Table 2: Operational condition for a representative air-to-air refueling situation. (*) = estimated.

Parameter	Symbol	Unit	Value
Height above sea level	h	m	2130
Temperature	T_∞	°C	1.16
Air density	ρ_∞	kg/m³	0.9933
Speed of sound	a_∞	m/s	332.6
Flight speed	V_∞	m/s	65.71
Flight Mach number	M_∞		0.1976
Aircraft mass	m_{airc}	t	130.0
Aircraft angle of attack	α_{airc}	deg	11.65
Glide ratio w. flaps	ϵ		6.68(*)
Propeller thrust	T_p	kN	47.73
Rotational speed	Ω_p	rad/s	88.2
Blade tip speed	$(\Omega R)_p$	m/s	235.5
Tip Mach number	$M_{tip,p}$		0.708
Helicopter mass	m_{hel}	t	17.0
Rotor disk tilt angle	α_S	deg	-12.0(*)
Rotational speed	Ω	rad/s	19.37
Blade tip speed	ΩR	m/s	213.1
Tip Mach number	M_{tip}		0.641
Flight speed ratio	μ_∞		0.3084
Rotor advance ratio	μ_0		0.3017
Rotor inflow ratio	μ_{z0}		0.0641

Table 3: Helicopter trim in undisturbed flow and propeller slipstream data.

Parameter	Symbol	Unit	Value
Collective control angle	Θ_{75}	deg	12.31
Longitudinal cyclic control angle	Θ_S	deg	-6.26
Lateral cyclic control angle	Θ_C	deg	0.0
Slipstream contraction ratio	R_∞/R_p		0.9228
Slipstream width ratio	$2\,R_\infty/R$		0.4480
Slipstream velocity	ΔV_∞	m/s	27.35
Slipstream velocity rato	$\Delta\mu_\infty$		0.1283
Perturbation advance ratio	$\Delta\mu$		0.1255
Inflow ratio perturbation	$\Delta\lambda$		0.0218
Combined inflow & advance ratio perturbation	$\Delta_{\mu\lambda}$		0.0194

2. Computation of the propeller slipstream velocity and contracted radius

The propeller-induced velocity inside its slipstream v_{ip} at a given flight speed (aerodynamically equivalent to a rotor in axial climb) needs to be evaluated, based on momentum theory as outlined in the classical literature. Aerodynamically, the propeller is a rotor in axial climb. For the investigation performed here the propeller is considered as an actuator disk and the helicopter rotor is considered having rigid, pre-twisted blades with the blade pitch angle as sole degree of freedom, because the focus here is on the aerodynamic aspect of trim disturbance and the disturbance rejection. First, the propeller's thrust T_p must be estimated, and this can roughly be based on the aircraft's lift = weight = $m_{airc}g$, its glide ratio ϵ, and the number of propellers N_p. Then, the hovering induced velocity v_{hp} and the associated induced inflow ratio λ_{hp} of that thrust can be computed, with R_p as the propeller radius, Ω_p its rotational frequency, g is the gravitational constant, and ρ the air density.

$$T_p \approx \frac{m_{airc}g}{\epsilon N_p}; \qquad v_{hp} = \sqrt{\frac{T_p}{2\rho\pi R_p^2}}; \qquad \lambda_{hp} = \frac{v_{hp}}{(\Omega R)_p} \tag{1}$$

Next, based on the axial inflow ratio λ_C of the propeller at the flight speed V_∞ of the tanker aircraft, the induced velocity v_{ip} in the propeller's disk at the speed of flight is resulting. The helicopter rotor, however, is subjected to the fully developed slipstream of the propeller, which has twice the induced velocity than within the propeller disk, leading to a perturbation tip speed ratio $\Delta\mu_\infty$. Following momentum theory, finally the perturbation advance ratio $\Delta\mu$ in the helicopter rotor disk and the perturbation inflow radio $\Delta\mu_z$ normal to the disk can be computed.

$$\lambda_C = \frac{V_\infty}{(\Omega R)_p}; \qquad \overline{\lambda_C} = \frac{\lambda_C}{2\lambda_{hp}}; \qquad \lambda_{ip} \approx \lambda_{hp}\left(\sqrt{\overline{\lambda_C}^2+1}-\overline{\lambda_C}\right); \qquad \Delta V_\infty = 2\lambda_{ip}(\Omega R)_p \tag{2}$$

Also, the propeller slipstream contraction can be estimated using momentum theory, and with it the maximum width of the strip subjected to the slipstream velocity in the helicopter rotor, which in this example amounts to 44.8 % of the rotor radius.

$$\frac{R_\infty}{R_p} = \sqrt{\frac{\overline{\lambda_C}+\sqrt{\overline{\lambda_C}^2+1}}{2\sqrt{\overline{\lambda_C}^2+1}}} = 0.923; \quad y_2 - y_1 = \frac{2R_\infty}{R} = 0.448 \tag{3}$$

Using data from Table 1, Table 2 and the above equations, a slipstream contraction ratio of $R_\infty/R_p = 0.9228$ can be computed, resulting in a slipstream width ratio of $2\,R_\infty/R = 0.448$. Therein, the slipstream velocity amounts to $\Delta V_\infty = 27.35$ m/s, or $\Delta\mu_\infty = 0.1283$.

3. Introduction to the perturbation integrals

Basis of the derivation is the blade element momentum theory with the usual simplifications, i.e. constant inflow, no tip or root losses, steady incompressible 2D aerodynamics without stall, etc. Starting point are the disturbance thrust and moment coefficient equations Eq. (8) of the main paper

$$\begin{Bmatrix} \Delta C_T^{(j)} \\ \Delta C_{Mx}^{(j)} \\ \Delta C_{My}^{(j)} \end{Bmatrix} = \frac{1}{2\pi} \sum_{i=1}^{N_{reg}} \int_{\psi_{beg}(i)}^{\psi_{end}(i)} \int_{r_{low}(i)}^{r_{up}(i)} \begin{Bmatrix} 1 \\ r\sin\psi \\ -r\cos\psi \end{Bmatrix} 2\Delta dC_T^{(j)}\, d\psi \qquad j = 2,3,4 \tag{4}$$

with the section thrust computation from Eq. (9) of the main paper.

$$\Delta dC_T^{(2)} = \frac{\sigma C_{l\alpha}}{2} \Delta\mu \left[\Delta\Theta_{75} 2\left[r\sin\psi + \mu\sin^2\psi \right] + \Delta\Theta_S 2\left[r\sin^2\psi + \mu\sin^3\psi \right] \right] dr$$

$$\Delta dC_T^{(3)} = \frac{\sigma C_{l\alpha}}{2} \Delta\mu \begin{bmatrix} \Theta_{tw}\left[2\left(r^2 - 0.75r\right)\sin\psi + c_\mu\left(r - 0.75\right)\sin^2\psi \right] \\ +\Theta_{75}\left(2r\sin\psi + c_\mu\sin^2\psi\right) \\ +\Theta_S\left(2r\sin^2\psi + c_\mu\sin^3\psi\right) \end{bmatrix} dr \tag{5}$$

$$\Delta dC_T^{(4)} = -\frac{\sigma C_{l\alpha}}{2} \left[\Delta\lambda r + \Delta_{\mu\lambda}\sin\psi \right] dr$$

with $\mu = \mu_0 + \Delta\mu$; $\quad \Delta\lambda = \Delta\mu_z + \Delta\lambda_i$; $\quad \Delta_{\mu\lambda} = \mu_0\Delta\lambda + \lambda\Delta\mu$; $\quad c_\mu = 2\mu_0 + \Delta\mu$

In the following sections, Cases I to V in Fig. 2 of the main paper are individually derived for the results needed in the disturbance rejection equations. All the upper and lower bounds of the different regions for all Cases 0 to V are listed in Table 4.

Table 4: Radial and azimuthal integration bounds for all regions of all cases.

Case	y_1	y_2	ψ_1	ψ_2	N_{reg}	Region	r_{low}	r_{up}	ψ_{beg}	ψ_{end}
I	≥ 0, < 1	≥ 1	$\tan^{-1}\frac{y_1}{\sqrt{1-y_1^2}}$	n.a.	1	1	$\frac{y_1}{\sin\psi}$	1	ψ_1	$\frac{\pi}{2}$
II	≥ 0, $< y_2$	$> y_1$, < 1	$\tan^{-1}\frac{y_1}{\sqrt{1-y_1^2}}$	$\tan^{-1}\frac{y_2}{\sqrt{1-y_2^2}}$	2	1	$\frac{y_1}{\sin\psi}$	1	ψ_1	ψ_2
						2	$\frac{y_1}{\sin\psi}$	$\frac{y_2}{\sin\psi}$	ψ_2	$\frac{\pi}{2}$
III	< 0, > -1	> 0, < 1	$\tan^{-1}\frac{y_1}{\sqrt{1-y_1^2}}$	$\tan^{-1}\frac{y_2}{\sqrt{1-y_2^2}}$	3	1	0	1	ψ_1	ψ_2
						2	0	$\frac{y_2}{\sin\psi}$	ψ_2	$\frac{\pi}{2}$
						3	0	$\frac{y_1}{\sin\psi}$	$-\frac{\pi}{2}$	ψ_1
IV	$< y_2$, > -1	≤ 0, $> y_1$	$\tan^{-1}\frac{y_1}{\sqrt{1-y_1^2}}$	$\tan^{-1}\frac{y_2}{\sqrt{1-y_2^2}}$	2	3	$\frac{y_2}{\sin\psi}$	$\frac{y_1}{\sin\psi}$	$-\frac{\pi}{2}$	ψ_1
						1	$\frac{y_2}{\sin\psi}$	1	ψ_1	ψ_2
V	≤ -1	≤ 0, > -1	n.a.	$\tan^{-1}\frac{y_2}{\sqrt{1-y_2^2}}$	1	1	$\frac{y_2}{\sin\psi}$	1	$-\frac{\pi}{2}$	ψ_2

The following integrals are needed:

$$\int \frac{1}{\sin^2 \psi} d\psi = -\cot \psi \qquad \int \frac{1}{\sin \psi} d\psi = \ln\left|\tan \frac{\psi}{2}\right| \qquad \int \sin \psi \, d\psi = -\cos \psi$$

$$\int \sin^2 \psi \, d\psi = \frac{1}{2}\left(\psi - \frac{\sin 2\psi}{2} \right) \qquad \int \sin^3 \psi \, d\psi = -\left(\cos \psi - \frac{\cos^3 \psi}{3} \right) \tag{6}$$

$$\int \sin^4 \psi \, d\psi = \frac{1}{2}\left(\frac{3}{4}\psi - \frac{\sin 2\psi}{2} + \frac{\sin 4\psi}{16} \right)$$

The final results include several terms of zero value, as are:

$$\sin(\pm\pi) = \sin(\pm 2\pi) = \cos \frac{\pm\pi}{2} = \cot \frac{\pm\pi}{2} = \ln\left|\tan \frac{\pm\pi}{4}\right| = 0 \tag{7}$$

4. Case I: The slipstream overlaps with the advancing side edge of the rotor

This case represents the overlap with the rotor advancing side's edge of the disk, which reduces the number of regions to integrate to one only: "Region 1". The results for Case II (see there) can be used by modifying the upper end of the azimuthal integration from ψ_2 to $\pi/2$.

2.1. Contributions to the thrust coefficient

For the second term, from Eq. (17)

$$\begin{aligned}\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \Delta\Theta_{75}\left[\begin{array}{l} -(1-2\mu y_1)\left(\cancel{\cos\frac{\pi}{2}} - \cos\psi_1\right) + \mu\left[\left(\frac{\pi}{2}-\psi_1\right) - \frac{\cancel{\sin\pi} - \sin 2\psi_1}{2}\right] \\ -y_1^2\left(\cancel{\ln\left|\tan\frac{\pi}{4}\right|} - \ln\left|\tan\frac{\psi_1}{2}\right|\right)\end{array}\right] \\ &+ \Delta\Theta_S\left[\begin{array}{l} \left(\frac{1}{2}-\mu y_1\right)\left[\left(\frac{\pi}{2}-\psi_1\right) - \frac{\cancel{\sin\pi} - \sin 2\psi_1}{2}\right] \\ -2\mu\left(\cancel{\cos\frac{\pi}{2}} - \cos\psi_1 - \frac{\cancel{\cos^3\frac{\pi}{2}} - \cos^3\psi_1}{3}\right) - y_1^2\left(\frac{\pi}{2}-\psi_1\right)\end{array}\right]\end{aligned} \tag{8}$$

For the third term, from Eq. (22)

$$\begin{aligned}\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \Theta_{tw}\left[\begin{array}{l} \frac{c_\mu}{2}\left(\frac{1}{2}-0.75\right)\left[\left(\frac{\pi}{2}-\psi_1\right) - \frac{\cancel{\sin\pi} - \sin 2\psi_1}{2}\right] \\ -\left[\frac{2}{3} - 0.75(1-c_\mu y_1)\right]\left(\cancel{\cos\frac{\pi}{2}} - \cos\psi_1\right) - \frac{c_\mu}{2}y_1^2\left(\frac{\pi}{2}-\psi_1\right) \\ +0.75 y_1^2\left(\cancel{\ln\left|\tan\frac{\pi}{4}\right|} - \ln\left|\tan\frac{\psi_1}{2}\right|\right) + \frac{2}{3}y_1^3\left(\cancel{\cot\frac{\pi}{2}} - \cot\psi_1\right)\end{array}\right] \\ &+ \Theta_{75}\left[\begin{array}{l} -(1-c_\mu y_1)\left(\cancel{\cos\frac{\pi}{2}} - \cos\psi_1\right) - y_1^2\left(\cancel{\ln\left|\tan\frac{\pi}{4}\right|} - \ln\left|\tan\frac{\psi_1}{2}\right|\right) \\ +\frac{c_\mu}{2}\left[\left(\frac{\pi}{2}-\psi_1\right) - \frac{\cancel{\sin\pi} - \sin 2\psi_1}{2}\right]\end{array}\right] \\ &+ \Theta_S\left[\begin{array}{l} -y_1^2\left(\frac{\pi}{2}-\psi_1\right) + \left(\frac{1}{2} - \frac{c_\mu}{2}y_1\right)\left[\left(\frac{\pi}{2}-\psi_1\right) - \frac{\cancel{\sin\pi} - \sin 2\psi_1}{2}\right] \\ -c_\mu\left[\left(\cancel{\cos\frac{\pi}{2}} - \cos\psi_1\right) - \frac{\cancel{\cos^3\frac{\pi}{2}} - \cos^3\psi_1}{3}\right]\end{array}\right]\end{aligned} \tag{9}$$

Finally, the fourth term, from Eq. (27)

$$\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = \Delta\lambda\left[-\frac{1}{2}\left(\frac{\pi}{2}-\psi_1\right)-\frac{1}{2}y_1^2\left(\cancel{\cot\frac{\pi}{2}}-\cot\psi_1\right)\right]$$

$$\Delta_{\mu\lambda}\left[y_1\left(\frac{\pi}{2}-\psi_1\right)+\left(\cancel{\cos\frac{\pi}{2}}-\cos\psi_1\right)\right] \tag{10}$$

2.2. Contributions to the rolling moment coefficient

The moment coefficient results are treated in the same manner as the thrust coefficient results. For the second term, from Eq. (32)

$$\frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\left[\begin{array}{l} \mu y_1^2\left(\cancel{\cos\frac{\pi}{2}}-\cos\psi_1\right)+\frac{1}{3}\left[\left(\frac{\pi}{2}-\psi_1\right)-\frac{\cancel{\sin\pi}-\sin 2\psi_1}{2}\right] \\ -\mu\left[\left(\cancel{\cos\frac{\pi}{2}}-\cos\psi_1\right)-\frac{\cancel{\cos^3\frac{\pi}{2}}-\cos^3\psi_1}{3}\right] \\ -\frac{2}{3}y_1^3\left(\cancel{\ln\left|\tan\frac{\pi}{4}\right|}-\ln\left|\tan\frac{\psi_1}{2}\right|\right) \end{array}\right]$$

$$+\Delta\Theta_S\left[\begin{array}{l} -\frac{2}{3}y_1^3\left(\frac{\pi}{2}-\psi_1\right)-\frac{\mu}{2}y_1^2\left[\left(\frac{\pi}{2}-\psi_1\right)-\frac{\cancel{\sin\pi}-\sin 2\psi_1}{2}\right] \\ -\frac{2}{3}\left[\left(\cancel{\cos\frac{\pi}{2}}-\cos\psi_1\right)-\frac{\cancel{\cos^3\frac{\pi}{2}}-\cos^3\psi_1}{3}\right] \\ +\frac{\mu}{2}\left[\frac{3}{4}\left(\frac{\pi}{2}-\psi_1\right)-\frac{\cancel{\sin\pi}-\sin 2\psi_1}{2}+\frac{\cancel{\sin 2\pi}-\sin 4\psi_1}{16}\right] \end{array}\right] \tag{11}$$

The third term, from Eq. (37)

$$\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Theta_{tw}\left[\begin{array}{l} -\frac{c_\mu}{3}y_1^3\left(\frac{\pi}{2}-\psi_1\right)-0.75\frac{c_\mu}{2}y_1^2\left(\cancel{\cos\frac{\pi}{2}}-\cos\psi_1\right) \\ -\left(\frac{2}{3}-0.75\right)\frac{c_\mu}{2}\left[\left(\cancel{\cos\frac{\pi}{2}}-\cos\psi_1\right)-\frac{\cancel{\cos^3\frac{\pi}{2}}-\cos^3\psi_1}{3}\right] \\ +0.75\frac{2}{3}y_1^3\left(\cancel{\ln\left|\tan\frac{\pi}{4}\right|}-\ln\left|\tan\frac{\psi_1}{2}\right|\right)+\frac{1}{2}y_1^4\left(\cancel{\cot\frac{\pi}{2}}-\cot\psi_1\right) \end{array}\right] \tag{12}$$

$$+\,\text{part 2}$$

$$\text{part }2=\Theta_{75}\left[\begin{array}{l}\dfrac{c_\mu}{2}y_1^2\left(\cancel{\cos\dfrac{\pi}{2}}-\cos\psi_1\right)+\dfrac{1}{3}\left[\left(\dfrac{\pi}{2}-\psi_1\right)-\dfrac{\cancel{\sin\pi}-\sin 2\psi_1}{2}\right]\\ -\dfrac{c_\mu}{2}\left[\left(\cancel{\cos\dfrac{\pi}{2}}-\cos\psi_1\right)-\dfrac{\cancel{\cos^3\dfrac{\pi}{2}}-\cos^3\psi_1}{3}\right]\\ -\dfrac{2}{3}y_1^3\left(\ln\left|\cancel{\tan\dfrac{\pi}{4}}\right|-\ln\left|\tan\dfrac{\psi_1}{2}\right|\right)\end{array}\right]$$

$$+\Theta_S\left[\begin{array}{l}-\dfrac{2}{3}y_1^3\left(\dfrac{\pi}{2}-\psi_1\right)-\dfrac{c_\mu}{4}y_1^2\left[\left(\dfrac{\pi}{2}-\psi_1\right)-\dfrac{\cancel{\sin\pi}-\sin 2\psi_1}{2}\right]\\ -\dfrac{2}{3}\left[\left(\cancel{\cos\dfrac{\pi}{2}}-\cos\psi_1\right)-\dfrac{\cancel{\cos^3\dfrac{\pi}{2}}-\cos^3\psi_1}{3}\right]\\ +\dfrac{c_\mu}{4}\left[\dfrac{3}{4}\left(\dfrac{\pi}{2}-\psi_1\right)-\dfrac{\cancel{\sin\pi}-\sin 2\psi_1}{2}+\dfrac{\cancel{\sin 2\pi}-\sin 4\psi_1}{16}\right]\end{array}\right]$$

And the fourth term, from Eq. (42):

$$\begin{aligned}\frac{\Delta C_{Mx}^{(4)}}{\dfrac{\sigma C_{l\alpha}}{2\pi}}&=\Delta\lambda\left[\frac{1}{3}\left(\cancel{\cos\frac{\pi}{2}}-\cos\psi_1\right)-\frac{1}{3}y_1^3\left(\cancel{\cot\frac{\pi}{2}}-\cot\psi_1\right)\right]\\&+\Delta_{\mu\lambda}\left[\frac{1}{2}y_1^2\left(\frac{\pi}{2}-\psi_1\right)-\frac{1}{4}\left[\left(\frac{\pi}{2}-\psi_1\right)-\frac{\cancel{\sin\pi}-\sin 2\psi_1}{2}\right]\right]\end{aligned} \tag{13}$$

5. Case II: The slipstream is within the advancing side of the rotor

Now $0 \leq y_1 < y_2 \leq 1$ and there is no overlap with the center of the rotor or the advancing side end of the disk. Two regions need to be integrated separately: "Region 1" ranging from $\psi_1 \geq 0$ to $\psi_2 < \pi/2$ and "Region 2" ranging from ψ_2 to $\pi/2$ with a different upper radial integration bound.

3.1. Contributions to the thrust coefficient

From Eqs. (4) and (5):

$$\Delta C_T^{(2)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_1}{\sin\psi}}^{1} 2\Delta dC_T^{(2)} + \int_{\psi_2}^{\pi/2}\int_{\frac{y_1}{\sin\psi}}^{\frac{y_2}{\sin\psi}} 2\Delta dC_T^{(2)}\right]d\psi \tag{14}$$

$$2\Delta dC_T^{(2)} = \sigma C_{l\alpha}\Delta\mu\left[2\Delta\Theta_{75}\left(r\sin\psi+\mu\sin^2\psi\right)+2\Delta\Theta_S\left(r\sin^2\psi+\mu\sin^3\psi\right)\right]dr$$

Performing the radial integration and substitution of the radial integration bounds, the azimuthal integration needs to be performed. This is more complicated, because the radial bounds introduce additional terms including the azimuth. For convenience, the integrals are sorted for the control perturbations.

$$\begin{aligned}\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \Delta\Theta_{75}\int_{\psi_1}^{\psi_2}\left[\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin\psi+2\mu\left(1-\frac{y_1}{\sin\psi}\right)\sin^2\psi\right]d\psi \\ &+\Delta\Theta_S\int_{\psi_1}^{\psi_2}\left[\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin^2\psi+2\mu\left(1-\frac{y_1}{\sin\psi}\right)\sin^3\psi\right]d\psi \\ &+\Delta\Theta_{75}\int_{\psi_2}^{\pi/2}\left[\left(\frac{y_2^2-y_1^2}{\sin^2\psi}\right)\sin\psi+2\mu\left(\frac{y_2-y_1}{\sin\psi}\right)\sin^2\psi\right]d\psi \\ &+\Delta\Theta_S\int_{\psi_2}^{\pi/2}\left[\left(\frac{y_2^2-y_1^2}{\sin^2\psi}\right)\sin^2\psi+2\mu\left(\frac{y_2-y_1}{\sin\psi}\right)\sin^3\psi\right]d\psi\end{aligned} \tag{15}$$

Sorting for powers of the Sine function,

$$\begin{aligned}\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \Delta\Theta_{75}\int_{\psi_1}^{\psi_2}\left[(1-2\mu y_1)\sin\psi+2\mu\sin^2\psi-y_1^2\frac{1}{\sin\psi}\right]d\psi \\ &+\Delta\Theta_S\int_{\psi_1}^{\psi_2}\left[(1-2\mu y_1)\sin^2\psi+2\mu\sin^3\psi-y_1^2\right]d\psi \\ &+\Delta\Theta_{75}\int_{\psi_2}^{\pi/2}\left[\left(y_2^2-y_1^2\right)\frac{1}{\sin\psi}+2\mu\left(y_2-y_1\right)\sin\psi\right]d\psi \\ &+\Delta\Theta_S\int_{\psi_2}^{\pi/2}\left[\left(y_2^2-y_1^2\right)+2\mu\left(y_2-y_1\right)\sin^2\psi\right]d\psi\end{aligned} \tag{16}$$

Solving the remaining integrals results in

$$\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\begin{bmatrix} -\left(1-2\mu y_1\right)\left(\cos\psi_2-\cos\psi_1\right)+\mu\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ -y_1^2\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right) \\ -2\mu\left(y_2-y_1\right)\left(\cos\frac{\pi}{2}-\cos\psi_2\right)+\left(y_2^2-y_1^2\right)\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right) \end{bmatrix}$$
$$+\Delta\Theta_S\begin{bmatrix} \left(\frac{1}{2}-\mu y_1\right)\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ -2\mu\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]-y_1^2\left(\psi_2-\psi_1\right) \\ +\left(y_2^2-y_1^2\right)\left(\frac{\pi}{2}-\psi_2\right)+\mu\left(y_2-y_1\right)\left[\left(\frac{\pi}{2}-\psi_2\right)-\frac{\sin\pi-\sin 2\psi_2}{2}\right] \end{bmatrix} \tag{17}$$

The result can formally be written as

$$\Delta C_T^{(2)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \end{bmatrix}\begin{Bmatrix} \Delta\Theta_{75} \\ \Delta\Theta_S \end{Bmatrix} \tag{18}$$

Note that $\ln\tan\pi/4 = \cos\pi/2 = \sin\pi = 0$. Now the third term is treated the same way. First, the term $\Delta dL^{(3)}$ from Eq. (5) is inserted into Eq. (4).

$$\Delta C_T^{(3)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_1}{\sin\psi}}^{1} 2\Delta dC_T^{(3)} + \int_{\psi_2}^{\pi/2}\int_{\frac{y_1}{\sin\psi}}^{\frac{y_2}{\sin\psi}} 2\Delta dC_T^{(3)}\right]d\psi$$
$$2\Delta dC_T^{(3)} = \sigma C_{l\alpha}\Delta\mu\begin{bmatrix} \Theta_{tw}\left[2\left(r^2-0.75r\right)\sin\psi+c_\mu\left(r-0.75\right)\sin^2\psi\right] \\ +\Theta_{75}\left(2r\sin\psi+c_\mu\sin^2\psi\right) \\ +\Theta_S\left(2r\sin^2\psi+c_\mu\sin^3\psi\right) \end{bmatrix}dr \tag{19}$$

Performing the radial integration and substitution of the radial integration bounds, the azimuthal integration is performed and terms are sorted for the basic control angles.

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Theta_{tw}\int_{\psi_1}^{\psi_2}\begin{bmatrix} \left[\frac{2}{3}\left(1-\frac{y_1^3}{\sin^3\psi}\right)-0.75\left(1-\frac{y_1^2}{\sin^2\psi}\right)\right]\sin\psi \\ +c_\mu\left[\frac{1}{2}\left(1-\frac{y_1^2}{\sin^2\psi}\right)-0.75\left(1-\frac{y_1}{\sin\psi}\right)\right]\sin^2\psi \end{bmatrix}d\psi$$
$$+\Theta_{75}\int_{\psi_1}^{\psi_2}\left[\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin\psi+c_\mu\left(1-\frac{y_1}{\sin\psi}\right)\sin^2\psi\right]d\psi$$
$$+\Theta_S\int_{\psi_1}^{\psi_2}\left[\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin^2\psi+c_\mu\left(1-\frac{y_1}{\sin\psi}\right)\sin^3\psi\right]d\psi$$
$$+\text{part 2} \tag{20}$$

$$\text{part 2} = \Theta_{tw} \int_{\psi_2}^{\pi/2} \left[\begin{array}{l} \left[\frac{2}{3}\left(\frac{y_2^3 - y_1^3}{\sin^3\psi} \right) - 0.75\left(\frac{y_2^2 - y_1^2}{\sin^2\psi} \right) \right] \sin\psi \\ + c_\mu \left[\frac{1}{2}\left(\frac{y_2^2 - y_1^2}{\sin^2\psi} \right) - 0.75\left(\frac{y_2 - y_1}{\sin\psi} \right) \right] \sin^2\psi \end{array} \right] d\psi$$

$$+ \Theta_{75} \int_{\psi_2}^{\pi/2} \left[\left(\frac{y_2^2 - y_1^2}{\sin^2\psi} \right) \sin\psi + c_\mu \left(\frac{y_2 - y_1}{\sin\psi} \right) \sin^2\psi \right] d\psi$$

$$+ \Theta_S \int_{\psi_2}^{\pi/2} \left[\left(\frac{y_2^2 - y_1^2}{\sin^2\psi} \right) \sin^2\psi + c_\mu \left(\frac{y_2 - y_1}{\sin\psi} \right) \sin^3\psi \right] d\psi$$

Sorting for powers of the Sine function,

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Theta_{tw} \int_{\psi_1}^{\psi_2} \left[\begin{array}{l} \left[\frac{2}{3} - 0.75\left(1 - c_\mu y_1\right) \right] \sin\psi + c_\mu \left(\frac{1}{2} - 0.75 \right) \sin^2\psi \\ -\frac{c_\mu}{2} y_1^2 + 0.75 y_1^2 \frac{1}{\sin\psi} - \frac{2}{3} y_1^3 \frac{1}{\sin^2\psi} \end{array} \right] d\psi$$

$$+ \Theta_{75} \int_{\psi_1}^{\psi_2} \left[\left(1 - c_\mu y_1\right) \sin\psi + c_\mu \sin^2\psi - y_1^2 \frac{1}{\sin\psi} \right] d\psi$$

$$+ \Theta_S \int_{\psi_1}^{\psi_2} \left[\left(1 - c_\mu y_1\right) \sin^2\psi + c_\mu \sin^3\psi - y_1^2 \right] d\psi$$

$$+ \Theta_{tw} \int_{\psi_2}^{\pi/2} \left[\begin{array}{l} \frac{c_\mu}{2}\left(y_2^2 - y_1^2\right) - c_\mu 0.75\left(y_2 - y_1\right)\sin\psi \\ -0.75\left(y_2^2 - y_1^2\right)\frac{1}{\sin\psi} + \frac{2}{3}\left(y_2^3 - y_1^3\right)\frac{1}{\sin^2\psi} \end{array} \right] d\psi$$

$$+ \Theta_{75} \int_{\psi_2}^{\pi/2} \left[c_\mu \left(y_2 - y_1\right)\sin\psi + \left(y_2^2 - y_1^2\right)\frac{1}{\sin\psi} \right] d\psi$$

$$+ \Theta_S \int_{\psi_2}^{\pi/2} \left[c_\mu \left(y_2 - y_1\right)\sin^2\psi + \left(y_2^2 - y_1^2\right) \right] d\psi \qquad (21)$$

Solving the remaining integrals results in

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Theta_{tw} \left[\begin{array}{l} \frac{c_\mu}{2}\left(\frac{1}{2} - 0.75 \right) \left[\left(\psi_2 - \psi_1\right) - \frac{\sin 2\psi_2 - \sin 2\psi_1}{2} \right] \\ -\left[\frac{2}{3} - 0.75\left(1 - c_\mu y_1\right) \right] \left(\cos\psi_2 - \cos\psi_1\right) - \frac{c_\mu}{2} y_1^2 \left(\psi_2 - \psi_1\right) \\ +0.75 y_1^2 \left(\ln\left|\tan\frac{\psi_2}{2}\right| - \ln\left|\tan\frac{\psi_1}{2}\right| \right) + \frac{2}{3} y_1^3 \left(\cot\psi_2 - \cot\psi_1\right) \\ +\frac{c_\mu}{2}\left(y_2^2 - y_1^2\right)\left(\frac{\pi}{2} - \psi_2 \right) + c_\mu 0.75\left(y_2 - y_1\right)\left(\cos\frac{\pi}{2} - \cos\psi_2 \right) \\ -0.75\left(y_2^2 - y_1^2\right)\left(\ln\left|\tan\frac{\pi}{4}\right| - \ln\left|\tan\frac{\psi_2}{2}\right| \right) - \frac{2}{3}\left(y_2^3 - y_1^3\right)\left(\cot\frac{\pi}{2} - \cot\psi_2 \right) \end{array} \right] \qquad (22)$$

$$+ \text{part 2}$$

$$
\text{part } 2 = \Theta_{75}\begin{bmatrix} -\left(1-c_\mu y_1\right)\left(\cos\psi_2-\cos\psi_1\right)-y_1^2\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right) \\ +\frac{c_\mu}{2}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ -c_\mu\left(y_2-y_1\right)\left(\cos\frac{\pi}{2}-\cos\psi_2\right)+\left(y_2^2-y_1^2\right)\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right) \end{bmatrix}
$$

$$
+\Theta_S\begin{bmatrix} -y_1^2\left(\psi_2-\psi_1\right)+\left(y_2^2-y_1^2\right)\left(\frac{\pi}{2}-\psi_2\right) \\ +\left(\frac{1}{2}-\frac{c_\mu}{2}y_1\right)\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ -c_\mu\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right] \\ +\frac{c_\mu}{2}\left(y_2-y_1\right)\left[\left(\frac{\pi}{2}-\psi_2\right)-\frac{\sin\pi-\sin 2\psi_2}{2}\right] \end{bmatrix}
$$

The result can formally be written as

$$
\Delta C_T^{(3)} = \begin{bmatrix} b_{11}^{(3)} & b_{12}^{(3)} & b_{13}^{(3)} \end{bmatrix}\begin{Bmatrix} \Theta_{tw} \\ \Theta_{75} \\ \Theta_S \end{Bmatrix} \tag{23}
$$

Note that $\cot\pi/4 = 1$. At last, the fourth term is treated the same way. First, the term $\Delta dL^{(4)}$ from Eq. (5) is inserted into Eq. (4) and then the radial integration is performed.

$$
\begin{aligned}
\Delta C_T^{(4)} &= \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_1}{\sin\psi}}^{1} 2\Delta dC_T^{(4)} + \int_{\psi_2}^{\pi/2}\int_{\frac{y_1}{\sin\psi}}^{\frac{y_2}{\sin\psi}} 2\Delta dC_T^{(4)}\right]d\psi \\
2\Delta dC_T^{(4)} &= -\sigma C_{l\alpha}\left[\Delta\lambda r+\Delta_{\mu\lambda}\sin\psi\right]dr \\
\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} &= -\int_{\psi_1}^{\psi_2}\left[\Delta\lambda\frac{1}{2}\left(1-\frac{y_1^2}{\sin^2\psi}\right)+\Delta_{\mu\lambda}\left(1-\frac{y_1}{\sin\psi}\right)\sin\psi\right]d\psi \\
&\quad -\int_{\psi_2}^{\pi/2}\left[\Delta\lambda\frac{1}{2}\frac{y_2^2-y_1^2}{\sin^2\psi}+\Delta_{\mu\lambda}\frac{y_2-y_1}{\sin\psi}\sin\psi\right]d\psi
\end{aligned} \tag{24}
$$

Sorting for powers of the Sine function,

$$
\begin{aligned}
\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} &= -\int_{\psi_1}^{\psi_2}\left[\left(\frac{\Delta\lambda}{2}-\Delta_{\mu\lambda}y_1\right)+\Delta_{\mu\lambda}\sin\psi-\frac{\Delta\lambda}{2}y_1^2\frac{1}{\sin^2\psi}\right]d\psi \\
&\quad -\int_{\psi_2}^{\pi/2}\left[\Delta_{\mu\lambda}\left(y_2-y_1\right)+\frac{\Delta\lambda}{2}\left(y_2^2-y_1^2\right)\frac{1}{\sin^2\psi}\right]d\psi
\end{aligned} \tag{25}
$$

Solving the remaining integrals results in

$$\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = -\begin{bmatrix} \left(\frac{\Delta\lambda}{2} - y_1\Delta_{\mu\lambda}\right)(\psi_2 - \psi_1) - \Delta_{\mu\lambda}\left(\cos\psi_2 - \cos\psi_1\right) + \frac{\Delta\lambda}{2} y_1^2\left(\cot\psi_2 - \cot\psi_1\right) \\ +\Delta_{\mu\lambda}\left(y_2 - y_1\right)\left(\frac{\pi}{2} - \psi_2\right) - \frac{\Delta\lambda}{2}\left(y_2^2 - y_1^2\right)\left(\cot\frac{\pi}{2} - \cot\psi_2\right) \end{bmatrix} \quad (26)$$

Putting the contributions together and sorting for the perturbations,

$$\begin{aligned} \frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} &= \Delta\lambda\left[-\frac{1}{2}(\psi_2 - \psi_1) - \frac{1}{2} y_1^2\left(\cot\psi_2 - \cot\psi_1\right) + \frac{1}{2}\left(y_2^2 - y_1^2\right)\left(\cot\frac{\pi}{2} - \cot\psi_2\right)\right] \\ &+ \Delta_{\mu\lambda}\left[y_1\left(\psi_2 - \psi_1\right) + \left(\cos\psi_2 - \cos\psi_1\right) - \left(y_2 - y_1\right)\left(\frac{\pi}{2} - \psi_2\right)\right] \end{aligned} \quad (27)$$

The result can formally be written as

$$\Delta C_T^{(4)} = \begin{bmatrix} b_{14}^{(4)} & b_{15}^{(4)} \end{bmatrix} \begin{Bmatrix} \Delta\lambda \\ \Delta_{\mu\lambda} \end{Bmatrix} \quad (28)$$

Note that $\cot\pi/2 = 0$.

3.2. Contributions to the rolling moment coefficient

The derivation of the moment expressions is given next. First, $\Delta dM^{(2)}$ from Eq. (5) is inserted into Eq. (4),

$$\begin{aligned} \Delta C_{Mx}^{(2)} &= \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_1}{\sin\psi}}^{1} 2r\Delta dC_T^{(2)} + \int_{\psi_2}^{\pi/2}\int_{\frac{y_1}{\sin\psi}}^{\frac{y_2}{\sin\psi}} 2r\Delta dC_T^{(2)}\right]\sin\psi\, d\psi \\ 2r\Delta dC_T^{(2)}\sin\psi &= \sigma C_{l\alpha}\Delta\mu\begin{bmatrix} \Delta\Theta_{75}2\left(r^2\sin^2\psi + \mu r\sin^3\psi\right) \\ +\Delta\Theta_S 2\left(r^2\sin^3\psi + \mu r\sin^4\psi\right)\end{bmatrix}dr \end{aligned} \quad (29)$$

Performing the radial integration and

substitution of the radial integration bounds, the azimuthal integration needs to be performed. For convenience, the integrals are sorted for the control perturbations.

$$\begin{aligned} \frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \Delta\Theta_{75}\int_{\psi_1}^{\psi_2}\left[\frac{2}{3}\left(1 - \frac{y_1^3}{\sin^3\psi}\right)\sin^2\psi + \mu\left(1 - \frac{y_1^2}{\sin^2\psi}\right)\sin^3\psi\right]d\psi \\ &+ \Delta\Theta_S\int_{\psi_1}^{\psi_2}\left[\frac{2}{3}\left(1 - \frac{y_1^3}{\sin^3\psi}\right)\sin^3\psi + \mu\left(1 - \frac{y_1^2}{\sin^2\psi}\right)\sin^4\psi\right]d\psi \\ &+ \Delta\Theta_{75}\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(\frac{y_2^3 - y_1^3}{\sin^3\psi}\right)\sin^2\psi + \mu\left(\frac{y_2^2 - y_1^2}{\sin^2\psi}\right)\sin^3\psi\right]d\psi \\ &+ \Delta\Theta_S\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(\frac{y_2^3 - y_1^3}{\sin^3\psi}\right)\sin^3\psi + \mu\left(\frac{y_2^2 - y_1^2}{\sin^2\psi}\right)\sin^4\psi\right]d\psi \end{aligned} \quad (30)$$

Sorting for powers of the Sine function,

$$\frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\int_{\psi_1}^{\psi_2}\left[-\mu y_1^2\sin\psi + \frac{2}{3}\sin^2\psi + \mu\sin^3\psi - \frac{2}{3}y_1^3\frac{1}{\sin\psi}\right]d\psi$$

$$\begin{aligned} &+\Delta\Theta_S\int_{\psi_1}^{\psi_2}\left[-\frac{2}{3}y_1^3 - \mu y_1^2\sin^2\psi + \frac{2}{3}\sin^3\psi + \mu\sin^4\psi\right]d\psi \\ &+\Delta\Theta_{75}\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(y_2^3 - y_1^3\right)\frac{1}{\sin\psi} + \mu\left(y_2^2 - y_1^2\right)\sin\psi\right]d\psi \\ &+\Delta\Theta_S\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(y_2^3 - y_1^3\right) + \mu\left(y_2^2 - y_1^2\right)\sin^2\psi\right]d\psi \end{aligned} \tag{31}$$

Solving the remaining integrals results in

$$\frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\begin{bmatrix} \mu y_1^2\left(\cos\psi_2 - \cos\psi_1\right) + \frac{1}{3}\left[\left(\psi_2 - \psi_1\right) - \frac{\sin 2\psi_2 - \sin 2\psi_1}{2}\right] \\ -\mu\left[\left(\cos\psi_2 - \cos\psi_1\right) - \frac{\cos^3\psi_2 - \cos^3\psi_1}{3}\right] \\ -\frac{2}{3}y_1^3\left(\ln\left|\tan\frac{\psi_2}{2}\right| - \ln\left|\tan\frac{\psi_1}{2}\right|\right) + \frac{2}{3}\left(y_2^3 - y_1^3\right)\left(\ln\left|\tan\frac{\pi}{4}\right| - \ln\left|\tan\frac{\psi_2}{2}\right|\right) \\ -\mu\left(y_2^2 - y_1^2\right)\left(\cos\frac{\pi}{2} - \cos\psi_2\right) \end{bmatrix}$$

$$+\Delta\Theta_S\begin{bmatrix} -\frac{2}{3}y_1^3\left(\psi_2 - \psi_1\right) - \frac{\mu}{2}y_1^2\left[\left(\psi_2 - \psi_1\right) - \frac{\sin 2\psi_2 - \sin 2\psi_1}{2}\right] \\ -\frac{2}{3}\left[\left(\cos\psi_2 - \cos\psi_1\right) - \frac{\cos^3\psi_2 - \cos^3\psi_1}{3}\right] \\ +\frac{\mu}{2}\left[\frac{3}{4}\left(\psi_2 - \psi_1\right) - \frac{\sin 2\psi_2 - \sin 2\psi_1}{2} + \frac{\sin 4\psi_2 - \sin 4\psi_1}{16}\right] \\ +\frac{2}{3}\left(y_2^3 - y_1^3\right)\left(\frac{\pi}{2} - \psi_2\right) + \frac{\mu}{2}\left(y_2^2 - y_1^2\right)\left[\left(\frac{\pi}{2} - \psi_2\right) - \frac{\sin\pi - \sin 2\psi_2}{2}\right] \end{bmatrix} \tag{32}$$

In matrix notation, this becomes

$$\Delta C_{Mx}^{(2)} = \begin{bmatrix} a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix}\begin{Bmatrix} \Delta\Theta_{75} \\ \Delta\Theta_S \end{Bmatrix} \tag{33}$$

Now the third term is treated the same way, by using $\Delta dM^{(3)}$ from Eq. (5).

$$\Delta C_{Mx}^{(3)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_1}{\sin\psi}}^{1} 2r\Delta dC_T^{(3)} + \int_{\psi_2}^{\pi/2}\int_{\frac{y_1}{\sin\psi}}^{\frac{y_2}{\sin\psi}} 2r\Delta dC_T^{(3)}\right]\sin\psi\, d\psi$$

$$2r\Delta dC_T^{(3)}\sin\psi = \sigma C_{l\alpha}\Delta\mu\begin{bmatrix} \Theta_{tw}\left[2\left(r^3 - 0.75r^2\right)\sin^2\psi + c_\mu\left(r^2 - 0.75r\right)\sin^3\psi\right] \\ +\Theta_{75}\left(2r^2\sin^2\psi + c_\mu r\sin^3\psi\right) \\ +\Theta_S\left(2r^2\sin^3\psi + c_\mu r\sin^4\psi\right) \end{bmatrix}dr \tag{34}$$

Performing the radial integration, substitution of the radial integration bounds and sorting for blade twist and control angles,

$$
\begin{aligned}
\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = {} & \Theta_{tw}\int_{\psi_1}^{\psi_2}\left[\begin{array}{l}\frac{1}{2}\left(1-\frac{y_1^4}{\sin^4\psi}\right)\sin^2\psi-0.75\frac{2}{3}\left(1-\frac{y_1^3}{\sin^3\psi}\right)\sin^2\psi\\ +\frac{c_\mu}{3}\left(1-\frac{y_1^3}{\sin^3\psi}\right)\sin^3\psi-0.75\frac{c_\mu}{2}\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin^3\psi\end{array}\right]d\psi \\
& +\Theta_{75}\int_{\psi_1}^{\psi_2}\left[\frac{2}{3}\left(1-\frac{y_1^3}{\sin^3\psi}\right)\sin^2\psi+\frac{c_\mu}{2}\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin^3\psi\right]d\psi \\
& +\Theta_S\int_{\psi_1}^{\psi_2}\left[\frac{2}{3}\left(1-\frac{y_1^3}{\sin^3\psi}\right)\sin^3\psi+\frac{c_\mu}{2}\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin^4\psi\right]d\psi \\
& +\Theta_{tw}\int_{\psi_2}^{\pi/2}\left[\begin{array}{l}\frac{1}{2}\left(\frac{y_2^4-y_1^4}{\sin^4\psi}\right)\sin^2\psi-0.75\frac{2}{3}\left(\frac{y_2^3-y_1^3}{\sin^3\psi}\right)\sin^2\psi\\ +\frac{c_\mu}{3}\left(\frac{y_2^3-y_1^3}{\sin^3\psi}\right)\sin^3\psi-0.75\frac{c_\mu}{2}\left(\frac{y_2^2-y_1^2}{\sin^2\psi}\right)\sin^3\psi\end{array}\right]d\psi \\
& +\Theta_{75}\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(\frac{y_2^3-y_1^3}{\sin^3\psi}\right)\sin^2\psi+\frac{c_\mu}{2}\left(\frac{y_2^2-y_1^2}{\sin^2\psi}\right)\sin^3\psi\right]d\psi \\
& +\Theta_S\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(\frac{y_2^3-y_1^3}{\sin^3\psi}\right)\sin^3\psi+\frac{c_\mu}{2}\left(\frac{y_2^2-y_1^2}{\sin^2\psi}\right)\sin^4\psi\right]d\psi
\end{aligned}
\tag{35}
$$

Sorting for powers of the Sine function,

$$
\begin{aligned}
\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = {} & \Theta_{tw}\int_{\psi_1}^{\psi_2}\left[\begin{array}{l}-\frac{c_\mu}{3}y_1^3+0.75\frac{c_\mu}{2}y_1^2\sin\psi+\cancel{\left(\frac{1}{2}-0.75\frac{2}{3}\right)}\sin^2\psi\\ +\left(\frac{2}{3}-0.75\right)\frac{c_\mu}{2}\sin^3\psi+0.75\frac{2}{3}y_1^3\frac{1}{\sin\psi}-\frac{1}{2}y_1^4\frac{1}{\sin^2\psi}\end{array}\right]d\psi \\
& +\Theta_{tw}\int_{\psi_2}^{\pi/2}\left[\begin{array}{l}\frac{c_\mu}{3}\left(y_2^3-y_1^3\right)-0.75\frac{c_\mu}{2}\left(y_2^2-y_1^2\right)\sin\psi\\ -0.75\frac{2}{3}\left(y_2^3-y_1^3\right)\frac{1}{\sin\psi}+\frac{1}{2}\left(y_2^4-y_1^4\right)\frac{1}{\sin^2\psi}\end{array}\right]d\psi \\
& +\Theta_{75}\int_{\psi_1}^{\psi_2}\left[-\frac{c_\mu}{2}y_1^2\sin\psi+\frac{2}{3}\sin^2\psi+\frac{c_\mu}{2}\sin^3\psi-\frac{2}{3}y_1^3\frac{1}{\sin\psi}\right]d\psi \\
& +\Theta_{75}\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(y_2^3-y_1^3\right)\frac{1}{\sin\psi}+\frac{c_\mu}{2}\left(y_2^2-y_1^2\right)\sin\psi\right]d\psi \\
& +\Theta_S\int_{\psi_1}^{\psi_2}\left[-\frac{2}{3}y_1^3-\frac{c_\mu}{2}y_1^2\sin^2\psi+\frac{2}{3}\sin^3\psi+\frac{c_\mu}{2}\sin^4\psi\right]d\psi \\
& +\Theta_S\int_{\psi_2}^{\pi/2}\left[\frac{2}{3}\left(y_2^3-y_1^3\right)+\frac{c_\mu}{2}\left(y_2^2-y_1^2\right)\sin^2\psi\right]d\psi
\end{aligned}
\tag{36}
$$

Solving the remaining integrals results in (note: $1/2-0.75\cdot 2/3=0$)

$$\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu}=\Theta_{tw}\begin{bmatrix}-\frac{c_\mu}{3}y_1^3(\psi_2-\psi_1)-0.75\frac{c_\mu}{2}y_1^2(\cos\psi_2-\cos\psi_1)\\ -\left(\frac{2}{3}-0.75\right)\frac{c_\mu}{2}\left[(\cos\psi_2-\cos\psi_1)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\ +0.75\frac{2}{3}y_1^3\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right)+\frac{1}{2}y_1^4(\cot\psi_2-\cot\psi_1)\\ +\frac{c_\mu}{3}\left(y_2^3-y_1^3\right)\left(\frac{\pi}{2}-\psi_2\right)+0.75\frac{c_\mu}{2}\left(y_2^2-y_1^2\right)\left(\cos\frac{\pi}{2}-\cos\psi_2\right)\\ -0.75\frac{2}{3}\left(y_2^3-y_1^3\right)\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)\\ -\frac{1}{2}\left(y_2^4-y_1^4\right)\left(\cot\frac{\pi}{2}-\cot\psi_2\right)\end{bmatrix}$$

$$+\Theta_{75}\begin{bmatrix}\frac{c_\mu}{2}y_1^2(\cos\psi_2-\cos\psi_1)+\frac{1}{3}\left[(\psi_2-\psi_1)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\ -\frac{c_\mu}{2}\left[(\cos\psi_2-\cos\psi_1)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\ -\frac{2}{3}y_1^3\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right)\\ +\frac{2}{3}\left(y_2^3-y_1^3\right)\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)\\ -\frac{c_\mu}{2}\left(y_2^2-y_1^2\right)\left(\cos\frac{\pi}{2}-\cos\psi_2\right)\end{bmatrix}$$

$$+\Theta_S\begin{bmatrix}-\frac{2}{3}y_1^3(\psi_2-\psi_1)-\frac{c_\mu}{4}y_1^2\left[(\psi_2-\psi_1)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\ -\frac{2}{3}\left[(\cos\psi_2-\cos\psi_1)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\ +\frac{c_\mu}{4}\left[\frac{3}{4}(\psi_2-\psi_1)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}+\frac{\sin 4\psi_2-\sin 4\psi_1}{16}\right]\\ +\frac{2}{3}\left(y_2^3-y_1^3\right)\left(\frac{\pi}{2}-\psi_2\right)+\frac{c_\mu}{4}\left(y_2^2-y_1^2\right)\left[\left(\frac{\pi}{2}-\psi_2\right)-\frac{\sin\pi-\sin 2\psi_2}{2}\right]\end{bmatrix} \quad (37)$$

In matrix notation, this becomes

$$\Delta C_{Mx}^{(3)}=\begin{bmatrix}b_{21}^{(3)} & b_{22}^{(3)} & b_{23}^{(3)}\end{bmatrix}\begin{Bmatrix}\Theta_{tw}\\ \Theta_{75}\\ \Theta_S\end{Bmatrix} \quad (38)$$

At last, the fourth term is treated the same way, using $\Delta dM^{(4)}$ from Eq. (5).

$$\Delta C_{Mx}^{(4)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_1}{\sin\psi}}^{1} 2r\Delta dC_T^{(4)} + \int_{\psi_2}^{\pi/2}\int_{\frac{y_1}{\sin\psi}}^{\frac{y_2}{\sin\psi}} 2r\Delta dC_T^{(4)}\right]\sin\psi\, d\psi \tag{39}$$

$$2r\Delta dC_T^{(4)}\sin\psi = -\sigma C_{l\alpha}\left[\Delta\lambda r^2\sin\psi + \Delta_{\mu\lambda} r\sin^2\psi\right]dr$$

Performing the radial integration and substitution of the radial integration bounds:

$$\begin{aligned}\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} &= -\Delta\lambda\left\{\int_{\psi_1}^{\psi_2}\left[\frac{1}{3}\left(1-\frac{y_1^3}{\sin^3\psi}\right)\sin\psi\right]d\psi + \int_{\psi_2}^{\pi/2}\left[\frac{1}{3}\left(\frac{y_2^3-y_1^3}{\sin^3\psi}\right)\sin\psi\right]d\psi\right\}\\ &-\Delta_{\mu\lambda}\left\{\int_{\psi_1}^{\psi_2}\left[\frac{1}{2}\left(1-\frac{y_1^2}{\sin^2\psi}\right)\sin^2\psi\right]d\psi + \int_{\psi_2}^{\pi/2}\left[\frac{1}{2}\left(\frac{y_2^2-y_1^2}{\sin^2\psi}\right)\sin^2\psi\right]d\psi\right\}\end{aligned} \tag{40}$$

Sorting for powers of the Sine function,

$$\begin{aligned}\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} &= -\Delta\lambda\frac{1}{3}\left\{\int_{\psi_1}^{\psi_2}\left[\sin\psi - y_1^3\frac{1}{\sin^2\psi}\right]d\psi + \int_{\psi_2}^{\pi/2}\left[\left(y_2^3-y_1^3\right)\frac{1}{\sin^2\psi}\right]d\psi\right\}\\ &-\Delta_{\mu\lambda}\frac{1}{2}\left\{\int_{\psi_1}^{\psi_2}\left[\sin^2\psi - y_1^2\right]d\psi + \int_{\psi_2}^{\pi/2}\left[\left(y_2^2-y_1^2\right)\right]d\psi\right\}\end{aligned} \tag{41}$$

Solving the remaining integrals results in

$$\begin{aligned}\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} &= \Delta\lambda\left[\begin{aligned}&\frac{1}{3}\left(\cos\psi_2-\cos\psi_1\right)-\frac{1}{3}y_1^3\left(\cot\psi_2-\cot\psi_1\right)\\ &+\frac{1}{3}\left(y_2^3-y_1^3\right)\left(\cot\frac{\pi}{2}-\cot\psi_2\right)\end{aligned}\right]\\ &+\Delta_{\mu\lambda}\left[\begin{aligned}&\frac{1}{2}y_1^2\left(\psi_2-\psi_1\right)-\frac{1}{4}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\ &-\frac{1}{2}\left(y_2^2-y_1^2\right)\left(\frac{\pi}{2}-\psi_2\right)\end{aligned}\right]\end{aligned} \tag{42}$$

In matrix notation, this becomes

$$\Delta C_{Mx}^{(4)} = \begin{bmatrix} b_{24}^{(4)} & b_{25}^{(4)}\end{bmatrix}\begin{Bmatrix}\Delta\lambda\\ \Delta_{\mu\lambda}\end{Bmatrix} \tag{43}$$

6. Case III: The slipstream covers the rotor center

This case represents the overlap with the rotor's center, with the first integral covering the retreating side from ψ_1 to the advancing side ψ_2, the second one from ψ_2 to $\pi/2$ and the third from $-\pi/2$ to ψ_1. Therefore, the total expression becomes longer as for Case II.

4.1. Contributions to the thrust coefficient

From Eqs. (4) and (5):

$$\Delta C_T^{(2)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_0^1 2\Delta dC_T^{(2)} + \int_{\psi_2}^{\pi/2}\int_0^{\frac{y_2}{\sin\psi}} 2\Delta dC_T^{(2)} + \int_{-\pi/2}^{\psi_1}\int_0^{\frac{y_1}{\sin\psi}} 2\Delta dC_T^{(2)}\right]d\psi \tag{44}$$

$$2\Delta dC_T^{(2)} = \sigma C_{l\alpha}\Delta\mu\left[\Delta\Theta_{75}2\left[r\sin\psi + \mu\sin^2\psi\right] + \Delta\Theta_S 2\left[r\sin^2\psi + \mu\sin^3\psi\right]\right]dr$$

Performing the radial integration and inserting the radial bounds:

$$\begin{aligned}\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = &\int_{\psi_1}^{\psi_2}\begin{bmatrix}\Delta\Theta_{75}\left[\sin\psi + 2\mu\sin^2\psi\right]\\ +\Delta\Theta_S\left[\sin^2\psi + 2\mu\sin^3\psi\right]\end{bmatrix}d\psi \\ &+\int_{\psi_2}^{\pi/2}\begin{bmatrix}\Delta\Theta_{75}\left[\frac{y_2^2}{\sin^2\psi}\sin\psi + 2\mu\frac{y_2}{\sin\psi}\sin^2\psi\right]\\ +\Delta\Theta_S\left[\frac{y_2^2}{\sin^2\psi}\sin^2\psi + 2\mu\frac{y_2}{\sin\psi}\sin^3\psi\right]\end{bmatrix}d\psi \\ &+\int_{-\pi/2}^{\psi_1}\begin{bmatrix}\Delta\Theta_{75}\left[\frac{y_1^2}{\sin^2\psi}\sin\psi + 2\mu\frac{y_1}{\sin\psi}\sin^2\psi\right]\\ +\Delta\Theta_S\left[\frac{y_1^2}{\sin^2\psi}\sin^2\psi + 2\mu\frac{y_1}{\sin\psi}\sin^3\psi\right]\end{bmatrix}d\psi\end{aligned} \tag{45}$$

Sorting for powers of the Sine function,

$$\begin{aligned}\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = &\int_{\psi_1}^{\psi_2}\begin{bmatrix}\Delta\Theta_{75}\left[\sin\psi + 2\mu\sin^2\psi\right]\\ +\Delta\Theta_S\left[\sin^2\psi + 2\mu\sin^3\psi\right]\end{bmatrix}d\psi \\ &+\int_{\psi_2}^{\pi/2}\begin{bmatrix}\Delta\Theta_{75}\left[y_2^2\frac{1}{\sin\psi} + 2\mu y_2\sin\psi\right]\\ +\Delta\Theta_S\left[y_2^2 + 2\mu y_2\sin^2\psi\right]\end{bmatrix}d\psi \\ &+\int_{-\pi/2}^{\psi_1}\begin{bmatrix}\Delta\Theta_{75}\left[y_1^2\frac{1}{\sin\psi} + 2\mu y_1\sin\psi\right]\\ +\Delta\Theta_S\left[y_1^2 + 2\mu y_1\sin^2\psi\right]\end{bmatrix}d\psi\end{aligned} \tag{46}$$

Solving the remaining integrals,

$$\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\begin{bmatrix} -\left(\cos\psi_2-\cos\psi_1\right)+\mu\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ +y_2^2\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)-2\mu y_2\left(\cos\frac{\pi}{2}-\cos\psi_2\right) \\ +y_1^2\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)-2\mu y_1\left(\cos\psi_1-\cos\frac{-\pi}{2}\right) \end{bmatrix}$$

$$+\Delta\Theta_S\begin{bmatrix} \frac{1}{2}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ -2\mu\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right] \\ +y_2^2\left(\frac{\pi}{2}-\psi_2\right)+\mu y_2\left[\left(\frac{\pi}{2}-\psi_2\right)-\frac{\sin\pi-\sin 2\psi_2}{2}\right] \\ +y_1^2\left(\psi_1-\frac{-\pi}{2}\right)+\mu y_1\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin\left(-\pi\right)}{2}\right] \end{bmatrix} \tag{47}$$

In matrix form

$$\Delta C_T^{(2)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \end{bmatrix}\begin{Bmatrix} \Delta\Theta_{75} \\ \Delta\Theta_S \end{Bmatrix} \tag{48}$$

The third contribution is

$$\Delta C_T^{(3)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_0^1 2\Delta dC_T^{(3)} + \int_{\psi_2}^{\pi/2}\int_0^{\frac{y_2}{\sin\psi}} 2\Delta dC_T^{(3)} + \int_{-\pi/2}^{\psi_1}\int_0^{\frac{y_1}{\sin\psi}} 2\Delta dC_T^{(3)}\right]d\psi$$

$$2\Delta dC_T^{(3)} = \sigma C_{l\alpha}\Delta\mu\begin{bmatrix} \Theta_{tw}\left[2\left(r^2-0.75r\right)\sin\psi+c_\mu\left(r-0.75\right)\sin^2\psi\right] \\ +\Theta_{75}\left(2r\sin\psi+c_\mu\sin^2\psi\right) \\ +\Theta_S\left(2r\sin^2\psi+c_\mu\sin^3\psi\right) \end{bmatrix}dr \tag{49}$$

Performing the radial integration and inserting integration bounds:

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \int_{\psi_1}^{\psi_2}\begin{bmatrix}\Theta_{tw}\left[\left(\frac{2}{3}-0.75\right)\sin\psi + c_\mu\left(\frac{1}{2}-0.75\right)\sin^2\psi\right] \\ +\Theta_{75}\left(\sin\psi + c_\mu\sin^2\psi\right)+\Theta_S\left(\sin^2\psi + c_\mu\sin^3\psi\right)\end{bmatrix}d\psi$$

$$+\int_{\psi_2}^{\pi/2}\begin{bmatrix}\Theta_{tw}\begin{bmatrix}\left(\frac{2}{3}\frac{y_2^3}{\sin^3\psi}-0.75\frac{y_2^2}{\sin^2\psi}\right)\sin\psi \\ +c_\mu\left(\frac{1}{2}\frac{y_2^2}{\sin^2\psi}-0.75\frac{y_2}{\sin\psi}\right)\sin^2\psi\end{bmatrix} \\ +\Theta_{75}\left(\frac{y_2^2}{\sin^2\psi}\sin\psi + c_\mu\frac{y_2}{\sin\psi}\sin^2\psi\right) \\ +\Theta_S\left(\frac{y_2^2}{\sin^2\psi}\sin^2\psi + c_\mu\frac{y_2}{\sin\psi}\sin^3\psi\right)\end{bmatrix}d\psi$$

$$+\int_{-\pi/2}^{\psi_1}\begin{bmatrix}\Theta_{tw}\begin{bmatrix}\left(\frac{2}{3}\frac{y_1^3}{\sin^3\psi}-0.75\frac{y_1^2}{\sin^2\psi}\right)\sin\psi \\ +c_\mu\left(\frac{1}{2}\frac{y_1^2}{\sin^2\psi}-0.75\frac{y_1}{\sin\psi}\right)\sin^2\psi\end{bmatrix} \\ +\Theta_{75}\left(\frac{y_1^2}{\sin^2\psi}\sin\psi + c_\mu\frac{y_1}{\sin\psi}\sin^2\psi\right) \\ +\Theta_S\left(\frac{y_1^2}{\sin^2\psi}\sin^2\psi + c_\mu\frac{y_1}{\sin\psi}\sin^3\psi\right)\end{bmatrix}d\psi \tag{50}$$

Sorting for powers of the Sine function,

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \int_{\psi_1}^{\psi_2}\begin{bmatrix}\Theta_{tw}\left[\left(\frac{2}{3}-0.75\right)\sin\psi + c_\mu\left(\frac{1}{2}-0.75\right)\sin^2\psi\right] \\ +\Theta_{75}\left(\sin\psi + c_\mu\sin^2\psi\right)+\Theta_S\left(\sin^2\psi + c_\mu\sin^3\psi\right)\end{bmatrix}d\psi$$
$$+\int_{\psi_2}^{\pi/2}\begin{bmatrix}\Theta_{tw}\left(\frac{2}{3}y_2^3\frac{1}{\sin^2\psi}-0.75y_2^2\frac{1}{\sin\psi}+\frac{c_\mu}{2}y_2^2-0.75c_\mu y_2\sin\psi\right) \\ +\Theta_{75}\left(y_2^2\frac{1}{\sin\psi}+c_\mu y_2\sin\psi\right)+\Theta_S\left(y_2^2+c_\mu y_2\sin^2\psi\right)\end{bmatrix}d\psi \tag{51}$$
$$+\int_{-\pi/2}^{\psi_1}\begin{bmatrix}\Theta_{tw}\left(\frac{2}{3}y_1^3\frac{1}{\sin^2\psi}-0.75y_1^2\frac{1}{\sin\psi}+\frac{c_\mu}{2}y_1^2-0.75c_\mu y_1\sin\psi\right) \\ +\Theta_{75}\left(y_1^2\frac{1}{\sin\psi}+c_\mu y_1\sin\psi\right)+\Theta_S\left(y_1^2+c_\mu y_1\sin^2\psi\right)\end{bmatrix}d\psi$$

Solving the remaining integrals,

$$
\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu}=\Theta_{tw}\begin{bmatrix}
-\left(\frac{2}{3}-0.75\right)\left(\cos\psi_2-\cos\psi_1\right)\\
+\frac{c_\mu}{2}\left(\frac{1}{2}-0.75\right)\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\
-\frac{2}{3}y_2^3\left(\cot\frac{\pi}{2}-\cot\psi_2\right)-0.75y_2^2\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)\\
+\frac{c_\mu}{2}y_2^2\left(\frac{\pi}{2}-\psi_2\right)+0.75c_\mu y_2\left(\cos\frac{\pi}{2}-\cos\psi_2\right)\\
-\frac{2}{3}y_1^3\left(\cot\psi_1-\cot\frac{-\pi}{2}\right)-0.75y_1^2\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)\\
+\frac{c_\mu}{2}y_1^2\left(\psi_1-\frac{-\pi}{2}\right)+0.75c_\mu y_1\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)
\end{bmatrix}
$$

$$
+\Theta_{75}\begin{bmatrix}
-\left(\cos\psi_2-\cos\psi_1\right)+\frac{c_\mu}{2}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\
+y_2^2\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)-c_\mu y_2\left(\cos\frac{\pi}{2}-\cos\psi_2\right)\\
+y_1^2\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)-c_\mu y_1\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)
\end{bmatrix}
$$

$$
+\Theta_S\begin{bmatrix}
\frac{1}{2}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\
-c_\mu\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\
+y_2^2\left(\frac{\pi}{2}-\psi_2\right)+\frac{c_\mu}{2}y_2\left[\left(\frac{\pi}{2}-\psi_2\right)-\frac{\sin\pi-\sin 2\psi_2}{2}\right]\\
+y_1^2\left(\psi_1-\frac{-\pi}{2}\right)+\frac{c_\mu}{2}y_1\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin(-\pi)}{2}\right]
\end{bmatrix}\tag{52}
$$

In matrix form

$$
\Delta C_T^{(3)}=\left[b_{11}^{(3)}\quad b_{12}^{(3)}\quad b_{13}^{(3)}\right]\begin{Bmatrix}\Theta_{tw}\\ \Theta_{75}\\ \Theta_S\end{Bmatrix}\tag{53}
$$

Finally, for the fourth term

$$
\Delta C_T^{(4)}=\frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_0^1 2\Delta dC_T^{(4)}+\int_{\psi_2}^{\pi/2}\int_0^{\frac{y_2}{\sin\psi}}2\Delta dC_T^{(4)}+\int_{-\pi/2}^{\psi_1}\int_0^{\frac{y_1}{\sin\psi}}2\Delta dC_T^{(4)}\right]d\psi\tag{54}
$$

$$
2\Delta dC_T^{(4)}=-\sigma C_{l\alpha}\left[\Delta\lambda r+\Delta_{\mu\lambda}\sin\psi\right]dr
$$

After integration and substitution of the radial integration bounds

$$\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = -\int_{\psi_1}^{\psi_2}\left[\Delta\lambda\frac{1}{2}+\Delta_{\mu\lambda}\sin\psi\right]d\psi - \int_{\psi_2}^{\pi/2}\left[\Delta\lambda\frac{1}{2}y_2^2\frac{1}{\sin^2\psi}+\Delta_{\mu\lambda}y_2\right]d\psi$$

$$-\int_{-\pi/2}^{\psi_1}\left[\Delta\lambda\frac{1}{2}y_1^2\frac{1}{\sin^2\psi}+\Delta_{\mu\lambda}y_1\right]d\psi \tag{55}$$

Solving the remaining integrals,

$$\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = \Delta\lambda\left[-\frac{1}{2}(\psi_2-\psi_1)+\frac{1}{2}y_2^2\left(\cot\frac{\pi}{2}-\cot\psi_2\right)+\frac{1}{2}y_1^2\left(\cot\psi_1-\cot\frac{-\pi}{2}\right)\right]$$

$$+\Delta_{\mu\lambda}\left[(\cos\psi_2-\cos\psi_1)-y_2\left(\frac{\pi}{2}-\psi_2\right)-y_1\left(\psi_1-\frac{-\pi}{2}\right)\right] \tag{56}$$

The result can formally be written as

$$\Delta C_T^{(4)} = \begin{bmatrix} b_{14}^{(4)} & b_{15}^{(4)} \end{bmatrix}\begin{Bmatrix}\Delta\lambda \\ \Delta_{\mu\lambda}\end{Bmatrix} \tag{57}$$

4.2. Contributions to the rolling moment coefficient

The moment coefficient results are treated in the same manner as the thrust coefficient results.

$$\Delta C_{Mx}^{(2)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_0^1 2r\Delta dC_T^{(2)} + \int_{\psi_2}^{\pi/2}\int_0^{\frac{y_2}{\sin\psi}} 2r\Delta dC_T^{(2)} + \int_{-\pi/2}^{\psi_1}\int_0^{\frac{y_1}{\sin\psi}} 2r\Delta dC_T^{(2)}\right]\sin\psi\, d\psi$$

$$2r\Delta dC_T^{(2)}\sin\psi = \sigma C_{l\alpha}\Delta\mu\begin{bmatrix}\Delta\Theta_{75}2\left[r^2\sin^2\psi+\mu r\sin^3\psi\right] \\ +\Delta\Theta_S 2\left[r^2\sin^3\psi+\mu r\sin^4\psi\right]\end{bmatrix}dr \tag{58}$$

Performing the radial integration and inserting radial bounds

$$\frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \int_{\psi_1}^{\psi_2}\left[\Delta\Theta_{75}\left(\frac{2}{3}\sin^2\psi+\mu\sin^3\psi\right)+\Delta\Theta_S\left(\frac{2}{3}\sin^3\psi+\mu\sin^4\psi\right)\right]d\psi$$

$$+\int_{\psi_2}^{\pi/2}\left[\Delta\Theta_{75}\left(\frac{2}{3}y_2^3\frac{1}{\sin\psi}+\mu y_2^2\sin\psi\right)+\Delta\Theta_S\left(\frac{2}{3}y_2^3+\mu y_2^2\sin^2\psi\right)\right]d\psi \tag{59}$$

$$+\int_{-\pi/2}^{\psi_1}\left[\Delta\Theta_{75}\left(\frac{2}{3}y_1^3\frac{1}{\sin\psi}+\mu y_1^2\sin\psi\right)+\Delta\Theta_S\left(\frac{2}{3}y_1^3+\mu y_1^2\sin^2\psi\right)\right]d\psi$$

Solving the remaining integrals and sorting for perturbation control angles

$$\frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\left[\begin{array}{l} \frac{1}{3}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ -\mu\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right] \\ +\frac{2}{3}y_2^3\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)-\mu y_2^2\left(\cos\frac{\pi}{2}-\cos\psi_2\right) \\ +\frac{2}{3}y_1^3\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)-\mu y_1^2\left(\cos\psi_1-\cos\frac{-\pi}{2}\right) \end{array}\right]$$

$$+\Delta\Theta_S\left[\begin{array}{l} -\frac{2}{3}\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right] \\ +\frac{\mu}{2}\left[\frac{3}{4}\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}+\frac{\sin 4\psi_2-\sin 4\psi_1}{16}\right] \\ +\frac{2}{3}y_2^3\left(\frac{\pi}{2}-\psi_2\right)+\frac{\mu}{2}y_2^2\left[\left(\frac{\pi}{2}-\psi_2\right)-\frac{\sin\pi-\sin 2\psi_2}{2}\right] \\ +\frac{2}{3}y_1^3\left(\psi_1-\frac{-\pi}{2}\right)+\frac{\mu}{2}y_1^2\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin\left(-\pi\right)}{2}\right] \end{array}\right] \tag{60}$$

After sorting for the perturbation control angles, this can be written formally as

$$\Delta C_{Mx}^{(2)} = \begin{bmatrix} a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix}\begin{Bmatrix} \Delta\Theta_{75} \\ \Delta\Theta_S \end{Bmatrix} \tag{61}$$

The third contribution is treated the same way.

$$\Delta C_{Mx}^{(3)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_0^1 2r\Delta dC_T^{(3)} + \int_{\psi_2}^{\pi/2}\int_0^{\frac{y_2}{\sin\psi}} 2r\Delta dC_T^{(3)} + \int_{-\pi/2}^{\psi_1}\int_0^{\frac{y_1}{\sin\psi}} 2r\Delta dC_T^{(3)}\right]\sin\psi\, d\psi$$

$$2r\Delta dC_T^{(3)}\sin\psi = \sigma C_{l\alpha}\Delta\mu\left[\begin{array}{l} \Theta_{tw}\left[2\left(r^3-0.75r^2\right)\sin^2\psi+c_\mu\left(r^2-0.75r\right)\sin^3\psi\right] \\ +\Theta_{75}\left(2r^2\sin^2\psi+c_\mu r\sin^3\psi\right) \\ +\Theta_S\left(2r^2\sin^3\psi+c_\mu r\sin^4\psi\right) \end{array}\right]dr \tag{62}$$

Performing the radial integration and inserting radial bounds

$$\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \int_{\psi_1}^{\psi_2}\left[\begin{array}{l} \Theta_{tw}\left[\cancel{\left(\frac{1}{2}-0.75\frac{2}{3}\right)}\sin^2\psi+\frac{c_\mu}{2}\left(\frac{2}{3}-0.75\right)\sin^3\psi\right] \\ +\Theta_{75}\left(\frac{2}{3}\sin^2\psi+\frac{c_\mu}{2}\sin^3\psi\right)+\Theta_S\left(\frac{2}{3}\sin^3\psi+\frac{c_\mu}{2}\sin^4\psi\right) \end{array}\right]d\psi$$

$$+\int_{\psi_2}^{\pi/2}\left[\begin{array}{l} \Theta_{tw}\left(\frac{1}{2}y_2^4\frac{1}{\sin^2\psi}-0.75\frac{2}{3}y_2^3\frac{1}{\sin\psi}+\frac{c_\mu}{3}y_2^3-0.75\frac{c_\mu}{2}y_2^2\sin\psi\right) \\ +\Theta_{75}\left(\frac{2}{3}y_2^3\frac{1}{\sin\psi}+\frac{c_\mu}{2}y_2^2\sin\psi\right)+\Theta_S\left(\frac{2}{3}y_2^3+\frac{c_\mu}{2}y_2^2\sin^2\psi\right) \end{array}\right]d\psi \tag{63}$$

$$+\,\text{part 2}$$

$$\text{part } 2=\int_{-\pi/2}^{\psi_1}\left[\begin{array}{l}\Theta_{tw}\left(\frac{1}{2}y_1^4\frac{1}{\sin^2\psi}-0.75\frac{2}{3}y_1^3\frac{1}{\sin\psi}+\frac{c_\mu}{3}y_1^3-0.75\frac{c_\mu}{2}y_1^2\sin\psi\right)\\+\Theta_{75}\left(\frac{2}{3}y_1^3\frac{1}{\sin\psi}+\frac{c_\mu}{2}y_1^2\sin\psi\right)+\Theta_S\left(\frac{2}{3}y_1^3+\frac{c_\mu}{2}y_1^2\sin^2\psi\right)\end{array}\right]d\psi$$

Solving the remaining integrals and sorting for the twist and trim control angles

$$\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu}=\Theta_{tw}\left[\begin{array}{l}-\frac{c_\mu}{2}\left(\frac{2}{3}-0.75\right)\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\-\frac{1}{2}y_2^4\left(\cot\frac{\pi}{2}-\cot\psi_2\right)-0.75\frac{2}{3}y_2^3\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)\\+\frac{c_\mu}{3}y_2^3\left(\frac{\pi}{2}-\psi_2\right)+0.75\frac{c_\mu}{2}y_2^2\left(\cos\frac{\pi}{2}-\cos\psi_2\right)\\-\frac{1}{2}y_1^4\left(\cot\psi_1-\cot\frac{-\pi}{2}\right)-0.75\frac{2}{3}y_1^3\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)\\+\frac{c_\mu}{3}y_1^3\left(\psi_1-\frac{-\pi}{2}\right)+0.75\frac{c_\mu}{2}y_1^2\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)\end{array}\right]$$

$$+\Theta_{75}\left[\begin{array}{l}\frac{1}{3}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\-\frac{c_\mu}{2}\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\+\frac{2}{3}y_2^3\left(\ln\left|\tan\frac{\pi}{4}\right|-\ln\left|\tan\frac{\psi_2}{2}\right|\right)-\frac{c_\mu}{2}y_2^2\left(\cos\frac{\pi}{2}-\cos\psi_2\right)\\+\frac{2}{3}y_1^3\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)-\frac{c_\mu}{2}y_1^2\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)\end{array}\right]$$

$$+\Theta_S\left[\begin{array}{l}-\frac{2}{3}\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\+\frac{c_\mu}{4}\left[\frac{3}{4}\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}+\frac{\sin 4\psi_2-\sin 4\psi_1}{16}\right]\\+\frac{2}{3}y_2^3\left(\frac{\pi}{2}-\psi_2\right)+\frac{c_\mu}{4}y_2^2\left[\left(\frac{\pi}{2}-\psi_2\right)-\frac{\sin\pi-\sin 2\psi_2}{2}\right]\\+\frac{2}{3}y_1^3\left(\psi_1-\frac{-\pi}{2}\right)+\frac{c_\mu}{4}y_1^2\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin\left(-\pi\right)}{2}\right]\end{array}\right] \tag{64}$$

In matrix notation, this becomes

$$\Delta C_{Mx}^{(3)}=\left[b_{21}^{(3)}\quad b_{22}^{(3)}\quad b_{23}^{(3)}\right]\begin{Bmatrix}\Theta_{tw}\\\Theta_{75}\\\Theta_S\end{Bmatrix} \tag{65}$$

The fourth contribution is treated the same way.

$$\Delta C_{Mx}^{(4)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_0^1 2r\Delta dC_T^{(4)} + \int_{\psi_2}^{\pi/2}\int_0^{\frac{y_2}{\sin\psi}} 2r\Delta dC_T^{(4)} + \int_{-\pi/2}^{\psi_1}\int_0^{\frac{y_1}{\sin\psi}} 2r\Delta dC_T^{(4)}\right]\sin\psi\, d\psi \tag{66}$$

$$2r\Delta dC_T^{(4)}\sin\psi = -\sigma C_{l\alpha}\left[\Delta\lambda r^2\sin\psi + \Delta_{\mu\lambda} r\sin^2\psi\right]dr$$

Performing the radial integration and inserting radial bounds

$$\begin{aligned}\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = &-\int_{\psi_1}^{\psi_2}\left[\Delta\lambda\frac{1}{3}\sin\psi + \Delta_{\mu\lambda}\frac{1}{2}\sin^2\psi\right]d\psi - \int_{\psi_2}^{\pi/2}\left[\Delta\lambda\frac{1}{3}y_2^3\frac{1}{\sin^2\psi} + \Delta_{\mu\lambda}\frac{1}{2}y_2^2\right]d\psi \\ &-\int_{-\pi/2}^{\psi_1}\left[\Delta\lambda\frac{1}{3}y_1^3\frac{1}{\sin^2\psi} + \Delta_{\mu\lambda}\frac{1}{2}y_1^2\right]d\psi\end{aligned} \tag{67}$$

Solving the remaining integrals and sorting for perturbation flow components

$$\begin{aligned}\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = &\Delta\lambda\left[\frac{1}{3}\left(\cos\psi_2 - \cos\psi_1\right) + \frac{1}{3}y_2^3\left(\cot\frac{\pi}{2} - \cot\psi_2\right) + \frac{1}{3}y_1^3\left(\cot\psi_1 - \cot\frac{-\pi}{2}\right)\right] \\ &+\Delta_{\mu\lambda}\left[-\frac{1}{4}\left[\left(\psi_2 - \psi_1\right) - \frac{\sin 2\psi_2 - \sin 2\psi_1}{2}\right] - \frac{1}{2}y_2^2\left(\frac{\pi}{2} - \psi_2\right) - \frac{1}{2}y_1^2\left(\psi_1 - \frac{-\pi}{2}\right)\right]\end{aligned} \tag{68}$$

In matrix notation, this becomes

$$\Delta C_{Mx}^{(4)} = \begin{bmatrix} b_{24}^{(4)} & b_{25}^{(4)}\end{bmatrix}\begin{Bmatrix}\Delta\lambda \\ \Delta_{\mu\lambda}\end{Bmatrix} \tag{69}$$

7. Case IV: The slipstream is within the retreating side of the rotor

This case represents the slipstream only affecting the retreating side of the rotor's disk, similar to Case II for the advancing side. Now only "Region 3" and "Region 1" are to be computed and the radial integration begins at $y_2/\sin\psi$ for both regions.

5.1. Contributions to the thrust coefficient

From Eqs. (4) and (5):

$$\Delta C_T^{(2)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_2}{\sin\psi}}^{1} 2\Delta dC_T^{(2)} + \int_{-\pi/2}^{\psi_1}\int_{\frac{y_2}{\sin\psi}}^{\frac{y_1}{\sin\psi}} 2\Delta dC_T^{(2)}\right]d\psi \tag{70}$$

$$2\Delta dC_T^{(2)} = \sigma C_{l\alpha}\Delta\mu\left[\Delta\Theta_{75}2\left(r\sin\psi+\mu\sin^2\psi\right)+\Delta\Theta_S 2\left(r\sin^2\psi+\mu\sin^3\psi\right)\right]dr$$

Performing the radial integration results in

$$\begin{aligned}\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \int_{\psi_1}^{\psi_2}\left[\begin{array}{l}\Delta\Theta_{75}\left[\sin\psi - y_2^2\frac{1}{\sin\psi}+2\mu\left(\sin^2\psi - y_2\sin\psi\right)\right]\\ +\Delta\Theta_S\left[\sin^2\psi - y_2^2+2\mu\left(\sin^3\psi - y_2\sin^2\psi\right)\right]\end{array}\right]d\psi\\ &+\int_{-\pi/2}^{\psi_1}\left[\begin{array}{l}\Delta\Theta_{75}\left[\left(y_1^2-y_2^2\right)\frac{1}{\sin\psi}+2\mu\left(y_1-y_2\right)\sin\psi\right]\\ +\Delta\Theta_S\left[\left(y_1^2-y_2^2\right)+2\mu\left(y_1-y_2\right)\sin^2\psi\right]\end{array}\right]d\psi\end{aligned} \tag{71}$$

Solving the remaining integrals and sorting for perturbation control angles

$$\begin{aligned}\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \Delta\Theta_{75}\left[\begin{array}{l}-\left(1-2\mu y_2\right)\left(\cos\psi_2-\cos\psi_1\right)-y_2^2\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right)\\ +\mu\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\ +\left(y_1^2-y_2^2\right)\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)\\ -2\mu\left(y_1-y_2\right)\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)\end{array}\right]\\ &+\Delta\Theta_S\left[\begin{array}{l}\left(\frac{1}{2}-\mu y_2\right)\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]-y_2^2\left(\psi_2-\psi_1\right)\\ -2\mu\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]\\ +\left(y_1^2-y_2^2\right)\left(\psi_1-\frac{-\pi}{2}\right)\\ +\mu\left(y_1-y_2\right)\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin\left(-\pi\right)}{2}\right]\end{array}\right]\end{aligned} \tag{72}$$

In matrix form

$$\Delta C_T^{(2)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \end{bmatrix} \begin{Bmatrix} \Delta\Theta_{75} \\ \Delta\Theta_S \end{Bmatrix} \quad (73)$$

The third contribution is

$$\Delta C_T^{(3)} = \frac{1}{2\pi} \left[\int_{\psi_1}^{\psi_2} \int_{\frac{y_2}{\sin\psi}}^{1} 2\Delta dC_T^{(3)} + \int_{-\pi/2}^{\psi_1} \int_{\frac{y_2}{\sin\psi}}^{\frac{y_1}{\sin\psi}} 2\Delta dC_T^{(3)} \right] d\psi$$

$$2\Delta dC_T^{(3)} = \sigma C_{l\alpha} \Delta\mu \begin{bmatrix} \Theta_{tw}\left[2\left(r^2 - 0.75r\right)\sin\psi + c_\mu\left(r - 0.75\right)\sin^2\psi\right] \\ +\Theta_{75}\left(2r\sin\psi + c_\mu \sin^2\psi\right) \\ +\Theta_S\left(2r\sin^2\psi + c_\mu \sin^3\psi\right) \end{bmatrix} dr \quad (74)$$

Performing the radial integration and sorting for powers of the Sine function:

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \int_{\psi_1}^{\psi_2} \begin{bmatrix} \Theta_{tw} \begin{bmatrix} \left(\frac{2}{3} - 0.75\left(1 - c_\mu y_2\right)\right)\sin\psi - \frac{2}{3} y_2^3 \frac{1}{\sin^2\psi} + 0.75 y_2^2 \frac{1}{\sin\psi} \\ -\frac{c_\mu}{2} y_2^2 + c_\mu\left(\frac{1}{2} - 0.75\right)\sin^2\psi \end{bmatrix} \\ +\Theta_{75}\left[\left(1 - c_\mu y_2\right)\sin\psi - y_2^2 \frac{1}{\sin\psi} + c_\mu \sin^2\psi\right] \\ +\Theta_S\left[\left(1 - c_\mu y_2\right)\sin^2\psi - y_2^2 + c_\mu \sin^3\psi\right] \end{bmatrix} d\psi$$

$$+ \int_{-\pi/2}^{\psi_1} \begin{bmatrix} \Theta_{tw} \begin{bmatrix} \frac{2}{3}\left(y_1^3 - y_2^3\right)\frac{1}{\sin^2\psi} - 0.75\left(y_1^2 - y_2^2\right)\frac{1}{\sin\psi} \\ +\frac{c_\mu}{2}\left(y_1^2 - y_2^2\right) - 0.75 c_\mu\left(y_1 - y_2\right)\sin\psi \end{bmatrix} \\ +\Theta_{75}\left[\left(y_1^2 - y_2^2\right)\frac{1}{\sin\psi} + c_\mu\left(y_1 - y_2\right)\sin\psi\right] \\ +\Theta_S\left[\left(y_1^2 - y_2^2\right) + c_\mu\left(y_1 - y_2\right)\sin^2\psi\right] \end{bmatrix} d\psi \quad (75)$$

Solving the remaining integrals

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu}=\Theta_{tw}\begin{bmatrix}-\left[\frac{2}{3}-0.75\left(1-c_\mu y_2\right)\right]\left(\cos\psi_2-\cos\psi_1\right)\\+\frac{2}{3}y_2^3\left(\cot\psi_2-\cot\psi_1\right)+0.75y_2^2\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right)\\-\frac{c_\mu}{2}y_2^2\left(\psi_2-\psi_1\right)+\frac{c_\mu}{2}\left(\frac{1}{2}-0.75\right)\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\-\frac{2}{3}\left(y_1^3-y_2^3\right)\left(\cot\psi_1-\cot\frac{-\pi}{2}\right)\\-0.75\left(y_1^2-y_2^2\right)\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)\\+\frac{c_\mu}{2}\left(y_1^2-y_2^2\right)\left(\psi_1-\frac{-\pi}{2}\right)+0.75c_\mu\left(y_1-y_2\right)\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)\end{bmatrix}$$

$$+\Theta_{75}\begin{bmatrix}-\left(1-c_\mu y_2\right)\left(\cos\psi_2-\cos\psi_1\right)-y_2^2\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right)\\+\frac{c_\mu}{2}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]\\+\left(y_1^2-y_2^2\right)\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)\\-c_\mu\left(y_1-y_2\right)\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)\end{bmatrix}$$

$$+\Theta_S\begin{bmatrix}\left(\frac{1}{2}-\frac{c_\mu}{2}y_2\right)\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]-y_2^2\left(\psi_2-\psi_1\right)\\-c_\mu\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]+\left(y_1^2-y_2^2\right)\left(\psi_1-\frac{-\pi}{2}\right)\\+\frac{c_\mu}{2}\left(y_1-y_2\right)\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin\left(-\pi\right)}{2}\right]\end{bmatrix} \quad (76)$$

In matrix form

$$\Delta C_T^{(3)}=\left[b_{11}^{(3)}\quad b_{12}^{(3)}\quad b_{13}^{(3)}\right]\begin{Bmatrix}\Theta_{tw}\\\Theta_{75}\\\Theta_S\end{Bmatrix} \quad (77)$$

Finally, for the fourth term

$$\Delta C_T^{(4)}=\frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_2}{\sin\psi}}^{1}2\Delta dC_T^{(4)}+\int_{-\pi/2}^{\psi_1}\int_{\frac{y_2}{\sin\psi}}^{\frac{y_1}{\sin\psi}}2\Delta dC_T^{(4)}\right]d\psi$$
$$2\Delta dC_T^{(4)}=-\sigma C_{l\alpha}\left[\Delta\lambda r+\Delta_{\mu\lambda}\sin\psi\right]dr \quad (78)$$

After radial integration and substitution of the radial integration bounds

$$\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = -\int_{\psi_1}^{\psi_2}\left[\Delta\lambda\frac{1}{2}\left(1-y_2^2\frac{1}{\sin^2\psi}\right)+\Delta_{\mu\lambda}\left(\sin\psi-y_2\right)\right]d\psi$$
$$-\int_{-\pi/2}^{\psi_1}\left[\Delta\lambda\frac{1}{2}\left(y_1^2-y_2^2\right)\frac{1}{\sin^2\psi}+\Delta_{\mu\lambda}\left(y_1-y_2\right)\right]d\psi \tag{79}$$

Solving the remaining integrals

$$\frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = \Delta\lambda\left[-\frac{1}{2}\left(\psi_2-\psi_1\right)-\frac{1}{2}y_2^2\left(\cot\psi_2-\cot\psi_1\right)+\frac{1}{2}\left(y_1^2-y_2^2\right)\left(\cot\psi_1-\cot\frac{-\pi}{2}\right)\right]$$
$$+\Delta_{\mu\lambda}\left[\left(\cos\psi_2-\cos\psi_1\right)+y_2\left(\psi_2-\psi_1\right)-\left(y_1-y_2\right)\left(\psi_1-\frac{-\pi}{2}\right)\right] \tag{80}$$

The result can formally be written as

$$\Delta C_T^{(4)} = \begin{bmatrix} b_{14}^{(4)} & b_{15}^{(4)} \end{bmatrix}\begin{Bmatrix} \Delta\lambda \\ \Delta_{\mu\lambda} \end{Bmatrix} \tag{81}$$

5.2. Contributions to the rolling moment coefficient

The moment coefficient results are treated in the same manner as the thrust coefficient results.

$$\Delta C_{Mx}^{(2)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_2}{\sin\psi}}^{1} 2r\Delta dC_T^{(2)} + \int_{-\pi/2}^{\psi_1}\int_{\frac{y_2}{\sin\psi}}^{\frac{y_1}{\sin\psi}} 2r\Delta dC_T^{(2)}\right]\sin\psi\, d\psi$$
$$2r\Delta dC_T^{(2)}\sin\psi = \sigma C_{l\alpha}\Delta\mu\begin{bmatrix}\Delta\Theta_{75}2\left[r^2\sin^2\psi+\mu r\sin^3\psi\right] \\ +\Delta\Theta_S 2\left[r^2\sin^3\psi+\mu r\sin^4\psi\right]\end{bmatrix}dr \tag{82}$$

Performing the radial integration and inserting radial bounds

$$\frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \int_{\psi_1}^{\psi_2}\begin{bmatrix}\Delta\Theta_{75}\left[\frac{2}{3}\left(\sin^2\psi-y_2^3\frac{1}{\sin\psi}\right)+\mu\left(\sin^3\psi-y_2^2\sin\psi\right)\right] \\ +\Delta\Theta_S\left[\frac{2}{3}\left(\sin^3\psi-y_2^3\right)+\mu\left(\sin^4\psi-y_2^2\sin^2\psi\right)\right]\end{bmatrix}d\psi$$
$$+\int_{-\pi/2}^{\psi_1}\begin{bmatrix}\Delta\Theta_{75}\left[\frac{2}{3}\left(y_1^3-y_2^3\right)\frac{1}{\sin\psi}+\mu\left(y_1^2-y_2^2\right)\sin\psi\right] \\ +\Delta\Theta_S\left[\frac{2}{3}\left(y_1^3-y_2^3\right)+\mu\left(y_1^2-y_2^2\right)\sin^2\psi\right]\end{bmatrix}d\psi \tag{83}$$

Solving the remaining integrals

$$\frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\begin{bmatrix} \frac{1}{3}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}-2y_2^3\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right)\right] \\ -\mu\left[\left(\cos\psi_2-\cos\psi_1\right)\left(1-y_2^2\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right] \\ +\frac{2}{3}\left(y_1^3-y_2^3\right)\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right) \\ -\mu\left(y_1^2-y_2^2\right)\left(\cos\psi_1-\cos\frac{-\pi}{2}\right) \end{bmatrix}$$

$$+\Delta\Theta_S\begin{bmatrix} -\frac{2}{3}\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]-\frac{2}{3}y_2^3\left(\psi_2-\psi_1\right) \\ +\frac{\mu}{2}\left[\frac{3}{4}\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}+\frac{\sin 4\psi_2-\sin 4\psi_1}{16}\right] \\ -\frac{\mu}{2}y_2^2\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right] \\ +\frac{2}{3}\left(y_1^3-y_2^3\right)\left(\psi_1-\frac{-\pi}{2}\right) \\ +\frac{\mu}{2}\left(y_1^2-y_2^2\right)\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin\left(-\pi\right)}{2}\right] \end{bmatrix} \tag{84}$$

This can be written formally as

$$\Delta C_{Mx}^{(2)} = \begin{bmatrix} a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix}\begin{Bmatrix} \Delta\Theta_{75} \\ \Delta\Theta_S \end{Bmatrix} \tag{85}$$

The third contribution is treated the same way.

$$\Delta C_{Mx}^{(3)} = \frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_2}{\sin\psi}}^{1} 2r\Delta dC_T^{(3)} + \int_{-\pi/2}^{\psi_1}\int_{\frac{y_2}{\sin\psi}}^{\frac{y_1}{\sin\psi}} 2r\Delta dC_T^{(3)}\right]\sin\psi\, d\psi$$

$$2r\Delta dC_T^{(3)}\sin\psi = \sigma C_{l\alpha}\Delta\mu\begin{bmatrix} \Theta_{tw}\left[2\left(r^3-0.75r^2\right)\sin^2\psi+c_\mu\left(r^2-0.75r\right)\sin^3\psi\right] \\ +\Theta_{75}\left(2r^2\sin^2\psi+c_\mu r\sin^3\psi\right) \\ +\Theta_S\left(2r^2\sin^3\psi+c_\mu r\sin^4\psi\right) \end{bmatrix}dr \tag{86}$$

Performing the radial integration and inserting radial bounds

$$\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \int_{\psi_1}^{\psi_2} \left[\begin{array}{l} \Theta_{tw}\left[\begin{array}{l} \cancel{\left(\frac{1}{2}-\frac{2}{3}0.75\right)}\sin^2\psi - \frac{1}{2}y_2^4\frac{1}{\sin^2\psi} + \frac{2}{3}0.75y_2^3\frac{1}{\sin\psi} \\ +\left(\frac{2}{3}-0.75\right)\frac{c_\mu}{2}\sin^3\psi - \frac{c_\mu}{3}y_2^3 + 0.75\frac{c_\mu}{2}y_2^2\sin\psi \end{array} \right] \\ +\Theta_{75}\left(\frac{2}{3}\sin^2\psi - \frac{2}{3}y_2^3\frac{1}{\sin\psi} + \frac{c_\mu}{2}\sin^3\psi - \frac{c_\mu}{2}y_2^2\sin\psi\right) \\ +\Theta_S\left(\frac{2}{3}\sin^3\psi - \frac{2}{3}y_2^3 + \frac{c_\mu}{2}\sin^4\psi - \frac{c_\mu}{2}y_2^2\sin^2\psi\right) \end{array} \right] d\psi \tag{87}$$

$$+\text{part } 2$$

$$\text{part } 2 = \int_{-\pi/2}^{\psi_1} \left[\begin{array}{l} \Theta_{tw}\left[\begin{array}{l} \frac{1}{2}\left(y_1^4 - y_2^4\right)\frac{1}{\sin^2\psi} - \frac{2}{3}0.75\left(y_1^3 - y_2^3\right)\frac{1}{\sin\psi} \\ +\frac{c_\mu}{3}\left(y_1^3 - y_2^3\right) - 0.75\frac{c_\mu}{2}\left(y_1^2 - y_2^2\right)\sin\psi \end{array} \right] \\ +\Theta_{75}\left[\frac{2}{3}\left(y_1^3 - y_2^3\right)\frac{1}{\sin\psi} + \frac{c_\mu}{2}\left(y_1^2 - y_2^2\right)\sin\psi\right] \\ +\Theta_S\left[\frac{2}{3}\left(y_1^3 - y_2^3\right) + \frac{c_\mu}{2}\left(y_1^2 - y_2^2\right)\sin^2\psi\right] \end{array} \right] d\psi$$

Solving the remaining integrals

$$\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Theta_{tw} \left[\begin{array}{l} \frac{1}{2}y_2^4\left(\cot\psi_2 - \cot\psi_1\right) + \frac{2}{3}0.75y_2^3\left(\ln\left|\tan\frac{\psi_2}{2}\right| - \ln\left|\tan\frac{\psi_1}{2}\right|\right) \\ -\left(\frac{2}{3}-0.75\right)\frac{c_\mu}{2}\left[\left(\cos\psi_2 - \cos\psi_1\right) - \frac{\cos^3\psi_2 - \cos^3\psi_1}{3}\right] \\ -\frac{c_\mu}{3}y_2^3\left(\psi_2 - \psi_1\right) - 0.75\frac{c_\mu}{2}y_2^2\left(\cos\psi_2 - \cos\psi_1\right) \\ -\frac{1}{2}\left(y_1^4 - y_2^4\right)\left(\cot\psi_1 - \cot\frac{-\pi}{2}\right) \\ -\frac{2}{3}0.75\left(y_1^3 - y_2^3\right)\left(\ln\left|\tan\frac{\psi_1}{2}\right| - \ln\left|\tan\frac{-\pi}{4}\right|\right) \\ +\frac{c_\mu}{3}\left(y_1^3 - y_2^3\right)\left(\psi_1 - \frac{-\pi}{2}\right) \\ +0.75\frac{c_\mu}{2}\left(y_1^2 - y_2^2\right)\left(\cos\psi_1 - \cos\frac{-\pi}{2}\right) \end{array} \right]$$

$$+\text{part } 2 \tag{88}$$

$$
\text{part } 2 = \Theta_{75}\left[\begin{array}{l}
\frac{1}{3}\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]-\frac{2}{3}y_2^3\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\ln\left|\tan\frac{\psi_1}{2}\right|\right)\\
-\frac{c_\mu}{2}\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]+\frac{c_\mu}{2}y_2^2\left(\cos\psi_2-\cos\psi_1\right)\\
+\frac{2}{3}\left(y_1^3-y_2^3\right)\left(\ln\left|\tan\frac{\psi_1}{2}\right|-\ln\left|\tan\frac{-\pi}{4}\right|\right)\\
-\frac{c_\mu}{2}\left(y_1^2-y_2^2\right)\left(\cos\psi_1-\cos\frac{-\pi}{2}\right)
\end{array}\right]
$$

$$
+\Theta_S\left[\begin{array}{l}
-\frac{2}{3}\left[\left(\cos\psi_2-\cos\psi_1\right)-\frac{\cos^3\psi_2-\cos^3\psi_1}{3}\right]-\frac{2}{3}y_2^3\left(\psi_2-\psi_1\right)\\
+\frac{c_\mu}{4}\left[\frac{3}{4}\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}+\frac{\sin 4\psi_2-\sin 4\psi_1}{16}\right]\\
-\frac{c_\mu}{4}y_2^2\left[\left(\psi_2-\psi_1\right)-\frac{\sin 2\psi_2-\sin 2\psi_1}{2}\right]+\frac{2}{3}\left(y_1^3-y_2^3\right)\left(\psi_1-\frac{-\pi}{2}\right)\\
+\frac{c_\mu}{4}\left(y_1^2-y_2^2\right)\left[\left(\psi_1-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_1-\sin\left(-\pi\right)}{2}\right]
\end{array}\right]
$$

After sorting for the perturbation control angles, this can be written formally as

$$
\Delta C_{Mx}^{(3)}=\left[b_{21}^{(3)}\quad b_{22}^{(3)}\quad b_{23}^{(3)}\right]\left\{\begin{array}{l}\Theta_{tw}\\ \Theta_{75}\\ \Theta_S\end{array}\right\} \tag{89}
$$

The fourth contribution is treated the same way.

$$
\begin{gathered}
\Delta C_{Mx}^{(4)}=\frac{1}{2\pi}\left[\int_{\psi_1}^{\psi_2}\int_{\frac{y_2}{\sin\psi}}^{1}2r\Delta dC_T^{(4)}+\int_{-\pi/2}^{\psi_1}\int_{\frac{y_2}{\sin\psi}}^{\frac{y_1}{\sin\psi}}2r\Delta dC_T^{(4)}\right]\sin\psi\, d\psi\\
2r\Delta dC_T^{(4)}\sin\psi=-\sigma C_{l\alpha}\left[\Delta\lambda r^2\sin\psi+\Delta_{\mu\lambda}r\sin^2\psi\right]dr
\end{gathered} \tag{90}
$$

Performing the radial integration and inserting radial bounds

$$
\begin{aligned}
\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}}=&-\int_{\psi_1}^{\psi_2}\left[\Delta\lambda\frac{1}{3}\left(\sin\psi-y_2^3\frac{1}{\sin^2\psi}\right)+\Delta_{\mu\lambda}\frac{1}{2}\left(\sin^2\psi-y_2^2\right)\right]d\psi\\
&-\int_{-\pi/2}^{\psi_1}\left[\Delta\lambda\frac{1}{3}\left(y_1^3-y_2^3\right)\frac{1}{\sin^2\psi}+\Delta_{\mu\lambda}\frac{1}{2}\left(y_1^2-y_2^2\right)\right]d\psi
\end{aligned} \tag{91}
$$

Solving the remaining integrals

$$\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} = \Delta\lambda \begin{bmatrix} \frac{1}{3}\left(\cos\psi_2 - \cos\psi_1\right) - \frac{1}{3} y_2^3 \left(\cot\psi_2 - \cot\psi_1\right) \\ + \frac{1}{3}\left(y_1^3 - y_2^3\right)\left(\cot\psi_1 - \cot\frac{-\pi}{2}\right) \end{bmatrix}$$
$$+ \Delta_{\mu\lambda} \begin{bmatrix} \frac{1}{2} y_2^2 \left(\psi_2 - \psi_1\right) - \frac{1}{4}\left[\left(\psi_2 - \psi_1\right) - \frac{\sin 2\psi_2 - \sin 2\psi_1}{2}\right] \\ - \frac{1}{2}\left(y_1^2 - y_2^2\right)\left(\psi_1 - \frac{-\pi}{2}\right) \end{bmatrix} \quad (92)$$

This can be written formally as

$$\Delta C_{Mx}^{(4)} = \begin{bmatrix} b_{24}^{(4)} & b_{25}^{(4)} \end{bmatrix} \begin{Bmatrix} \Delta\lambda \\ \Delta_{\mu\lambda} \end{Bmatrix} \quad (93)$$

8. Case V: The slipstream overlaps with the retreating edge of the rotor

This case represents the overlap with the rotor retreating side's edge of the disk, which reduces the number of regions to integrate to one only: "Region 1". The results for Case IV (see there) can be used by modifying the lower end of the azimuthal integration from ψ_1 to $-\pi/2$ and the azimuthal range is therefore from $-\pi/2$ to ψ_2, while the lower radial bound is $y_2/\sin\psi$, the upper is 1.

6.1. Contributions to the thrust coefficient

For the second contribution, from Eq. (72):

$$\frac{\Delta C_T^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Delta\Theta_{75}\left[\begin{array}{l} -\left(1-2\mu y_2\right)\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) - y_2^2\left(\ln\left|\tan\frac{\psi_2}{2}\right| - \cancel{\ln\left|\tan\frac{-\pi}{4}\right|}\right) \\ +\mu\left[\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2}\right] \end{array}\right]$$

$$+\Delta\Theta_S\left[\begin{array}{l} \left(\frac{1}{2} - \mu y_2\right)\left[\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2}\right] - y_2^2\left(\psi_2 - \frac{-\pi}{2}\right) \\ -2\mu\left[\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) - \frac{\cos^3\psi_2 - \cancel{\cos^3\frac{-\pi}{2}}}{3}\right] \end{array}\right] \qquad (94)$$

The third contribution, from Eq. (76)

$$\frac{\Delta C_T^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} = \Theta_{tw}\left[\begin{array}{l} -\left[\frac{2}{3} - 0.75\left(1 - c_\mu y_2\right)\right]\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) \\ +\frac{2}{3}y_2^3\left(\cot\psi_2 - \cancel{\cot\frac{-\pi}{2}}\right) + 0.75 y_2^2\left(\ln\left|\tan\frac{\psi_2}{2}\right| - \cancel{\ln\left|\tan\frac{-\pi}{4}\right|}\right) \\ -\frac{c_\mu}{2}y_2^2\left(\psi_2 - \frac{-\pi}{2}\right) \\ +\frac{c_\mu}{2}\left(\frac{1}{2} - 0.75\right)\left[\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2}\right] \end{array}\right]$$

$$+\Theta_{75}\left[\begin{array}{l} -\left(1 - c_\mu y_2\right)\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) - y_2^2\left(\ln\left|\tan\frac{\psi_2}{2}\right| - \cancel{\ln\left|\tan\frac{-\pi}{4}\right|}\right) \\ +\frac{c_\mu}{2}\left[\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2}\right] \end{array}\right] \qquad (95)$$

$$+\text{part } 2$$

$$\text{part } 2 = \Theta_S \left[\begin{array}{l} \left(\frac{1}{2} - \frac{c_\mu}{2} y_2\right)\left[\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2}\right] - y_2^2\left(\psi_2 - \frac{-\pi}{2}\right) \\ -c_\mu\left[\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) - \frac{\cos^3\psi_2 - \cancel{\cos^3\frac{-\pi}{2}}}{3}\right] \end{array} \right]$$

The fourth contribution, from Eq. (80)

$$\begin{aligned} \frac{\Delta C_T^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}} &= \Delta\lambda\left[-\frac{1}{2}\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{1}{2} y_2^2\left(\cot\psi_2 - \cancel{\cot\frac{-\pi}{2}}\right)\right] \\ &+ \Delta_{\mu\lambda}\left[y_2\left(\psi_2 - \frac{-\pi}{2}\right) + \left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right)\right] \end{aligned} \tag{96}$$

6.2. Contributions to the rolling moment coefficient

For the second contribution, from Eq. (84):

$$\begin{aligned} \frac{\Delta C_{Mx}^{(2)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu} &= \Delta\Theta_{75}\left[\begin{array}{l} \frac{1}{3}\left[\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2}\right] \\ -\frac{2}{3} y_2^3\left(\ln\left|\tan\frac{\psi_2}{2}\right| - \cancel{\ln\left|\tan\frac{-\pi}{4}\right|}\right) \\ -\mu\left[\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) - \frac{\cos^3\psi_2 - \cancel{\cos^3\frac{-\pi}{2}}}{3}\right] \\ +\mu y_2^2\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) \end{array}\right] \\ &+ \Delta\Theta_S\left[\begin{array}{l} -\frac{2}{3}\left[\left(\cos\psi_2 - \cancel{\cos\frac{-\pi}{2}}\right) - \frac{\cos^3\psi_2 - \cancel{\cos^3\frac{-\pi}{2}}}{3}\right] - \frac{2}{3} y_2^3\left(\psi_2 - \frac{-\pi}{2}\right) \\ +\frac{\mu}{2}\left[\frac{3}{4}\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2} + \frac{\sin 4\psi_2 - \cancel{\sin(-2\pi)}}{16}\right] \\ -\frac{\mu}{2} y_2^2\left[\left(\psi_2 - \frac{-\pi}{2}\right) - \frac{\sin 2\psi_2 - \cancel{\sin(-\pi)}}{2}\right] \end{array}\right] \end{aligned} \tag{97}$$

For the third contribution, from Eq. (88)

$$\frac{\Delta C_{Mx}^{(3)}}{\frac{\sigma C_{l\alpha}}{2\pi}\Delta\mu}=\Theta_{tw}\left[\begin{array}{l}\frac{1}{2}y_2^4\left(\cot\psi_2-\cancel{\cot\frac{-\pi}{2}}\right)+\frac{2}{3}0.75y_2^3\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\cancel{\ln\left|\tan\frac{-\pi}{4}\right|}\right)\\ -\left(\frac{2}{3}-0.75\right)\frac{c_\mu}{2}\left[\left(\cos\psi_2-\cancel{\cos\frac{-\pi}{2}}\right)-\frac{\cos^3\psi_2-\cancel{\cos^3\frac{-\pi}{2}}}{3}\right]\\ -\frac{c_\mu}{3}y_2^3\left(\psi_2-\frac{-\pi}{2}\right)-0.75\frac{c_\mu}{2}y_2^2\left(\cos\psi_2-\cancel{\cos\frac{-\pi}{2}}\right)\end{array}\right] \tag{98}$$

$$+\text{part 2}$$

$$\text{part 2}=\Theta_{75}\left[\begin{array}{l}\frac{1}{3}\left[\left(\psi_2-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_2-\cancel{\sin(-\pi)}}{2}\right]-\frac{2}{3}y_2^3\left(\ln\left|\tan\frac{\psi_2}{2}\right|-\cancel{\ln\left|\tan\frac{-\pi}{4}\right|}\right)\\ -\frac{c_\mu}{2}\left[\left(\cos\psi_2-\cancel{\cos\frac{-\pi}{2}}\right)-\frac{\cos^3\psi_2-\cancel{\cos^3\frac{-\pi}{2}}}{3}\right]+\frac{c_\mu}{2}y_2^2\left(\cos\psi_2-\cancel{\cos\frac{-\pi}{2}}\right)\end{array}\right]$$

$$+\Theta_S\left[\begin{array}{l}-\frac{2}{3}\left[\left(\cos\psi_2-\cancel{\cos\frac{-\pi}{2}}\right)-\frac{\cos^3\psi_2-\cancel{\cos^3\frac{-\pi}{2}}}{3}\right]-\frac{2}{3}y_2^3\left(\psi_2-\frac{-\pi}{2}\right)\\ +\frac{c_\mu}{4}\left[\frac{3}{4}\left(\psi_2-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_2-\cancel{\sin(-\pi)}}{2}+\frac{\sin 4\psi_2-\cancel{\sin(-2\pi)}}{16}\right]\\ -\frac{c_\mu}{4}y_2^2\left[\left(\psi_2-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_2-\cancel{\sin(-\pi)}}{2}\right]\end{array}\right]$$

For the fourth contribution, from Eq. (92)

$$\begin{aligned}\frac{\Delta C_{Mx}^{(4)}}{\frac{\sigma C_{l\alpha}}{2\pi}}&=\Delta\lambda\left[\frac{1}{3}\left(\cos\psi_2-\cancel{\cos\frac{-\pi}{2}}\right)-\frac{1}{3}y_2^3\left(\cot\psi_2-\cancel{\cot\frac{-\pi}{2}}\right)\right]\\ &+\Delta_{\mu\lambda}\left[\frac{1}{2}y_2^2\left(\psi_2-\frac{-\pi}{2}\right)-\frac{1}{4}\left[\left(\psi_2-\frac{-\pi}{2}\right)-\frac{\sin 2\psi_2-\cancel{\sin(-\pi)}}{2}\right]\right]\end{aligned} \tag{99}$$