Supplemental Materials I

To show that the time- and state-dependent Hermitian Hamiltonian $\hat{H}(t)$ given by Eq. (2) is a linear operator, we only need to show that for arbitrary state vectors $|a(t)\rangle$ and $|b(t)\rangle$, and complex numbers α and β, we always have

$$\hat{H}(t)(\alpha\,|a(t)\rangle + \beta\,|b(t)\rangle) = \alpha\hat{H}(t)\,|a(t)\rangle + \beta\hat{H}(t)\,|b(t)\rangle\,. \tag{11}$$

This is evident by

$$\begin{aligned}\hat{H}(t)(\alpha\,|a(t)\rangle + \beta\,|b(t)\rangle) &= \left[i(|\partial_t\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)| - |\tilde{\psi}(t)\rangle\langle\partial_t\tilde{\psi}(t)|) + \dot{\phi}(t)\mathbb{1}\right](\alpha\,|a(t)\rangle + \beta\,|b(t)\rangle)\\ &= \alpha\left(i|\partial_t\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)|\right)|a(t)\rangle - \alpha\left(i|\tilde{\psi}(t)\rangle\langle\partial_t\tilde{\psi}(t)|\right)|a(t)\rangle + \alpha\dot{\phi}(t)\mathbb{1}\,|a(t)\rangle\\ &\quad + \beta\left(i|\partial_t\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)|\right)|b(t)\rangle - \beta\left(i|\tilde{\psi}(t)\rangle\langle\partial_t\tilde{\psi}(t)|\right)|b(t)\rangle + \beta\dot{\phi}(t)\mathbb{1}\,|b(t)\rangle\\ &= \alpha\hat{H}(t)\,|a(t)\rangle + \beta\hat{H}(t)\,|b(t)\rangle\,.\end{aligned} \tag{12}$$

Hence $H(t)$ as given by Eq. (2) is a linear operator.

Supplemental Materials II

The measuring Hamiltonian may not be unique. Two totally different Hamiltonians with different spectra may result in exactly the same quantum evolution. For example, both $\hat{H} = -\hat{\sigma}_z$ and $\hat{H}(t) = \frac{1}{3}\mathbb{1} - \frac{2}{9}\sqrt{2}\cos(2t)\hat{\sigma}_x + \frac{2}{9}\sqrt{2}\sin(2t)\hat{\sigma}_y - \frac{8}{9}\hat{\sigma}_z$ drive $|\psi(t)\rangle = \frac{e^{it}}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}e^{-it}\,|1\rangle$, and there is no way to tell which one is the actual Hamiltonian responsible for the driving just from (hypothetically) observing the quantum trajectory of $|\psi(t)\rangle$. This implies that the measuring Hamiltonian we obtained from Eq. (2)-(4) may not be the actual Hamiltonian responsible for the wave function collapse. Is there a way to find the true measuring Hamiltonian? With only the information from the quantum trajectories $|\psi(t)\rangle$, the answer is negative. Nevertheless, we prove the following theorem.

Theorem. Let $|\psi(t)\rangle \neq 0$ be an arbitrary pure state at time t. Let V be the linear space formed by all Hermitian operators $\hat{T}(t)$ satisfying $\hat{T}(t)\,|\psi(t)\rangle = 0$. If one can find at least one Hamiltonian $\hat{H}(t)$ satisfying $\hat{H}(t)\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$, then one can find all possible Hamiltonians $\hat{H}'(t)$ satisfying $\hat{H}'(t)\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$. In particular, $\hat{H}'(t)$ can be given by $\hat{H}'(t) = \hat{H}(t) + \hat{T}(t)$ where $\hat{T}(t)$ is arbitrarily chosen from V.

Proof. Let $\hat{H}(t)$ be a fixed Hamiltonian satisfying $\hat{H}(t)\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$. Let $H'(t)$ be an arbitrary Hamiltonian that satisfies $\hat{H}'(t)\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$. Since $\hat{H}(t)\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$, we conclude $(\hat{H}'(t) - \hat{H}(t))\,|\psi(t)\rangle = 0$. By defining $T(t) \equiv \hat{H}'(t) - \hat{H}(t)$, we have $T(t) \in V$.

On the other hand, let $\hat{H}(t)$ be a fixed Hamiltonian satisfying $\hat{H}(t)\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$ and let $\hat{T}(t)$ be an arbitrary Hermitian operator such that $\hat{T}(t) \in V$. By definition, $\hat{T}(t)\,|\psi(t)\rangle = 0$, therefore $(\hat{T}(t) + \hat{H}(t))\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$. By defining $H'(t) \equiv \hat{T}(t) + \hat{H}(t)$, we see that $\hat{H}'(t)\,|\psi(t)\rangle = i\,|\partial_t\psi(t)\rangle$. □

Since the unknowns in $\hat{T}$(t) are n^2 for n-dimensional Hilbert space, and the corresponding Schrödinger equation contains n equations, we have that $\dim V = n^2 - n \geq 2$, where $n = 2, 3, 4, \ldots$. In other words, there are plenty of linearly independent $\hat{T}(t)$ that satisfies $\hat{T}(t)\,|\psi(t)\rangle = 0$ when n becomes large.

The above theorem implies that even though we may never be able to know the actual measuring Hamiltonian, we can be assured that it must be one of the possible $\hat{H}'(t)$, if we are able to solve for all $\hat{T}(t)$. Moreover, all such Hamiltonians $H'(t)$ should have the same energy variance equal to $\langle\partial_t\psi(t)|\partial_t\psi(t)\rangle - |\,\langle\psi(t)|\partial_t\psi(t)\rangle\,|^2$.

Supplemental Materials III

Consider the measurement on an arbitrary n-level system initially at $|\psi(0)\rangle = \sum_i^n a_i(0)\,|i\rangle$ where $\sum_i^n |a_i(0)|^2 = 1$. Similar to the two-dimensional case, let the measurement operator be $\hat{\mathcal{M}}_{\delta t} = \sum_i^n M_i(t)|i\rangle\langle i|$ where $M_i(t) = (\delta t/2\pi\tau)^{1/4}\exp[-\frac{\delta t}{4\tau}(r_t - \lambda_i)^2]$. The λ_i represents the i-th eigenvalue corresponding to the i-th eigenstate of all possible outcomes. By using again Eq. (2)-(4), it can be shown that the measuring Hamiltonian responsible for the wave function collapse is given by (do not confuse the imaginary number i with the summation index i),

$$[H(t)]_{ij} = i\frac{a_i(t)a_j^*(t)}{4\tau}\sum_k |a_k(t)|^2\left(\eta_{ik}+\eta_{kj}\right), \tag{13}$$

where $\eta_{ij} \equiv (\lambda_i - \lambda_j)(2r_t - \lambda_i - \lambda_j)$. The instantaneous energy variance $[\Delta H(t)]^2$ is calculated to be

$$[\Delta H(t)]^2 = \sum_j^n \left[\frac{|a_j(t)|^2}{16\tau^2}\left(\sum_i^n |a_i(t)|^2\eta_{ij}\right)^2\right]. \tag{14}$$

Below we provide a detailed calculation regarding the above results.

Let $|\psi(t)\rangle = \sum_i^n a_i(t)|i\rangle$ be an arbitrary pure state for an n-level system, where $\sum_i^n |a_i(t)|^2 = 1$ so that it is normalized. Let $\hat{\mathcal{M}}_{\delta t} = \sum_i^n M_i(t)|i\rangle\langle i|$ be the measurement operator where $M_i(t) = (\delta t/2\pi\tau)^{1/4}\exp\left[-\frac{\delta t}{4\tau}(r_t-\lambda_i)^2\right]$ and λ_i is the i-th eigenvalue corresponding to i-th eigenstate off all possible states when the von Neumann measurement is finished. According to Eq. (3)(4), We need to calculate

$$\begin{aligned}\lim_{\delta t\to 0}\frac{|\psi(t+\delta t)\rangle - |\psi(t)\rangle}{\delta t} &= \lim_{\delta t\to 0}\left(\frac{\hat{\mathcal{M}}_{\delta t}}{\sqrt{\langle\psi(t)|\hat{\mathcal{M}}_{\delta t}^\dagger\hat{\mathcal{M}}_{\delta t}|\psi(t)\rangle}} - 1\right)|\psi(t)\rangle/\delta t \\ &= \lim_{\delta t\to 0}\left[\sum_j^n\left(\frac{M_j(t)}{\sqrt{\sum_i^n |a_i(t)|^2|M_i(t)|^2}} - 1\right)|j\rangle\langle j|\right]|\psi(t)\rangle/\delta t.\end{aligned} \tag{15}$$

Since

$$\lim_{\delta t\to 0}\sqrt{\sum_i^n |a_i(t)|^2|M_i(t)|^2} = \left(\frac{\delta t}{2\pi\tau}\right)^{1/4} - \frac{\delta t}{4\tau}\left(\frac{\delta t}{2\pi\tau}\right)^{1/4}\sum_i |a_i(t)|^2(r_t-\lambda_i)^2 + \mathcal{O}(\delta t^2), \tag{16}$$

$$\lim_{\delta t\to 0} M_i(t) = \left(\frac{\delta t}{2\pi\tau}\right)^{1/4} - \frac{\delta t}{4\tau}\left(\frac{\delta t}{2\pi\tau}\right)^{1/4}(r_t-\lambda_i)^2 + \mathcal{O}(\delta t^2), \tag{17}$$

we have

$$\lim_{\delta t\to 0}\frac{M_j(t)}{\sqrt{\sum_i^n |a_i(t)|^2|M_i(t)|^2}} = 1 - \frac{\delta t}{4\tau}\sum_{i\neq j}^n |a_i(t)|^2\eta_{ij} + \mathcal{O}(\delta t^2), \tag{18}$$

where $\eta_{ij} \equiv (\lambda_i - \lambda_j)(2r_t - \lambda_i - \lambda_j)$, hence

$$\lim_{\delta t\to 0}\left(\frac{\hat{\mathcal{M}}_{\delta t}}{\sqrt{\langle\psi(t)|\hat{\mathcal{M}}_{\delta t}^\dagger\hat{\mathcal{M}}_{\delta t}|\psi(t)\rangle}} - 1\right)/\delta t = -\sum_j\left(\frac{1}{4\tau}\sum_i^n |a_i(t)|^2\eta_{ij} + \mathcal{O}(\delta t)\right)|j\rangle\langle j|, \tag{19}$$

therefore

$$|\partial_t\psi(t)\rangle_j = -a_j(t)\left(\frac{1}{4\tau}\sum_i^n |a_i(t)|^2\eta_{ij}\right)|j\rangle. \tag{20}$$

On the other hand, since

$$[\langle\psi(t)|\partial_t\psi(t)\rangle]_j = -|a_j(t)|^2\left(\frac{1}{4\tau}\sum_i^n |a_i(t)|^2\eta_{ij}\right) \tag{21}$$

we have

$$\langle\psi(t)|\partial_t\psi(t)\rangle = \sum_j -|a_j(t)|^2\left(\frac{1}{4\tau}\sum_i^n |a_i(t)|^2\eta_{ij}\right) = -\sum_{i,j}\frac{1}{4\tau}|a_i(t)|^2|a_j(t)|^2\eta_{ij} = 0, \tag{22}$$

where we have used the fact that $\eta_{ij} = -\eta_{ji}$ and $\eta_{ii} = 0$. Since $\langle\psi(t)|\partial_t\psi(t)\rangle = 0$, we have $|\tilde{\psi}(t)\rangle = |\psi(t)\rangle$. Now that we have obtained the expression of $|\tilde{\psi}(t)\rangle$ and $|\partial_t\tilde{\psi}(t)\rangle$, by Eq. (2), we can obtain the measuring Hamiltonian

$$[H(t)]_{ij} = i\frac{a_i(t)a_j(t)^*}{4\tau}\sum_k |a_k(t)|^2\left(\eta_{ki}+\eta_{kj}\right) \tag{23}$$

and energy uncertainty

$$\Delta H(t) = \langle\psi(t)|\, H^2(t)\, |\psi(t)\rangle - (\langle\psi(t)|\, H(t)\, |\psi(t)\rangle)^2 = \|\, |\partial_t\psi(t)\rangle\, \| = \sqrt{\sum_j^n\left[\frac{|a_j(t)|^2}{16\tau^2}\left(\sum_i^n |a_i(t)|^2\eta_{ij}\right)^2\right]}. \tag{24}$$

Supplemental Materials IV

Below, we will show that for an n-level system, the Bloch vector representation v_H of the Hamiltonian $\hat{H}(t)$ obtained via Eq. (2) is always orthogonal to the Bloch vector $\vec{v}_\rho$ of $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ for arbitrary $|\psi(t)\rangle$. In addition, we show that that the effects of $\hat{H}(t)$ on $|\psi(t)\rangle$ is making $|\psi(t)\rangle$ rotate around the Bloch vector $\vec{v}_H$ of $\hat{H}(t)$.

To start with, for an arbitrary Hermitian operator T, it can be written under the bases of the generators $\{\Lambda_i\}$ of SU(n) group,

$$T = \lambda_0\mathbb{1} + \sum_{i=1}^{n^2-1}\lambda_i\Lambda_i \tag{S1}$$

such that we can define its generalized Bloch vector $\vec{v}_T = (\lambda_1\ \lambda_2\ \ldots\lambda_{n^2-1})^T$. The generators $\{\Lambda_i\}$, for example, can be the generalized Gell-Mann matrices. In the cases where λ_0 in unimportant or implicitly assumed, the Bloch vector $\vec{v}_T$ fully characterize the Hermitian operator T.

Recall that we need to show $\vec{v}_\rho\cdot\vec{v}_H = 0$. This can be done via the brute force of calculation, i.e. decomposing the measuring Hamiltonian given by Eq. (7) and the density matrix $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ in terms of the bases of generalized Gell-Mann matrices [43] to obtain $\vec{v}_\rho$ and $\vec{v}_H$, and then verify that indeed it is $\vec{v}_\rho\cdot\vec{v}_H = 0$. It turns out that

$$\vec{v}_\rho = (\{\lambda_s^{ij}\},\{\lambda_a^{ij}\},\{\lambda^i\}) \tag{S2}$$

$$\vec{v}_H = \frac{1}{4\tau}(\{\lambda_a^{ij}\Sigma_{ij}\},\{-\lambda_s^{ij}\Sigma_{ij}\},0,0,\cdots,0) \tag{S3}$$

where

$$\Lambda_s^{ij} \equiv |j\rangle\langle k| + |k\rangle\langle j|, \quad 1\leqslant j<k\leqslant d \tag{S4}$$

$$\Lambda_a^{jk} \equiv -\mathrm{i}|j\rangle\langle k| + \mathrm{i}|k\rangle\langle j|, \quad 1\leqslant j<k\leqslant d \tag{S5}$$

$$\Sigma_{ij} = \sum_k^n \lambda^k\left(\eta_{ki}+\eta_{kj}\right) \tag{S6}$$

where $\eta_{ij} \equiv (E_i - E_j)(2r - E_i - E_j)$. The E_i denotes the i-th eigenvalue corresponding to the i-th eigenstate after finishing the von Neumann measurement on $|\psi(t)\rangle$. The λ_s^{ij}, λ_a^{ij}, and λ^i are the coefficients of the Gell-Mann matrices Λ_s^{ij}, Λ_a^{ij} and Λ^i [43], respectively, where

$$\Lambda^l = \sqrt{\frac{2}{l(l+1)}}\left(\sum_{j=1}^l |j\rangle\langle j| - l|l+1\rangle\langle l+1|\right), \quad 1\leqslant l\leqslant d-1. \tag{S7}$$

It is straightforward to verify that $\vec{v}_\rho \cdot \vec{v}_H = 0$.

Another simpler way to prove $\vec{v}_\rho \cdot \vec{v}_H = 0$ is by proving $\hat{\rho}(t)$ is orthogonal to $\hat{H}(t)$. This is equivalent to proving $\mathrm{Tr}[\hat{\rho}(t)\hat{H}(t)] = \langle \hat{H}(t) \rangle_t = 0$, which is always true as shown in Eq. (22).

To prove that the effects of $\hat{H}(t)$ on $|\psi(t)\rangle$ is to make $|\psi(t)\rangle$ rotate around $\vec{v}_H$, we note that by Eq. (2),

$$\hat{H}^2(t) = |\partial_t \psi(t)\rangle \langle \partial_t \psi(t)| + \omega^2(t) |\psi(t)\rangle \langle \psi(t)| , \tag{S8}$$

$$\hat{H}^3(t) = \omega^2(t) \hat{H}(t), \tag{S9}$$

where

$$\omega^2(t) = \langle \partial_t \psi(t) | \partial_t \psi(t) \rangle , \tag{S10}$$

hence there exists an recursing pattern for the power of $\hat{H}(t)$. By making use of this pattern, and assuming for the moment that $\hat{H}(t)$ to be a constant, we have

$$\begin{aligned} U(T) = \exp(-iH(t)T) &= 1 - iH(t)T + \frac{1}{2!}(-i)^2 H^2(t) T^2 + \frac{1}{3!}(-i)^3 H^3(t) T^3 + \dots \\ &= 1 - iH(t)T - \frac{1}{2!}H^2(t)T^2 - \frac{1}{3!} i\omega^2(t) H(t) T^3 + \frac{1}{4!}\omega^2(t) H^2(t) T^4 - \frac{1}{5!} i\omega^4(t) H(t) T^5 + \dots . \end{aligned} \tag{S11}$$

such that

$$\begin{aligned} |\psi(t+T)\rangle &= U(T) |\psi(t)\rangle \\ &= |\psi(t)\rangle + |\partial_t \psi(t)\rangle T - \frac{1}{2!}\omega^2(t) |\psi(t)\rangle T^2 + \frac{1}{3!}\omega^2(t) |\partial_t\psi(t)\rangle T^3 + \frac{1}{4!}\omega^4 |\psi(t)\rangle T^4 + \cdots \\ &= \alpha |\psi(t)\rangle + \beta |\partial_t \psi(t)\rangle \end{aligned} \tag{S12}$$

where α and β are some real numbers that we do not need to solve explicitly. Since $\langle \psi(t+T)| \hat{H}(t) |\psi(t+T)\rangle = 0$, it indicates $\vec{v}_H$ is orthogonal to $\vec{v}_{\rho_{t+T}}$.

Of course, $\hat{H}(t)$ could be time-dependent, so that the direct application of Eq. (S11) may be inappropriate. In that case, we can let T be an infinitesimally small number δt so that the above argument is valid during each infinitesimally small time interval $[t, t+\delta t], [t+\delta t, t+2\delta t], \cdots$. The scenario then becomes that $\vec{v}_\rho$ is always orthogonal to and rotates around $\vec{v}_H$, and meanwhile $\vec{v}_H$ itself is changing its pointing direction in the Bloch sphere.

Supplemental Materials V

Let the initial and final state of a qubit be denoted by $\hat{\rho}(0) = \frac{1}{2}(\mathbb{1} + x_\mathrm{I}\hat{\sigma}_x + y_\mathrm{I}\hat{\sigma}_y + z_\mathrm{I}\hat{\sigma}_z)$ and $\rho(T) = \frac{1}{2}(\mathbb{1} + x_\mathrm{F}\hat{\sigma}_x + y_\mathrm{F}\hat{\sigma}_y + z_\mathrm{F}\hat{\sigma}_z)$, respectively. Consider the case where the Hamiltonian of the qubit is given by $\hat{H} = \frac{\epsilon}{2}\hat{\sigma}_z - \frac{\Delta}{2}\hat{\sigma}_x$ and the measuring operator $\hat{\mathcal{M}}_{\delta t}$ is given by $\hat{\mathcal{M}}_{\delta t} = (\delta t/2\pi\tau)^{1/4} \exp\left[-\frac{\delta t}{4\tau}\left(r_t \mathbb{1} - \hat{\sigma}_z\right)^2\right]$. The qubit being continuously measured will then evolve under both the effect from its own Hamiltonian and measuring processes, such that

$$\hat{\rho}(t+\delta t) = \frac{\hat{\mathcal{U}}_{\delta t}\hat{\rho}(t)\hat{\mathcal{U}}_{\delta t}^\dagger}{\mathrm{Tr}\left[\hat{\mathcal{U}}_{\delta t}\hat{\rho}(t)\hat{\mathcal{U}}_{\delta t}^\dagger\right]} \quad \text{where} \quad \hat{\mathcal{U}}_{\delta t} \equiv e^{-\frac{i}{\hbar}\hat{H}\delta t}\hat{\mathcal{M}}_{\delta t}, \tag{S13}$$

from which one can obtain a master equation

$$\partial_t \hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \frac{r_t}{2\tau}\{\hat{\sigma}_z, \hat{\rho}\} - \frac{r_t}{\tau}\langle \hat{\sigma}_z \rangle \hat{\rho}. \tag{S14}$$

It is shown in [41] that the most probable path of the qubit subject to the above master equation is given by

$$\begin{aligned} \bar{x}(t) &= \frac{x_I \cos\epsilon t - y_I \sin\epsilon t}{\cosh \bar{r}t/\tau + z_I \sinh \bar{r}t/\tau} \\ \bar{y}(t) &= \frac{y_I \cos\epsilon t + x_I \sin\epsilon t}{\cosh \bar{r}t/\tau + z_I \sinh \bar{r}t/\tau} \\ \bar{z}(t) &= \frac{z_I \cosh \bar{r}t/\tau + \sinh \bar{r}t/\tau}{\cosh \bar{r}t/\tau + z_I \sinh \bar{r}t/\tau}, \end{aligned} \tag{S15}$$

$$\text{where} \quad \bar{r} = \frac{\tau}{T}\tanh^{-1}\left(\frac{z_I - z_F}{z_I z_F - 1}\right) \quad \text{if} \quad \Delta = 0. \tag{S16}$$

To calculate the effective Hamiltonian $\hat{H}_{\text{eff}}(t)$ responsible for the most probable path, let us consider the case where $\Delta = 0$ and $x_{\text{I}} = 1, y_{\text{I}} = 0$, and $z_{\text{I}} = 0$, which corresponds to the pure initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The density matrix at time t then is

$$\rho(t) = \frac{1}{2}(\mathbb{1} + \bar{x}(t)\hat{\sigma}_x + \bar{y}(t)\hat{\sigma}_y + \bar{z}(t)\hat{\sigma}_z). \tag{S17}$$

Let $|\varphi_i(t)\rangle$ be the normalized instantaneous eigenvector of $\rho(t)$, from which we can obtained $|\tilde{\varphi}_i(t)\rangle$ defined by $|\tilde{\varphi}_i(t)\rangle \equiv e^{i\phi(t)}|\varphi_i(t)\rangle$ where $\phi(t) \equiv \int -i\langle\partial_t\varphi_i(t)|\varphi_i(t)\rangle\, dt$. The effective Hamiltonian $\hat{H}_{\text{eff}}(t)$ driving $\rho(t)$ then can be given by [34]

$$\begin{aligned}\hat{H}_{\text{eff}}(t) &= \frac{i}{2}\sum_{i=1}^{2}(|\partial_t\tilde{\varphi}_i(t)\rangle\langle\tilde{\varphi}_i(t)| - |\tilde{\varphi}_i(t)\rangle\langle\partial_t\tilde{\varphi}_i(t)|) \\ &= \begin{pmatrix} \frac{\epsilon}{2}\operatorname{sech}^2\alpha(t, z_{\text{F}})) & \frac{ie^{-it\epsilon}}{2T}(\tanh^{-1} z_F + iT\epsilon\tanh\alpha(t, z_{\text{F}}))\operatorname{sech}(\alpha(t, z_{\text{F}})) \\ \frac{-ie^{it\epsilon}}{2T}(\tanh^{-1} z_F - iT\epsilon\tanh(\alpha(t, z_{\text{F}}))\operatorname{sech}(\alpha(t, z_{\text{F}})) & -\frac{\epsilon}{2}\operatorname{sech}^2(\alpha(t, z_{\text{F}})) \end{pmatrix}\end{aligned} \tag{S18}$$

where

$$\alpha(t, z_{\text{F}}) = \frac{t}{T}\tanh^{-1} z_{\text{F}}. \tag{S19}$$

It is worth noting that the instantaneous energy variance of $\hat{H}_{\text{eff}}(t)$ is given by

$$[\Delta H_{\text{eff}}(t)]^2 = \text{Tr}(\rho(t)\hat{H}^2_{\text{eff}}(t)) - \text{Tr}(\rho(t)\hat{H}_{\text{eff}}(t))^2 = \frac{(T^2\epsilon^2 + [\tanh^{-1} z_{\text{F}}]^2)\operatorname{sech}^2(\alpha(t, z_{\text{F}}))}{4T^2}, \tag{S20}$$

which indicates the smaller the T and the larger the z_{F} are, the larger the $[\Delta H_{\text{eff}}(t)]^2$ is.

Supplemental Materials VI

In the following, we will show more details in calculations of Example III in the main text.

Let the initial wave function $\psi(x, 0)$ at time $t = 0$ be peaked at x_0,

$$\psi(x, 0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}(x - x_0)^2\right), \tag{S21}$$

such that we can obtain $|\psi(0)\rangle$ by $|\psi(0)\rangle = \int \psi(x, 0)\,|x\rangle\, dx$. Let the measurement operator $\hat{\mathcal{M}}_{\delta t}$ be

$$\hat{\mathcal{M}}_{\delta t} = \int (\delta t/2\pi\tau)^{1/4}\exp\left[-\frac{\delta t}{4\tau}(r_t - x)^2\right]|x\rangle\langle x|\, dx. \tag{S22}$$

We need to calculate

$$|\partial_t\psi(0)\rangle = \lim_{\delta t\to 0}\frac{|\psi(\delta t)\rangle - |\psi(0)\rangle}{\delta t} = \lim_{\delta t\to 0}\frac{\left(\frac{\hat{\mathcal{M}}_{\delta t}}{\sqrt{\int\langle\psi(0)|\hat{\mathcal{M}}^\dagger_{\delta t}\hat{\mathcal{M}}_{\delta t}|\psi(0)\rangle}} - 1\right)}{\delta t}|\psi(0)\rangle \tag{S23}$$

Since

$$\lim_{\delta t\to 0}\left(\frac{\hat{\mathcal{M}}_{\delta t}}{\sqrt{\langle\psi(0)|\,\hat{\mathcal{M}}^\dagger_{\delta t}\hat{\mathcal{M}}_{\delta t}\,|\psi(0)\rangle}} - 1\right)/\delta t = \int \frac{\hbar + 2m\omega(x - x_0)(2r_t - x - x_0)}{8m\omega\tau}|x\rangle\langle x|\, dx + \mathcal{O}(\delta t), \tag{S24}$$

by Eq. (2)(3) we have

$$|\partial_t\psi(0)\rangle = \int \underbrace{\frac{\hbar + 2m\omega(x - x_0)(2r_t - x - x_0)}{8m\omega\tau}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}(x - x_0)^2\right)}_{\dot{\psi}(x,0)}|x\rangle\, dx. \tag{S25}$$

Since $\langle\psi(0)|\partial_t\psi(0)\rangle = 0$, we have again $|\tilde{\psi}(0)\rangle = |\psi(0)\rangle$. Therefore, according to Eq. (2), the measuring Hamiltonian at time $t = 0$ is given by

$$\begin{aligned}\hat{H}(0) &= i\hbar(|\partial_t\tilde{\psi}(0)\rangle\langle\tilde{\psi}(0)| - |\tilde{\psi}(0)\rangle\langle\partial_t\tilde{\psi}(0)|) \\ &= i\hbar\int(\dot{\psi}(x,0)\psi(x',0) - \psi(x,0)\dot{\psi}(x',0))\,|x\rangle\,\langle x'|\,dxdx' \\ &= i\hbar\int\frac{\eta_{xx'}}{4\tau}\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega((x-x_0)^2+(x'-x_0)^2)}{2\hbar}\right)|x\rangle\,\langle x'|\,dxdx'\end{aligned} \tag{S26}$$

where $\eta_{xx'} \equiv (x-x')(2r_t - x - x')$. We can then calculate instantaneous energy uncertainty,

$$\Delta H(0) = \|\hbar\,|\partial_t\psi(0)\rangle\,\| = \sqrt{\int|\hbar\dot{\psi}(x,0)|^2dx} = \sqrt{\frac{\hbar\,(\hbar+4m\omega(r_t-x_0)^2)}{32m^2\omega^2\tau^2}}\hbar \tag{S27}$$

and the wave function at time $t = \delta t$,

$$|\psi(\delta t)\rangle = \int\frac{1}{\Delta X(\delta t)\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\langle X(\delta t)\rangle}{\Delta X(\delta t)}\right)^2\right]|x\rangle\,dx, \tag{S28}$$

which is still a Gaussian, with mean position

$$\langle X(\delta t)\rangle = \frac{r_t\hbar\,\delta t + 2m\tau\omega x_0}{\hbar\,\delta t + 2m\tau\omega} \tag{S29}$$

and position uncertainty,

$$\Delta X(\delta t)^2 = \langle X^2(\delta t)\rangle - \langle X(\delta t)\rangle^2 = \frac{\hbar\tau}{\hbar\,\delta t + 2m\tau\omega}. \tag{S30}$$

On the other hand, since

$$\Delta X(0)^2 = \frac{\hbar}{2m\omega}, \tag{S31}$$

hence

$$\delta\Delta X(0)^2 = \frac{\hbar\tau}{\hbar\delta t + 2m\tau\omega} - \frac{\hbar}{2m\omega} \tag{S32}$$

such that

$$\frac{\partial([\Delta X(0)]^2)}{\partial t} = \lim_{\delta t\to 0}\frac{\delta([\Delta X(0)]^2)}{\delta t} = -\frac{\hbar^2}{4m^2\omega^2\tau} = -[\Delta X(0)^2]^2\frac{1}{\tau} \tag{S33}$$

which becomes an ordinary differential equation. It can be solved as

$$\Delta X(t)^2 = \frac{\Delta X(0)^2\tau}{\Delta X(0)^2 t + \tau} \tag{S34}$$

The above calculation implies that if we apply sequential $\hat{\mathcal{M}}_{\delta t}$ on $|\psi(0)\rangle$, the wave function at any later time will still be a Gaussian, and only its mean and variance changes, i.e.

$$|\psi(t)\rangle \simeq \int\frac{1}{\Delta X_t\sqrt{2\pi}}\exp\left(-\frac{(x-\langle x_t\rangle)^2}{2\Delta X_t^2}\right)|x\rangle\,dx. \tag{S35}$$

Since we can relate ω with $\Delta X(t)$ by

$$\omega = \frac{\hbar}{2m\Delta X(t)^2}. \tag{S36}$$

Plugging in the expression to $\Delta H(0)$, we obtain $\Delta H(t)$

$$\Delta H(t) = \frac{\sqrt{2(r_t-x_0)^2\Delta X_t^2 + \Delta X_t^4}}{2\sqrt{2}\tau}\hbar, \tag{S37}$$

At the strong measurement limit $\tau \to 0$, we have

$$\lim_{\tau\to 0}\Delta H(t) = \frac{|r_t - x_t|}{2\sqrt{\tau t}}\hbar + \mathcal{O}(\tau). \tag{S38}$$