

Supplementary Information:

Quantum Zeno-like Paradox for Position Measurements

Xabier Oianguren-Asua* and Roderich Tumulka

*Mathematics Institute, Eberhard Karls University Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany.*

Mathematical Proofs

Since Theorem 1 is a special case of Theorem 2, it suffices to prove the latter. It is well known that because Q has finite volume, $L^\infty(Q) \subseteq L^2(Q) \subseteq L^1(Q)$.

Definition S1. Given $f \in L^1(Q)$ and a grid scheme $\{B_j^{(n)}\}_{j=1}^{N_n}$ for Q , we define the “discretization of f by the n -th grid” to be the function $f_n : Q \rightarrow \mathbb{C}$,

$$f_n(x) := \sum_{j=1}^{N_n} \underbrace{\frac{1}{|B_j^{(n)}|} \int_{y \in B_j^{(n)}} f(y) dy}_{\text{average of } f \text{ in } B_j^{(n)}} \mathbb{1}_{B_j^{(n)}}(x) \quad \text{for } x \in Q. \quad (\text{S1})$$

(Informally, it is the “bar chart” taking at each point of $B_j^{(n)}$ the average of f in $B_j^{(n)}$ as value.)

Lemma S1. *Let there be an arbitrary $f \in L^\infty(Q)$ and let $\{B_j^{(n)}\}_{j=1}^{N_n}$ be an arbitrary grid scheme for Q . Then, $f_n \in L^\infty(Q)$ and $\|f_n - f\|_{L^2} \xrightarrow{n \rightarrow \infty} 0$.*

Proof. First, note that for each $x \in Q$ and $n \in \mathbb{N}$, because $\{B_j^{(n)}\}_{j=1}^{N_n}$ is a partition of Q , there exists a unique $j_x^n \in \{1, \dots, N_n\}$ such that $x \in B_{j_x^n}^{(n)}$.

Next, for each $x \in Q$,

$$|f_n(x)| = \left| \sum_{j=1}^{N_n} \frac{\mathbb{1}_{B_j^{(n)}}(x)}{|B_j^{(n)}|} \int_{y \in B_j^{(n)}} f(y) dy \right| \quad (\text{S2})$$

$$\leq \sum_{j=1}^{N_n} \frac{\mathbb{1}_{B_j^{(n)}}(x)}{|B_j^{(n)}|} \int_{y \in B_j^{(n)}} |f(y)| dy \quad (\text{S3})$$

$$\stackrel{(a)}{=} \frac{1}{|B_{j_x^n}^{(n)}|} \int_{y \in B_{j_x^n}^{(n)}} |f(y)| dy \quad (\text{S4})$$

$$\leq \|f\|_{L^\infty}, \quad (\text{S5})$$

and hence $\|f_n\|_{L^\infty} \leq \|f\|_{L^\infty}$, such that $f_n \in L^\infty(Q)$, and a fortiori $f_n \in L^2(Q)$.

Now recall that for each $x \in Q$, $B_{j_x^n}^{(n)} = \prod_{k=1}^n I_{j_x^n, k}^{(n)}$ for some intervals $I_{j_x^n, k}^{(n)} \subseteq [0, 1)$ of length in $[\frac{1}{Cn}, \frac{1}{n}]$. This implies that the longest diagonal of $B_{j_x^n}^{(n)}$ has at most length $\frac{\sqrt{d}}{n}$ and thus that $B_{j_x^n}^{(n)}$ is contained in the open ball of radius $R_n := \frac{2\sqrt{d}}{n}$ around x , which we denote by $V_{R_n}(x)$. Moreover, for the n -independent number $K := \frac{1}{C^d(2\sqrt{d})^d |V_1(0)|}$,

*E-mail: xabier.oianguren-asua@math.uni-tuebingen.de

^(a)By definition, $\mathbb{1}_{B_j^{(n)}}(x) = 0$ for all $j \in \{1, \dots, N_n\}$ except for $j = j_x^n$, where $\mathbb{1}_{B_{j_x^n}^{(n)}}(x) = 1$.

$$|B_{j_x}^{(n)}| \stackrel{(b)}{\geq} \frac{1}{(Cn)^d} = K \left(\frac{2\sqrt{d}}{n} \right)^d |V_1(0)| = K(R_n)^d |V_1(0)| = K|V_{R_n}(x)|. \quad (\text{S6})$$

This condition, together with the fact that f is integrable^(c) —which by Theorem 1.6.19 in Ref. [7] implies that almost every (a. e.) $x \in Q$ is a Lebesgue point— allow us to apply Ex. 1.6.15 in Ref. [7], which is a corollary of the Lebesgue differentiation theorem. This corollary proves that for a. e. $x \in Q$,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{|B_{j_x}^{(n)}|} \int_{y \in B_{j_x}^{(n)}} f(y) dy \right) = f(x). \quad (\text{S7})$$

Now, note the following.

(i) Point-wise, for a. e. $x \in Q$,

$$|f_n(x) - f(x)| = \left| \sum_{j=1}^{N_n} \frac{\mathbb{1}_{B_j^{(n)}}(x)}{|B_j^{(n)}|} \int_{y \in B_j^{(n)}} f(y) dy - f(x) \right| \quad (\text{S8})$$

$$\stackrel{(a)}{=} \left| \frac{1}{|B_{j_x}^{(n)}|} \int_{y \in B_{j_x}^{(n)}} f(y) dy - f(x) \right| \xrightarrow[n \rightarrow \infty]{(\text{S7})} 0. \quad (\text{S9})$$

(ii) The constant (and hence L^∞) function $g(x) \equiv 2\|f\|_{L^\infty}$ satisfies that for a. e. $x \in Q$,

$$|f_n(x) - f(x)| \leq \frac{1}{|B_{j_x}^{(n)}|} \int_{y \in B_{j_x}^{(n)}} |f(y)| dy + |f(x)| \leq 2\|f\|_{L^\infty}. \quad (\text{S10})$$

By virtue of (i) and (ii), g is a dominating function in $L^2(Q)$ that grants the application of the dominated convergence theorem in the last step of the following chain of equations:

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^2}^2 = \lim_{n \rightarrow \infty} \int_{x \in Q} |f_n(x) - f(x)|^2 dx = 0. \quad (\text{S11})$$

□

We are now ready to prove the result for Q in the special case that $\hat{\rho} = |\psi\rangle\langle\psi|$ is pure and both ψ and ϕ are (essentially) bounded functions rather than general L^2 functions.

Lemma S2. *Given arbitrary $\psi, \phi \in L^\infty(Q)$ and an arbitrary grid scheme $\{B_j^{(n)}\}_{j=1}^{N_n}$ for Q :*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 = 0. \quad (\text{S12})$$

Proof. First, define $f(x) := \overline{\phi(x)}\psi(x)$ for $x \in Q$. Then $f \in L^\infty(Q)$ because for a. e. $x \in Q$, $|f(x)| = |\phi(x)| |\psi(x)| \leq \|\phi\|_{L^\infty} \|\psi\|_{L^\infty}$. Hence, $f \in L^2(Q)$ and moreover, we can apply Lemma S1 to f to get that $\|f_n - f\|_{L^2} \xrightarrow[n \rightarrow \infty]{} 0$. This implies by the reverse triangle inequality that $\|f_n\|_{L^2} \rightarrow \|f\|_{L^2}$ and thus $\|f_n\|_{L^2}^2 \rightarrow \|f\|_{L^2}^2$.

^(c)By assumption, $f \in L^\infty(Q)$, which implies that $f \in L^1(Q)$.

Now,

$$\|f_n\|_{L^2}^2 = \int_{x \in Q} \left| \sum_{j=1}^{N_n} \frac{\mathbb{1}_{B_j^{(n)}}(x)}{|B_j^{(n)}|} \underbrace{\int_{y \in B_j^{(n)}} f(y) dy}_{\int_{y \in B_j^{(n)}} \overline{\phi(y)} \psi(y) dy} \right|^2 dx \quad (\text{S13})$$

$$= \int_{x \in Q} \left| \sum_{j=1}^{N_n} \frac{\mathbb{1}_{B_j^{(n)}}(x)}{|B_j^{(n)}|} \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 dx \quad (\text{S14})$$

[use that $\{\mathbb{1}_{B_j^{(n)}}\}_{j=1}^{N_n}$ have disjoint supports]

$$= \sum_{j=1}^{N_n} \int_{x \in Q} \left| \frac{\mathbb{1}_{B_j^{(n)}}(x)}{|B_j^{(n)}|} \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 dx \quad (\text{S15})$$

$$= \sum_{j=1}^{N_n} \frac{|B_j^{(n)}|}{|B_j^{(n)}|^2} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 \quad (\text{S16})$$

$$= \sum_{j=1}^{N_n} \frac{1}{|B_j^{(n)}|} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2. \quad (\text{S17})$$

But then $\lim_{n \rightarrow \infty} \|f_n\|_{L^2}^2 = \|f\|_{L^2}^2$ means that the sequence $\left(\sum_{j=1}^{N_n} \frac{1}{|B_j^{(n)}|} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 \right)_{n \in \mathbb{N}}$ converges to the finite number $\|f\|_{L^2}^2$. Using this result to obtain Eq. (S20) below, together with the fact that $(1/n^d)_{n \in \mathbb{N}}$ converges to 0 and the continuity of the product of complex numbers,

$$\sum_{j=1}^{N_n} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 = \sum_{j=1}^{N_n} \frac{|B_j^{(n)}|}{|B_j^{(n)}|} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 \quad (\text{S18})$$

$$\stackrel{(12)}{\leq} \frac{1}{n^d} \sum_{j=1}^{N_n} \frac{1}{|B_j^{(n)}|} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 \quad (\text{S19})$$

$$\xrightarrow{n \rightarrow \infty} 0 \cdot \|f\|_{L^2}^2 = 0. \quad (\text{S20})$$

□

With that, we can prove the spatial quantum Zeno effect for any pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ on $L^2(Q)$.

Lemma S3. *Given arbitrary $\psi, \phi \in L^2(Q)$ and an arbitrary grid scheme $\{B_j^{(n)}\}_{j=1}^{N_n}$ for Q ,*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 = 0. \quad (\text{S21})$$

Proof. It is well known that the smooth compactly supported functions $\mathcal{C}_0^\infty(Q)$ are dense in $L^2(Q)$ (see Prop. 8.17 in Ref. [8] for the case of $\mathbb{R}^d$ —with the right identifications, one can restrict this result to Q). In particular, all functions on $\mathcal{C}_0^\infty(Q)$ are continuous maps with compact support and thus, they are bounded functions. As such, $\mathcal{C}_0^\infty(Q) \subseteq L^\infty(Q)$, proving that $L^\infty(Q)$ is also dense in $L^2(Q)$.

By this denseness, for an arbitrary $\varepsilon > 0$, one can successively find $\psi_\varepsilon, \phi_\varepsilon \in L^\infty(Q)$ such that $\|\psi - \psi_\varepsilon\|^2 \leq \frac{\varepsilon}{6\|\phi\|^2}$ and $\|\phi - \phi_\varepsilon\|^2 \leq \frac{\varepsilon}{12\|\psi_\varepsilon\|^2}$. In particular, by Lemma S2, $\sum_{j=1}^{N_n} \frac{1}{|B_j^{(n)}|} |\langle \phi_\varepsilon | \hat{P}_j^{(n)} \psi_\varepsilon \rangle|^2$

$\xrightarrow{n \rightarrow \infty} 0$, so there is $n_\varepsilon \in \mathbb{N}$ such that $\sum_{j=1}^{N_n} \frac{1}{|B_j^{(n)}|} \left| \langle \phi_\varepsilon | \hat{P}_j^{(n)} \psi_\varepsilon \rangle \right|^2 \leq \frac{\varepsilon}{12}$ for all $n \geq n_\varepsilon$. Hence, for all $n \geq n_\varepsilon$,

$$\sum_{j=1}^{N_n} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 \stackrel{(d)}{\leq} \sum_{j=1}^{N_n} 2 \left(\left| \langle \phi | \hat{P}_j^{(n)} (\psi - \psi_\varepsilon) \rangle \right|^2 + \left| \langle \phi | \hat{P}_j^{(n)} \psi_\varepsilon \rangle \right|^2 \right) \quad (\text{S22})$$

$$\stackrel{(d)}{\leq} \sum_{j=1}^{N_n} 2 \left(\left| \langle \hat{P}_j^{(n)} \phi | \psi - \psi_\varepsilon \rangle \right|^2 + 2 \left| \langle \phi - \phi_\varepsilon | \hat{P}_j^{(n)} \psi_\varepsilon \rangle \right|^2 + 2 \left| \langle \phi_\varepsilon | \hat{P}_j^{(n)} \psi_\varepsilon \rangle \right|^2 \right) \quad (\text{S23})$$

$$\leq 2 \underbrace{\sum_{j=1}^{N_n} \left\| \hat{P}_j^{(n)} \phi \right\|^2}_{=\|\phi\|^2 \text{ by (e)}} \|\psi - \psi_\varepsilon\|^2 + 4 \|\phi - \phi_\varepsilon\|^2 \underbrace{\sum_{j=1}^{N_n} \left\| \hat{P}_j^{(n)} \psi_\varepsilon \right\|^2}_{=\|\psi_\varepsilon\|^2 \text{ by (e)}} + 4 \sum_{j=1}^{N_n} \frac{|B_j^{(n)}|}{|B_j^{(n)}|} \left| \langle \phi_\varepsilon | \hat{P}_j^{(n)} \psi_\varepsilon \rangle \right|^2 \quad (\text{S24})$$

[now use the definition of $\psi_\varepsilon, \phi_\varepsilon$ and Eq. (12)]

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 4 \frac{1}{n^d} \sum_{j=1}^{N_n} \frac{1}{|B_j^{(n)}|} \left| \langle \phi_\varepsilon | \hat{P}_j^{(n)} \psi_\varepsilon \rangle \right|^2 \stackrel{(n \geq n_\varepsilon)}{\leq} \frac{2\varepsilon}{3} + \frac{1}{n^d} \frac{\varepsilon}{3} \leq \varepsilon. \quad (\text{S25})$$

But by definition, this is to say that $\sum_{j=1}^{N_n} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 \xrightarrow{n \rightarrow \infty} 0$. $\square$

The generalization of Lemma S3 to translated and scaled cubes (namely, to arbitrary rectangles in $\mathbb{R}^d$) is straightforward.

Next, we are going to split $\mathbb{R}^d$ into countably many cubes and equip each of them with a grid scheme, making together a family of “position measurement grids” for the whole $\mathbb{R}^d$.

Definition S2. Given a translated unit cube $\tilde{Q} := \prod_{k=1}^d [a_k, a_k + 1)$ for some $a_k \in \mathbb{R}$, we define a *grid scheme* for $\tilde{Q}$ to be the set of translations $\tilde{V}_j^{(n)} := V_j^{(n)} + (a_1, \dots, a_d)$ of an arbitrary grid scheme for $[0, 1)^d$, $\{V_j^{(n)}\}_{j=1}^{N_n}$.

Then, we define a *grid scheme* for $\mathbb{R}^d$ to be a partition of $\mathbb{R}^d$ into translated unit cubes $\{Q_\ell\}_{\ell \in \mathbb{N}}$ and a choice of *grid scheme* per Q_ℓ , say, $\{\tilde{V}_{\ell,j}^{(n)}\}_{\ell \in \mathbb{N}, j \in \{1, \dots, N_\ell^{(n)}\}}$ ($n \in \mathbb{N}$). Since for each $n \in \mathbb{N}$, $\{V_{\ell,j}^{(n)}\}_{\ell \in \mathbb{N}, j \in \{1, \dots, N_\ell^{(n)}\}}$ is a countable set, we will write $\{V_{\ell,j}^{(n)}\}_{\ell \in \mathbb{N}, j \in \{1, \dots, N_\ell^{(n)}\}} = \{B_j^{(n)}\}_{j \in \mathbb{N}}$ and define $J_\ell^{(n)} := \{j \in \mathbb{N} \mid B_j^{(n)} \subseteq Q_\ell\}$ (such that $|J_\ell^{(n)}| = N_\ell^{(n)}$). $\blacklozenge$

For each fixed $n \in \mathbb{N}$, $\{B_j^{(n)}\}_{j \in \mathbb{N}}$ is trivially a partition of $\mathbb{R}^d$ into rectangles of edges in the order of $1/n$. The only restriction we put by the inductive definition is that the interior of no $B_j^{(n)}$ is allowed to intersect the boundary of a cube Q_ℓ . Although this made the last definition a bit cumbersome, we imposed it to make the following statements and their proofs notationally simpler.

^(d) Successively, we add and subtract $\langle \phi | \hat{P}_j^{(n)} \psi_\varepsilon \rangle$ and then $\langle \phi_\varepsilon | \hat{P}_j^{(n)} \psi_\varepsilon \rangle$, using after each such operation that for all $a, b \in \mathbb{C}$, $|a - b|^2 = |a|^2 + |b|^2 - 2\text{Re}\{\bar{a}b\} \leq |a|^2 + |b|^2 + 2|a||b| \leq 2|a|^2 + 2|b|^2$.

^(e) Let $\eta \in L^2(Q)$ be arbitrary. Then, using that $\hat{P}_j^{(n)}$ are self-adjoint projectors and the obvious property $\hat{I} = \sum_{j=1}^{N_n} \hat{P}_j^{(n)}$, we get that $\sum_{j=1}^{N_n} \left\| \hat{P}_j^{(n)} \eta \right\|^2 = \langle \eta | \sum_{j=1}^{N_n} \hat{P}_j^{(n)} \eta \rangle = \langle \eta | \eta \rangle = \|\eta\|^2$.

Lemma S4. Given arbitrary $\psi, \phi \in L^2(\mathbb{R}^d)$ and an arbitrary grid scheme $\{B_j^{(n)}\}_{j \in \mathbb{N}}$ for $\mathbb{R}^d$:

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 = 0. \quad (\text{S26})$$

Proof. First, note the following observation. Given a measurable infinite set $A \subseteq \mathbb{R}^d$, a partition $\{A_M\}_{M \in \mathbb{N}}$ for A and a non-negative integrable function $f : \mathbb{R}^d \rightarrow [0, +\infty]$,

- (i) for all $x \in A$, $\sum_{j=1}^M \mathbb{1}_{A_j}(x)f(x)$ equals $\mathbb{1}_A(x)f(x)$ from some M onward,^(f)
- (ii) $\sum_{j=1}^M \mathbb{1}_{A_j}f$ is a monotonously increasing sequence of measurable functions.

Consequently, the monotone convergence theorem implies that:

$$\lim_{M \rightarrow \infty} \sum_{j=1}^M \left\| \mathbb{1}_{A_j} f \right\|_{L^1(\mathbb{R})} = \lim_{M \rightarrow \infty} \int_{x \in \mathbb{R}^d} \sum_{j=1}^M \mathbb{1}_{A_j}(x) f(x) dx = \int_{x \in \mathbb{R}^d} \mathbb{1}_A(x) f(x) dx = \left\| \mathbb{1}_A f \right\|_{L^1(\mathbb{R}^d)}. \quad (\text{S27})$$

This proves the last step of the following.

$$\sum_{j=1}^M \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 \leq \|\phi\|^2 \sum_{j=1}^M \left\| \hat{P}_j^{(n)} \psi \right\|^2 \quad (\text{S28})$$

$$= \|\phi\|^2 \sum_{j=1}^M \left\| \mathbb{1}_{B_j^{(n)}} |\psi|^2 \right\|_{L^1} \xrightarrow[M \rightarrow \infty]{(\text{S27})} \|\phi\|^2 \|\psi\|^2. \quad (\text{S29})$$

Hence, for each $n \in \mathbb{N}$, the symbol $\sum_{j=1}^{\infty} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2$ in the Lemma's statement is finite and well defined. In particular, it is an absolutely convergent series, which implies that one can add its terms in finite “chunks” before adding them together and the result will still be the same. That is, using the notation of Definition S2,

$$\sum_{j=1}^{\infty} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 = \sum_{\ell=1}^{\infty} \sum_{j \in J_{\ell}^{(n)}} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2. \quad (\text{S30})$$

Now, define $h_n(\ell) := \sum_{j \in J_{\ell}^{(n)}} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2$ and note the following.

- (i) Point-wise, for each fixed $\ell \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} h_n(\ell) = \lim_{n \rightarrow \infty} \sum_{j \in J_{\ell}^{(n)}} \left| \langle \phi | \hat{P}_j^{(n)} \psi \rangle \right|^2 \stackrel{(\text{Lemma S3})}{=} 0. \quad (\text{S31})$$

- (ii) For each $\ell \in \mathbb{N}$ and independently of $n \in \mathbb{N}$,

$$|h_n(\ell)| \leq \|\phi\|^2 \sum_{j \in J_{\ell}^{(n)}} \left\| \hat{P}_j^{(n)} \psi \right\|^2 \stackrel{(\text{S27})}{=} \|\phi\|^2 \left\| \mathbb{1}_{Q_{\ell}} |\psi|^2 \right\|_{L^1(\mathbb{R}^d)} =: g(\ell). \quad (\text{S32})$$

In particular, denoting by $d\nu$ the counting measure of $\mathbb{N}$,

$$\|g\|_{L^1(\mathbb{N}, d\nu)} = \|\phi\|^2 \lim_{M \rightarrow \infty} \sum_{\ell=1}^M \left\| \mathbb{1}_{Q_{\ell}} |\psi|^2 \right\|_{L^1(\mathbb{R}^d)} \stackrel{(\text{S27})}{=} \|\phi\|^2 \|\psi\|^2 < +\infty. \quad (\text{S33})$$

Hence, g is a dominating function in $L^1(\mathbb{N}, d\nu)$ for the sequence of functions $h_n : \mathbb{N} \rightarrow \mathbb{R}$.

^(f)Namely, for all M if $x \notin A$ and otherwise, after the M such that $x \in A_M$.

By the dominated convergence theorem, (i) and (ii) imply that

$$\lim_{n \rightarrow \infty} \int_{\ell \in \mathbb{N}} h_n(\ell) d\nu = 0. \quad (\text{S34})$$

Putting it all together,

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 \stackrel{(\text{S30})}{=} \lim_{n \rightarrow \infty} \sum_{\ell=1}^{\infty} \sum_{j \in J_{\ell}^{(n)}} |\langle \phi | \hat{P}_j^{(n)} \psi \rangle|^2 \stackrel{(\text{by def.})}{=} \lim_{n \rightarrow \infty} \int_{\ell \in \mathbb{N}} h_n(\ell) d\nu \stackrel{(\text{S34})}{=} 0. \quad (\text{S35})$$

□

Finally, let us lift these results to the case of arbitrary density matrices.

Theorem 2. *For any positive trace-class operator $\hat{\rho}$ on $L^2(Q)$, any vector $\phi \in L^2(Q)$, and any grid scheme $\{B_j^{(n)}\}_{j=1}^{N_n}$ for Q ,*

$$\lim_{n \rightarrow \infty} \left\langle \phi \left| \sum_{j=1}^{N_n} \hat{P}_j^{(n)} \hat{\rho} \hat{P}_j^{(n)} \phi \right. \right\rangle = 0. \quad (\text{S36})$$

The same statement holds if we replace Q by $\mathbb{R}^d$ (consequently setting $N_n = +\infty$).

Proof. We give the proof for Q and $\mathbb{R}^d$ together. The only difference will be whether N_n is finite or infinite for every $n \in \mathbb{N}$.

By Theorems 1.1 and 1.2 in Ref. [9], because $\hat{\rho}$ is a self-adjoint and trace-class (hence, compact) operator, there exists an ONB of eigenvectors $\{\psi_{\ell}\}_{\ell \in \mathbb{N}} \subset L^2(Q)$ (resp. in $L^2(\mathbb{R}^d)$) with associated eigenvalues $p_{\ell} \geq 0$, such that,

$$\hat{\rho} = \sum_{\ell=1}^{\infty} p_{\ell} |\psi_{\ell}\rangle \langle \psi_{\ell}|, \quad (\text{S37})$$

where the series is taken in the operator norm. In particular, by definition, the trace-norm of $\hat{\rho}$, $\|\hat{\rho}\|_{tr}$, equals $\sum_{\ell=1}^{\infty} p_{\ell}$. Using the sequential continuity of the inner product to get Eqs. (S38) and (S40),

$$\left\langle \phi \left| \sum_{j=1}^{N_n} \hat{P}_j^{(n)} \hat{\rho} \hat{P}_j^{(n)} \phi \right. \right\rangle = \sum_{j=1}^{N_n} \left\langle \phi \left| \hat{P}_j^{(n)} \hat{\rho} \hat{P}_j^{(n)} \phi \right. \right\rangle \quad (\text{S38})$$

$$\stackrel{(\text{S37})}{=} \sum_{j=1}^{N_n} \sum_{\ell=1}^{\infty} p_{\ell} \left\langle \phi \left| \hat{P}_j^{(n)} \psi_{\ell} \right. \right\rangle \left\langle \psi_{\ell} \left| \hat{P}_j^{(n)} \phi \right. \right\rangle \quad (\text{S39})$$

$$= \sum_{j=1}^{N_n} \sum_{\ell=1}^{\infty} p_{\ell} \left| \left\langle \phi \left| \hat{P}_j^{(n)} \psi_{\ell} \right. \right\rangle \right|^2 \quad (\text{S40})$$

[considering the series as iterated integrals in the counting measure $d\nu$ of $\mathbb{N}$, since the integrand is non-negative, we can apply the Tonelli theorem (2.37 in Ref. [8]) to switch the order of summation]

$$= \sum_{\ell=1}^{\infty} \sum_{j=1}^{N_n} p_{\ell} \left| \left\langle \phi \left| \hat{P}_j^{(n)} \psi_{\ell} \right. \right\rangle \right|^2 \quad (\text{S41})$$

[we define $h_n(\ell) := \sum_{j=1}^{N_n} p_{\ell} \left| \left\langle \phi \left| \hat{P}_j^{(n)} \psi_{\ell} \right. \right\rangle \right|^2$ and write the resulting series as an integral]

$$= \int_{\ell \in \mathbb{N}} h_n(\ell) d\nu. \quad (\text{S42})$$

Finally, note that:

(i) Point-wise, for each fixed $\ell \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} h_n(\ell) \stackrel{(\text{by def.})}{=} p_\ell \lim_{n \rightarrow \infty} \sum_{j=1}^{N_n} \left| \langle \phi | \hat{P}_j^{(n)} \psi_\ell \rangle \right|^2 \stackrel{(\text{Lemmas S3 \& S4})}{=} 0. \quad (\text{S43})$$

(ii) For each $\ell \in \mathbb{N}$ and independently of $n \in \mathbb{N}$,

$$|h_n(\ell)| \stackrel{(\text{Cauchy-Schwarz})}{\leq} p_\ell \|\phi\|^2 \sum_{j=1}^{N_n} \left\| \hat{P}_j^{(n)} \psi_\ell \right\|^2 \stackrel{(\text{S27})}{=} p_\ell \|\phi\|^2 \|\psi_\ell\|^2 \stackrel{(\|\psi_\ell\|=1)}{=} p_\ell \|\phi\|^2. \quad (\text{S44})$$

In particular, defining $g(\ell) := p_\ell \|\phi\|^2$,

$$\|g\|_{L^1(\mathbb{N}, d\nu)} = \|\phi\|^2 \sum_{\ell=1}^{\infty} |p_\ell| = \|\phi\|^2 \|\hat{\rho}\|_{tr} < +\infty. \quad (\text{S45})$$

Hence, g is a dominating function in $L^1(\mathbb{N}, d\nu)$ for the sequence of functions h_n .

By (i) and (ii), we can use the dominated convergence theorem to get the last step of the following:

$$\lim_{n \rightarrow \infty} \left\langle \phi \left| \sum_{j=1}^{N_n} \hat{P}_j^{(n)} \hat{\rho} \hat{P}_j^{(n)} \phi \right. \right\rangle \stackrel{(\text{S42})}{=} \lim_{n \rightarrow \infty} \int_{\ell \in \mathbb{N}} h_n(\ell) d\nu = 0. \quad (\text{S46})$$

□

References

- [1] B. Misra and E. Sudarshan, “The Zeno’s paradox in quantum theory,” *Journal of Mathematical Physics*, vol. 18, pp. 756–763, 1977.
- [2] R. Tumulka, *Foundations of quantum mechanics*. Springer, 2022.
- [3] I. M. Gelfand and G. E. Shilov, *Generalized functions, Vol. 4: applications of harmonic analysis*. Academic Press, 1964.
- [4] M. Gadella and F. Gómez, “On the Mathematical Basis of the Dirac Formulation of Quantum Mechanics,” *International Journal of Theoretical Physics*, vol. 42, no. 10, pp. 2225–2254, 2003.
- [5] D. Burgarth, P. Facchi, and R. Hillier, “Control of quantum noise: On the role of dilations,” *Annales Henri Poincaré*, vol. 24, no. 1, pp. 325–347, 2023.
- [6] M. Keyl, D. Schlingemann, and R. Werner, “Infinitely entangled states,” *Quantum Information & Computation*, vol. 3, no. 4, pp. 281–306, 2003.
- [7] T. Tao, *An Introduction to Measure Theory*. American Mathematical Society, 2011.
- [8] G. B. Folland, *Real Analysis: Modern Techniques and their Applications*. John Wiley & Sons, 1999.
- [9] B. Simon, *Trace Ideals and Their Applications* (Lecture note series vol. 35). Cambridge University Press, 1979.