Supplement to: Efficient WENO-Based Prolongation Strategies for Divergence-Preserving Vector Fields

By

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Supplement A

We describe a very efficient finite volume WENO reconstruction in 2D and 3D at third, fourth and fifth orders. The 2D reconstruction should be used to construct the moments of the normal component of the field within a face. The 3D reconstruction can be used to construct the charge densities.

A.1) r=3 Finite-Volume WENO Reconstruction in 2D

The reconstruction polynomial at third order is given by

$$u(x,y) \;=\; u_{0,0} + u_x\; x + u_y\; y \;+\; u_{xx}\left(x^2 - 1/12\right) \;+\; u_{yy}\left(y^2 - 1/12\right) \;+\; u_{xy}\; x\, y \quad . \tag{A.1}$$

The $u_{0,0}$ denotes the value in the zone of interest, and the integer subscripts of "u" will denote shifts relative to the central zone where the reconstruction is to be carried out. When "u" has subscripts of "x", "y" or "z", they will denote spatial moments. The u_x and u_{xx} terms can be obtained one-dimensional WENO-AO methods applied in the x-direction from Balsara, Garain and Shu [8]. The u_y and u_{yy} terms can be obtained by a similar application of WENO-AO in the y-direction. Therefore, we only have to describe the process of obtaining the cross-term in eqn. (A.1). Four possible evaluations of the u_{xy} term can then be obtained by taking all the known moments in the zone "0,0" and including any one of the four zone averaged values $\mathrm{u}_{1,1}, \mathrm{u}_{-1,1}, \mathrm{u}_{1,-1}$ and $\mathrm{u}_{-1,-1}$ from the zones that lie along the diagonals of the zone of interest. We catalogue the four possible evaluations of the cross-term u_{xy} below:

$$
\begin{aligned}
u_{xy} &= u_{1,1} - u_{0,0} - u_x - u_y - u_{xx} - u_{yy} \quad ; \\
u_{xy} &= - u_{1,-1} + u_{0,0} + u_x - u_y + u_{xx} + u_{yy} \quad ; \\
u_{xy} &= - u_{-1,1} + u_{0,0} - u_x + u_y + u_{xx} + u_{yy} \quad ; \\
u_{xy} &= u_{-1,-1} - u_{0,0} + u_x + u_y - u_{xx} - u_{yy} \quad .
\end{aligned} \tag{A.2}
$$

The four possible cross-terms in the equation above, correspond to the choice of four possible stencils. The smoothness indicator for eqn. (A.1), written as a sum of perfect squares, is given by

$$
IS = u_x^2 + u_y^2 + \frac{13}{3} u_{xx}^2 + \frac{13}{3} u_{yy}^2 + \frac{7}{6} u_{xy}^2 \quad . \tag{A.3}
$$

This completes our description of third order finite volume (r=3) WENO reconstruction in 2D.

A.2) r=4 Finite-Volume WENO Reconstruction in 2D

The reconstruction polynomial at fourth order is given by

$$
\begin{aligned}
u(x,y) \; = \; & u_{0,0} + u_x \, x + u_y \, y \; + \; u_{xx} \left(x^2 - 1/12 \right) \; + \; u_{yy} \left(y^2 - 1/12 \right) \; + \; u_{xy} \; x \, y \\
& + u_{xxx} \left(x^3 - 3x/20 \right) + u_{yyy} \left(y^3 - 3y/20 \right) + u_{xxy} \left(x^2 - 1/12 \right) y \; + \; u_{xyy} x \left(y^2 - 1/12 \right) \quad .
\end{aligned} \tag{A.4}
$$

One-dimensional WENO-AO can be used to obtain the u_x, u_{xx} and u_{xxx} terms and also the u_y, u_{yy} and u_{yyy} terms. Fig. A.1 shows us five possible stencils that can each be used to evaluate the u_{xy}, u_{xxy} and u_{xyy} cross-terms. The central stencil was added for stability reasons and has a linear weight that is 100 to 400 times larger than the linear weights of the other four directionally-biased stencils. Our choice of five stencils in Fig. A.1 gives us five possible evaluations of the cross-terms u_{xy}, u_{xxy} and u_{xyy} which we catalogue below. For the stencil S_1 we obtain

$$
\begin{aligned}
u_{xy} &= (60 u_{1,1} - 10 u_{1,2} - 10 u_{2,1} - 40 u_{0,0} - 30 u_x - 30 u_y - 10 u_{xx} - 10 u_{yy} + 27 u_{xxx} + 27 u_{yyy}) / 20 \quad ; \\
u_{xxy} &= (- 20 u_{1,1} + 10 u_{2,1} + 10 u_{0,0} + 10 u_y - 20 u_{xx} + 10 u_{yy} - 60 u_{xxx} + 11 u_{yyy}) / 20 \quad ; \\
u_{xyy} &= (- 20 u_{1,1} + 10 u_{1,2} + 10 u_{0,0} + 10 u_x + 10 u_{xx} - 20 u_{yy} + 11 u_{xxx} - 60 u_{yyy}) / 20 \quad .
\end{aligned} \tag{A.5}
$$

For the stencil S_2 we obtain

$$u_{xy}=(-60u_{-1,1}+10u_{-1,2}+10u_{-2,1}+40u_{0,0}-30u_x+30u_y+10u_{xx}+10u_{yy}+27u_{xxx}-27u_{yyy})/20 \;;$$
$$u_{xxy}=(-20u_{-1,1}+10u_{-2,1}+10u_{0,0}+10u_y-20u_{xx}+10u_{yy}+60u_{xxx}+11u_{yyy})/20 \;;$$
$$u_{xyy}=(20u_{-1,1}-10u_{-1,2}-10u_{0,0}+10u_x-10u_{xx}+20u_{yy}+11u_{xxx}+60u_{yyy})/20 \;.$$

(A.6)

For the stencil S_3 we obtain

$$u_{xy}=(-60u_{1,-1}+10u_{1,-2}+10u_{2,-1}+40u_{0,0}+30u_x-30u_y+10u_{xx}+10u_{yy}-27u_{xxx}+27u_{yyy})/20 \;;$$
$$u_{xxy}=(20u_{1,-1}-10u_{2,-1}-10u_{0,0}+10u_y+20u_{xx}-10u_{yy}+60u_{xxx}+11u_{yyy})/20 \;; \qquad \text{(A.7)}$$
$$u_{xyy}=(-20u_{1,-1}+10u_{1,-2}+10u_{0,0}+10u_x+10u_{xx}-20u_{yy}+11u_{xxx}+60u_{yyy})/20 \;.$$

For the stencil S_4 we obtain

$$u_{xy}=(60u_{-1,-1}-10u_{-2,-1}-10u_{-1,-2}-40u_{0,0}+30u_x+30u_y-10u_{xx}-10u_{yy}-27u_{xxx}-27u_{yyy})/20 \;;$$
$$u_{xxy}=(20u_{-1,-1}-10u_{-2,-1}-10u_{0,0}+10u_y+20u_{xx}-10u_{yy}-60u_{xxx}+11u_{yyy})/20 \;; \qquad \text{(A.8)}$$
$$u_{xyy}=(20u_{-1,-1}-10u_{-1,-2}-10\,u_{0,0}+10u_x-10u_{xx}+20u_{yy}+11u_{xxx}-60u_{yyy})/20 \;.$$

For the central S_5 stencil we obtain

$$u_{xy}=(u_{-1,-1}-u_{-1,1}-u_{1,-1}+u_{1,1})/4 \;;$$
$$u_{xxy}=(-5u_{-1,-1}+5u_{-1,1}-5u_{1,-1}+5u_{1,1}-22u_{yyy}-20u_y)/20 \;; \qquad \text{(A.9)}$$
$$u_{xyy}=(-5u_{-1,-1}-5u_{-1,1}+5u_{1,-1}+5u_{1,1}-22u_{xxx}-20u_x)/20 \;.$$

The smoothness indicator for eqn. (A.4), written as a sum of perfect squares, is given by

$$IS=\left(u_x+u_{xxx}/10\right)^2+\left(u_y+u_{yyy}/10\right)^2+\frac{13}{3}u_{xx}^2+\frac{13}{3}u_{yy}^2+\frac{781}{20}u_{xxx}^2+\frac{781}{20}u_{yyy}^2$$
$$+\frac{7}{6}u_{xy}^2+\frac{47}{10}u_{xxy}^2+\frac{47}{10}u_{xyy}^2 \;. \qquad \text{(A.10)}$$

This completes our description of fourth order finite volume (r=4) WENO reconstruction in 2D.

A.3) r=5 Finite-Volume WENO Reconstruction in 2D

The reconstruction polynomial at fifth order is given by

$$
\begin{aligned}
u(x,y) \;=\; & u_{0,0} + u_x\, x + u_y\, y \;+\; u_{xx}\left(x^2 - 1/12\right) \;+\; u_{yy}\left(y^2 - 1/12\right) \;+\; u_{xy}\; x\, y \\
& + u_{xxx}\left(x^3 - 3x/20\right) + u_{yyy}\left(y^3 - 3y/20\right) + u_{xxy}\left(x^2 - 1/12\right) y \;+\; u_{xyy} x\left(y^2 - 1/12\right) \\
& + u_{xxxx}\left(x^4 - 3x^2/14 + 3/560\right) + u_{yyyy}\left(y^4 - 3y^2/14 + 3/560\right) + u_{xxxy}\left(x^3 - 3x/20\right) y \\
& + u_{xyyy}\; x\left(y^3 - 3y/20\right) + u_{xxyy}\left(x^2 - 1/12\right)\left(y^2 - 1/12\right) \;\; .
\end{aligned}
\tag{A.11}
$$

One-dimensional WENO-AO can be used to obtain the u_x, u_{xx}, u_{xxx} and u_{xxxx} terms and also the u_y, u_{yy}, u_{yyy} and u_{yyyy} terms. Fig. A.2 shows us nine possible stencils that can each be used to evaluate the $u_{xy}, u_{xxy}, u_{xyy}, u_{xxxy}, u_{xyyy}$ and u_{xxyy} cross-terms. The central stencil was added for stability reasons and has a linear weight that is 100 to 400 times larger than the linear weights of the other four directionally-biased stencils. Our choice of nine stencils in Fig. A.2 gives us nine possible evaluations of the cross-terms $u_{xy}, u_{xxy}, u_{xyy}, u_{xxxy}, u_{xyyy}$ and u_{xxyy} which we catalogue below. For the stencil S_1 we obtain

$$
\begin{aligned}
u_{xy} = & (-12110 u_{0,0} + 24780 u_{1,1} - 8190 u_{1,2} + 1330 u_{1,3} - 8190 u_{2,1} + 1050 u_{2,2} + 1330 u_{3,1} - 7630 u_x \\
& - 1330 u_{xx} + 2527 u_{xxx} - 11790 u_{xxxx} - 7630 u_y - 1330 u_{yy} + 2527 u_{yyy} - 11790 u_{yyyy}) / 4200 \;\; ; \\
u_{xxy} = & (175 u_{0,0} - 490 u_{1,1} + 70 u_{1,2} + 350 u_{2,1} - 35 u_{2,2} - 70 u_{3,1} - 210 u_{xx} - 210 u_{xxx} + 990 u_{xxxx} \\
& + 140 u_y + 70 u_{yy} - 56 u_{yyy} - 330 u_{yyyy}) / 140 \;\; ; \\
u_{xyy} = & (175 u_{0,0} - 490 u_{1,1} + 350 u_{1,2} - 70 u_{1,3} + 70 u_{2,1} - 35 u_{2,2} + 140 u_x + 70 u_{xx} - 56 u_{xxx} - 330 u_{xxxx} \\
& - 210 u_{yy} - 210 u_{yyy} + 990 u_{yyyy}) / 140 \;\; ; \\
\\
u_{xxxy} = & (-70 u_{0,0} + 210 u_{1,1} - 210 u_{2,1} + 70 u_{3,1} - 420 u_{xxx} - 2520 u_{xxxx} - 70 u_y - 70 u_{yy} \\
& - 77 u_{yyy} - 90 u_{yyyy}) / 420 \;\; ; \\
u_{xyyy} = & (-70 u_{0,0} + 210 u_{1,1} - 210 u_{1,2} + 70 u_{1,3} - 70 u_x - 70 u_{xx} - 77 u_{xxx} - 90 u_{xxxx} \\
& - 420 u_{yyy} - 2520 u_{yyyy}) / 420 \;\; ; \\
u_{xxyy} = & (-7 u_{0,0} + 28 u_{1,1} - 14 u_{1,2} - 14 u_{2,1} + 7 u_{2,2} + 14 u_{xx} + 42 u_{xxx} + 102 u_{xxxx} + 14 u_{yy} \\
& + 42 u_{yyy} + 102 u_{yyyy}) / 28 \;\; .
\end{aligned}
\tag{A.12}
$$

For the stencil S_2 we obtain

$$u_{xy}=(12110u_{0,0}-24780u_{-1,1}+8190u_{-1,2}-1330u_{-1,3}+8190u_{-2,1}-1050u_{-2,2}-1330u_{-3,1}-7630u_{x}$$
$$+1330u_{xx}+2527u_{xxx}+11790u_{xxxx}+7630u_{y}+1330u_{yy}-2527u_{yyy}+11790u_{yyyy})/4200\ ;$$
$$u_{xxy}=(175u_{0,0}-490u_{-1,1}+70u_{-1,2}+350u_{-2,1}-35u_{-2,2}-70u_{-3,1}-210u_{xx}+210u_{xxx}+990u_{xxxx}$$
$$+140u_{y}+70u_{yy}-56u_{yyy}-330u_{yyyy})/140\ ;$$
$$u_{xyy}=(-175u_{0,0}+490u_{-1,1}-350u_{-1,2}+70u_{-1,3}-70u_{-2,1}+35u_{-2,2}+140u_{x}-70u_{xx}-56u_{xxx}+330u_{xxxx}$$
$$+210u_{yy}+210u_{yyy}-990u_{yyyy})/140\ ;$$

$$u_{xxxy}=(70u_{0,0}-210u_{-1,1}+210u_{-2,1}-70u_{-3,1}-420u_{xxx}+2520u_{xxxx}+70u_{y}+70u_{yy}$$
$$+77u_{yyy}+90u_{yyyy})/420\ ;$$
$$u_{xyyy}=(70u_{0,0}-210u_{-1,1}+210u_{-1,2}-70u_{-1,3}-70u_{x}+70u_{xx}-77u_{xxx}+90u_{xxxx}$$
$$+420u_{yyy}+2520u_{yyyy})/420\ ; \tag{A.13}$$
$$u_{xxyy}=(-7u_{0,0}+28u_{-1,1}-14u_{-1,2}-14u_{-2,1}+7u_{-2,2}+14u_{xx}-42u_{xxx}+102u_{xxxx}+14u_{yy}$$
$$+42u_{yyy}+102u_{yyyy})/28\ .$$

For the stencil S_3 we obtain

$$u_{xy}=(-12110u_{0,0}+24780u_{-1,-1}-8190u_{-1,-2}+1330u_{-1,-3}-8190u_{-2,-1}+1050u_{-2,-2}+1330u_{-3,-1}$$
$$+7630u_{x}-1330u_{xx}-2527u_{xxx}-11790u_{xxxx}+7630u_{y}-1330u_{yy}-2527u_{yyy}-11790u_{yyyy})/4200\ ;$$
$$u_{xxy}=(-175u_{0,0}+490u_{-1,-1}-70u_{-1,-2}-350u_{-2,-1}+35u_{-2,-2}+70u_{-3,-1}+210u_{xx}-210u_{xxx}-990u_{xxxx}$$
$$+140u_{y}-70u_{yy}-56u_{yyy}+330u_{yyyy})/140\ ;$$
$$u_{xyy}=(-175u_{0,0}+490u_{-1,-1}-350u_{-1,-2}+70u_{-1,-3}-70u_{-2,-1}+35u_{-2,-2}+140u_{x}-70u_{xx}-56u_{xxx}+330u_{xxxx}$$
$$+210u_{yy}-210u_{yyy}-990u_{yyyy})/140\ ;$$

$$u_{xxxy}=(-70u_{0,0}+210u_{-1,-1}-210u_{-2,-1}+70u_{-3,-1}+420u_{xxx}-2520u_{xxxx}+70u_{y}-70u_{yy}$$
$$+77u_{yyy}-90u_{yyyy})/420\ ;$$
$$u_{xyyy}=(-70u_{0,0}+210u_{-1,-1}-210u_{-1,-2}+70u_{-1,-3}+70u_{x}-70u_{xx}+77u_{xxx}-90u_{xxxx}$$
$$+420u_{yyy}-2520u_{yyyy})/420\ ; \tag{A.14}$$
$$u_{xxyy}=(-7u_{0,0}+28u_{-1,-1}-14u_{-1,-2}-14u_{-2,-1}+7u_{-2,-2}+14u_{xx}-42u_{xxx}+102u_{xxxx}+14u_{yy}$$
$$-42u_{yyy}+102u_{yyyy})/28\ .$$

For the stencil S_4 we obtain

$$u_{xy} = (12110u_{0,0} - 24780u_{1,-1} + 8190u_{1,-2} - 1330u_{1,-3} + 8190u_{2,-1} - 1050u_{2,-2} - 1330u_{3,-1} + 7630u_x + 1330u_{xx} - 2527u_{xxx} + 11790u_{xxxx} - 7630u_y + 1330u_{yy} + 2527u_{yyy} + 11790u_{yyyy}) / 4200 \;;$$

$$u_{xxy} = (-175u_{0,0} + 490u_{1,-1} - 70u_{1,-2} - 350u_{2,-1} + 35u_{2,-2} + 70u_{3,-1} + 210u_{xx} + 210u_{xxx} - 990u_{xxxx} + 140u_y - 70u_{yy} - 56u_{yyy} + 330u_{yyyy}) / 140 \;;$$

$$u_{xyy} = (175u_{0,0} - 490u_{1,-1} + 350u_{1,-2} - 70u_{1,-3} + 70u_{2,-1} - 35u_{2,-2} + 140u_x + 70u_{xx} - 56u_{xxx} - 330u_{xxxx} - 210u_{yy} + 210u_{yyy} + 990u_{yyyy}) / 140 \;;$$

$$\begin{aligned}
u_{xxxy} &= (70u_{0,0} - 210u_{1,-1} + 210u_{2,-1} - 70u_{3,-1} + 420u_{xxx} + 2520u_{xxxx} - 70u_y + 70u_{yy} - 77u_{yyy} + 90u_{yyyy}) / 420 \;; \\
u_{xyyy} &= (70u_{0,0} - 210u_{1,-1} + 210u_{1,-2} - 70u_{1,-3} + 70u_x + 70u_{xx} + 77u_{xxx} + 90u_{xxxx} - 420u_{yyy} + 2520u_{yyyy}) / 420 \;; \\
u_{xxyy} &= (-7u_{0,0} + 28u_{1,-1} - 14u_{1,-2} - 14u_{2,-1} + 7u_{2,-2} + 14u_{xx} + 42u_{xxx} + 102u_{xxxx} + 14u_{yy} - 42u_{yyy} + 102u_{yyyy}) / 28 \;.
\end{aligned} \tag{A.15}$$

For the central S_5 stencil we obtain

$$u_{xy} = (-1470u_{0,0} - 1330u_{-1,1} - 1330u_{1,-1} + 8820u_{1,1} - 2870u_{1,2} - 2870u_{2,1} + 1050u_{2,2} - 2310u_x + 3990u_{xx} + 8379u_{xxx} + 26970u_{xxxx} - 2310u_y + 3990u_{yy} + 8379u_{yyy} + 26970u_{yyyy}) / 4200 \;;$$

$$u_{xxy} = (-105u_{0,0} + 70u_{-1,1} - 70u_{1,1} + 70u_{1,2} + 70u_{2,1} - 35u_{2,2} - 210u_{xx} - 210u_{xxx} - 690u_{xxxx} - 140u_y - 210u_{yy} - 364u_{yyy} - 690u_{yyyy}) / 140 \;;$$

$$u_{xyy} = (-105u_{0,0} + 70u_{1,-1} - 70u_{1,1} + 70u_{1,2} + 70u_{2,1} - 35u_{2,2} - 140u_x - 210u_{xx} - 364u_{xxx} - 690u_{xxxx} - 210u_{yy} - 210u_{yyy} - 690u_{yyyy}) / 140 \;;$$

$$\begin{aligned}
u_{xxxy} &= (210u_{0,0} - 70u_{-1,1} - 210u_{1,1} + 70u_{2,1} - 420u_{xxx} - 840u_{xxxx} + 210u_y + 210u_{yy} + 231u_{yyy} + 270u_{yyyy}) / 420 \;; \\
u_{xyyy} &= (210u_{0,0} - 70u_{1,-1} - 210u_{1,1} + 70u_{1,2} + 210u_x + 210u_{xx} + 231u_{xxx} + 270u_{xxxx} - 420u_{yyy} - 840u_{yyyy}) / 420 \;; \\
u_{xxyy} &= (-7u_{0,0} + 28u_{1,1} - 14u_{1,2} - 14u_{2,1} + 7u_{2,2} + 14u_{xx} + 42u_{xxx} + 102u_{xxxx} + 14u_{yy} + 42u_{yyy} + 102u_{yyyy}) / 28 \;.
\end{aligned} \tag{A.16}$$

For the central S_6 stencil we obtain

$$u_{xy}=(1470u_{0,0}+1330u_{-1,-1}-8820u_{-1,1}+2870u_{-1,2}+2870u_{-2,1}-1050u_{-2,2}+1330u_{1,1}-2310u_{x}$$
$$-3990u_{xx}+8379u_{xxx}-26970u_{xxxx}+2310u_{y}-3990u_{yy}-8379u_{yyy}-26970u_{yyyy})/4200\ ;$$
$$u_{xxy}=(-105u_{0,0}-70u_{-1,1}+70u_{-1,2}+70u_{-2,1}-35u_{-2,2}+70u_{1,1}-210u_{xx}+210u_{xxx}-690u_{xxxx}$$
$$-140u_{y}-210u_{yy}-364u_{yyy}-690u_{yyyy})/140\ ;$$
$$u_{xyy}=(105u_{0,0}-70u_{-1,-1}+70u_{-1,1}-70u_{-1,2}-70u_{-2,1}+35u_{-2,2}-140u_{x}+210u_{xx}-364u_{xxx}+690u_{xxxx}$$
$$+210u_{yy}+210u_{yyy}+690u_{yyyy})/140\ ;$$

$$u_{xxxy}=(-210u_{0,0}+210u_{-1,1}-70u_{-2,1}+70u_{1,1}-420u_{xxx}+840u_{xxxx}-210u_{y}-210u_{yy}$$
$$-231u_{yyy}-270u_{yyyy})/420\ ;$$
$$u_{xyyy}=(-210u_{0,0}+70u_{-1,-1}+210u_{-1,1}-70u_{-1,2}+210u_{x}-210u_{xx}+231u_{xxx}-270u_{xxxx}$$
$$+420u_{yyy}+840u_{yyyy})/420\ ; \tag{A.17}$$
$$u_{xxyy}=(-7u_{0,0}+28u_{-1,1}-14u_{-1,2}-14u_{-2,1}+7u_{-2,2}+14u_{xx}-42u_{xxx}+102u_{xxxx}+14u_{yy}$$
$$+42u_{yyy}+102u_{yyyy})/28\ .$$

For the central S_7 stencil we obtain

$$u_{xy}=(-1470u_{0,0}+8820u_{-1,-1}-2870u_{-1,-2}-1330u_{-1,1}-2870u_{-2,-1}+1050u_{-2,-2}-1330u_{1,-1}+2310u_{x}$$
$$+3990u_{xx}-8379u_{xxx}+26970u_{xxxx}+2310u_{y}+3990u_{yy}-8379u_{yyy}+26970u_{yyyy})/4200\ ;$$
$$u_{xxy}=(105u_{0,0}+70u_{-1,-1}-70u_{-1,-2}-70u_{-2,-1}+35u_{-2,-2}-70u_{1,-1}+210u_{xx}-210u_{xxx}+690u_{xxxx}$$
$$-140u_{y}+210u_{yy}-364u_{yyy}+690u_{yyyy})/140\ ;$$
$$u_{xyy}=(105u_{0,0}+70u_{-1,-1}-70u_{-1,-2}-70u_{-1,1}-70u_{-2,-1}+35u_{-2,-2}-140u_{x}+210u_{xx}-364u_{xxx}+690u_{xxxx}$$
$$+210u_{yy}-210u_{yyy}+690u_{yyyy})/140\ ;$$

$$u_{xxxy}=(210u_{0,0}-210u_{-1,-1}+70u_{-2,-1}-70u_{1,-1}+420u_{xxx}-840u_{xxxx}-210u_{y}+210u_{yy}$$
$$-231u_{yyy}+270u_{yyyy})/420\ ;$$
$$u_{xyyy}=(210u_{0,0}-210u_{-1,-1}+70u_{-1,-2}-70u_{-1,1}-210u_{x}+210u_{xx}-231u_{xxx}+270u_{xxxx}$$
$$+420u_{yyy}-840u_{yyyy})/420\ ; \tag{A.18}$$
$$u_{xxyy}=(-7u_{0,0}+28u_{-1,-1}-14u_{-1,-2}-14u_{-2,-1}+7u_{-2,-2}+14u_{xx}-42u_{xxx}+102u_{xxxx}+14u_{yy}$$
$$-42u_{yyy}+102u_{yyyy})/28\ .$$

For the central S_8 stencil we obtain

$$u_{xy} = (1470u_{0,0} + 1330u_{-1,-1} - 8820u_{1,-1} + 2870u_{1,-2} + 1330u_{1,1} + 2870u_{2,-1} - 1050u_{2,-2} + 2310u_x$$
$$- 3990u_{xx} - 8379u_{xxx} - 26970u_{xxxx} - 2310u_y - 3990u_{yy} + 8379u_{yyy} - 26970u_{yyyy}) / 4200 \ ;$$
$$u_{xxy} = (105u_{0,0} - 70u_{-1,-1} + 70u_{1,-1} - 70u_{1,-2} - 70u_{2,-1} + 35u_{2,-2} + 210u_{xx} + 210u_{xxx} + 690u_{xxxx}$$
$$- 140u_y + 210u_{yy} - 364u_{yyy} + 690u_{yyyy}) / 140 \ ;$$
$$u_{xyy} = (-105u_{0,0} - 70u_{1,-1} + 70u_{1,-2} + 70u_{1,1} + 70u_{2,-1} - 35u_{2,-2} - 140u_x - 210u_{xx} - 364u_{xxx} - 690u_{xxxx}$$
$$- 210u_{yy} + 210u_{yyy} - 690u_{yyyy}) / 140 \ ;$$

$$u_{xxxy} = (-210u_{0,0} + 70u_{-1,-1} + 210u_{1,-1} - 70u_{2,-1} + 420u_{xxx} + 840u_{xxxx} + 210u_y - 210u_{yy}$$
$$+ 231u_{yyy} - 270u_{yyyy}) / 420 \ ;$$
$$u_{xyyy} = (-210u_{0,0} + 210u_{1,-1} - 70u_{1,-2} + 70u_{1,1} - 210u_x - 210u_{xx} - 231u_{xxx} - 270u_{xxxx}$$
$$- 420u_{yyy} + 840u_{yyyy}) / 420 \ ; \tag{A.19}$$
$$u_{xxyy} = (-7u_{0,0} + 28u_{1,-1} - 14u_{1,-2} - 14u_{2,-1} + 7u_{2,-2} + 14u_{xx} + 42u_{xxx} + 102u_{xxxx} + 14u_{yy}$$
$$- 42u_{yyy} + 102u_{yyyy}) / 28 \ .$$

For the central S9 stencil we obtain

$$u_{xy} = 13u_{-1,-1} / 30 - 11u_{-1,-2} / 240 - 13u_{-1,1} / 30 + 11u_{-1,2} / 240 - 11u_{-2,-1} / 240 + 11u_{-2,1} / 240$$
$$- 13u_{1,-1} / 30 + 11u_{1,-2} / 240 + 13u_{1,1} / 30 - 11u_{1,2} / 240 + 11u_{2,-1} / 240 - 11u_{2,1} / 240 \ ;$$
$$u_{xxy} = (-u_{-1,-1} + u_{-1,1} - u_{1,-1} + u_{1,1}) / 4 - u_y - 11u_{yyy} / 10 \ ;$$
$$u_{xyy} = (-u_{-1,-1} - u_{-1,1} + u_{1,-1} + u_{1,1}) / 4 - u_x - 11u_{xxx} / 10 \ ;$$

$$u_{xxxy} = -u_{-1,-1} / 12 + u_{-1,1} / 12 + u_{-2,-1} / 24 - u_{-2,1} / 24 + u_{1,-1} / 12 - u_{1,1} / 12 - u_{2,-1} / 24 + u_{2,1} / 24 \ ;$$
$$u_{xyyy} = -u_{-1,-1} / 12 + u_{-1,-2} / 24 + u_{-1,1} / 12 - u_{-1,2} / 24 + u_{1,-1} / 12 - u_{1,-2} / 24 - u_{1,1} / 12 + u_{1,2} / 24 \ ;$$
$$u_{xxyy} = -u_{0,0} + u_{-1,-1} / 4 + u_{-1,1} / 4 + u_{1,-1} / 4 + u_{1,1} / 4 - u_{xx} - 9u_{xxxx} / 7 - u_{yy} - 9u_{yyyy} / 7 \ .$$
$$\tag{A.20}$$

The smoothness indicator for eqn. (A.11), written as a sum of perfect squares, is given by

$$
\begin{aligned}
IS &= (u_x + u_{xxx}/10)^2 + \frac{13}{3}(u_{xx} + 123u_{xxxx}/455)^2 + \frac{781}{20}u_{xxx}^2 \\
&+ \frac{1421461}{2275}u_{xxxx}^2 + (u_y + u_{yyy}/10)^2 + \frac{13}{3}(u_{yy} + 123u_{yyyy}/455)^2 \\
&+ \frac{781}{20}u_{yyy}^2 + \frac{1421461}{2275}u_{yyyy}^2 + \frac{7}{6}(u_{xy} + 13u_{xxxy}/140 + 13u_{xyyy}/140)^2 \\
&+ \frac{47}{10}u_{xxy}^2 + \frac{47}{10}u_{xyy}^2 + \frac{88841}{2100}(u_{xxxy}^2 + u_{xyyy}^2) + \frac{1}{16800}(u_{xxxy} - u_{xyyy})^2 + \frac{5083}{270}u_{xxyy}^2 \quad .
\end{aligned} \tag{A.21}
$$

This completes our description of finite volume r=5 WENO reconstruction in 2D.

A.4) r=4 WENO Reconstruction in 3D

Up to third order, there are no cross terms that simultaneously involve “x”, “y” and “z” moments. Therefore, our description of the differences introduced by three-dimensional reconstruction start at fourth order. At fourth order, the reconstruction polynomial has one term, u_{xyz} which simultaneously involves “x”, “y” and “z” moments. The reconstruction polynomial at fourth order is given by

$$
\begin{aligned}
u(x,y,z) = u_{0,0,0} &+ u_x x + u_y y + u_z z + u_{xx}(x^2 - 1/12) + u_{yy}(y^2 - 1/12) + u_{zz}(z^2 - 1/12) \\
&+ u_{xy}xy + u_{yz}yz + u_{xz}xz + u_{xxx}(x^3 - 3x/20) + u_{yyy}(y^3 - 3y/20) + u_{zzz}(z^3 - 3z/20) \\
&+ u_{xxy}(x^2 - 1/12)y + u_{xxz}(x^2 - 1/12)z + u_{xyy}x(y^2 - 1/12) + u_{yyz}(y^2 - 1/12)z \\
&+ u_{xzz}x(z^2 - 1/12) + u_{yzz}y(z^2 - 1/12) + u_{xyz}xyz \quad .
\end{aligned} \tag{A.22}
$$

One-dimensional WENO-AO can be used to obtain the u_x, u_{xx} and u_{xxx} terms and also the u_y, u_{yy} and u_{yyy} terms and also the u_z, u_{zz} and u_{zzz} terms. Furthermore, the two-dimensional WENO-AO can be used to obtain the u_{xy}, u_{xxy}, u_{xyy}, u_{yz}, u_{yyz}, u_{yzz}, u_{xz}, u_{xxz} and u_{xzz} cross-terms. As with Figs. A.1 and A.2, we use three integer subscripted variables to denote zones in three dimensions. The integer subscripts denote the relative position of the zone we are considering relative to the zone of interest, which is always denoted by $u_{0,0,0}$. There are eight possible stencils that can each be used to evaluate the three-dimensional cross term u_{xyz} which we catalogue below.

For stencil S_1 we obtain

$$u_{xyz} = u_{1,1,1} - \frac{11}{10}u_{zzz} - u_{xzz} - u_{zz} - u_{yzz} - u_{yyz} - u_{xxz} - u_{xz} - u_{z} - u_{yz}$$
$$- u_{yy} - \frac{11}{10}u_{yyy} - u_{xyy} - u_{xy} - u_{xxy} - u_{y} - u_{x} - u_{0,0,0} - \frac{11}{10}u_{xxx} - u_{xx} \quad . \tag{A.23}$$

For stencil S_2 we obtain

$$u_{xyz} = -u_{-1,1,1} + \frac{11}{10}u_{zzz} - u_{xzz} + u_{zz} + u_{yzz} + u_{yyz} + u_{xxz} - u_{xz} + u_{z} + u_{yz}$$
$$+ u_{yy} + \frac{11}{10}u_{yyy} - u_{xyy} - u_{xy} + u_{xxy} + u_{y} - u_{x} + u_{0,0,0} - \frac{11}{10}u_{xxx} + u_{xx} \quad . \tag{A.24}$$

For stencil S_3 we obtain

$$u_{xyz} = -u_{1,-1,1} + \frac{11}{10}u_{zzz} + u_{xzz} + u_{zz} - u_{yzz} + u_{yyz} + u_{xxz} + u_{xz} + u_{z} - u_{yz}$$
$$+ u_{yy} - \frac{11}{10}u_{yyy} + u_{xyy} - u_{xy} - u_{xxy} - u_{y} + u_{x} + u_{0,0,0} + \frac{11}{10}u_{xxx} + u_{xx} \quad . \tag{A.25}$$

For stencil S_4 we obtain

$$u_{xyz} = -u_{1,1,-1} - \frac{11}{10}u_{zzz} + u_{xzz} + u_{zz} + u_{yzz} - u_{yyz} - u_{xxz} - u_{xz} - u_{z} - u_{yz}$$
$$+ u_{yy} + \frac{11}{10}u_{yyy} + u_{xyy} + u_{xy} + u_{xxy} + u_{y} + u_{x} + u_{0,0,0} + \frac{11}{10}u_{xxx} + u_{xx} \quad . \tag{A.26}$$

For stencil S_5 we obtain

$$u_{xyz} = u_{-1,-1,1} - \frac{11}{10}u_{zzz} + u_{xzz} - u_{zz} + u_{yzz} - u_{yyz} - u_{xxz} + u_{xz} - u_{z} + u_{yz}$$
$$- u_{yy} + \frac{11}{10}u_{yyy} + u_{xyy} - u_{xy} + u_{xxy} + u_{y} + u_{x} - u_{0,0,0} + \frac{11}{10}u_{xxx} - u_{xx} \quad . \tag{A.27}$$

For stencil S_6 we obtain

$$u_{xyz} = u_{-1,1,-1} + \frac{11}{10}u_{zzz} + u_{xzz} - u_{zz} - u_{yzz} + u_{yyz} + u_{xxz} - u_{xz} + u_{z} + u_{yz}$$
$$- u_{yy} - \frac{11}{10}u_{yyy} + u_{xyy} + u_{xy} - u_{xxy} - u_{y} + u_{x} - u_{0,0,0} + \frac{11}{10}u_{xxx} - u_{xx} \quad . \tag{A.28}$$

For stencil S_7 we obtain

$$
\begin{aligned}
u_{xyz} = & u_{1,-1,-1} + \frac{11}{10}u_{zzz} - u_{xzz} - u_{zz} + u_{yzz} + u_{yyz} + u_{xxz} + u_{xz} + u_{z} - u_{yz} \\
& - u_{yy} + \frac{11}{10}u_{yyy} - u_{xyy} + u_{xy} + u_{xxy} + u_{y} - u_{x} - u_{0,0,0} - \frac{11}{10}u_{xxx} - u_{xx} \quad .
\end{aligned}
\tag{A.29}
$$

For stencil S_8 we obtain

$$
\begin{aligned}
u_{xyz} = & -u_{-1,-1,-1} - \frac{11}{10}u_{zzz} - u_{xzz} + u_{zz} - u_{yzz} - u_{yyz} - u_{xxz} + u_{xz} - u_{z} + u_{yz} \\
& + u_{yy} - \frac{11}{10}u_{yyy} - u_{xyy} + u_{xy} - u_{xxy} - u_{y} - u_{x} + u_{0,0,0} - \frac{11}{10}u_{xxx} + u_{xx} \quad .
\end{aligned}
\tag{A.30}
$$

The smoothness measure for each of the eight stencils is given by

$$
\begin{aligned}
IS = & (u_x + u_{xxx}/10)^2 + 13u_{xx}^2/3 + 781u_{xxx}^2/20 + (u_y + u_{yyy}/10)^2 + 13u_{yy}^2/3 + 781u_{yyy}^2/20 \\
& + (u_z + u_{zzz}/10)^2 + 13u_{zz}^2/3 + 781u_{zzz}^2/20 + 7u_{xy}^2/6 + 7u_{yz}^2/6 + 7u_{xz}^2/6 \\
& + 47u_{xxy}^2/10 + 47u_{xyy}^2/10 + 47u_{xxz}^2/10 + 47u_{xzz}^2/10 + 47u_{yyz}^2/10 + 47u_{yzz}^2/10 + 61u_{xyz}^2/48 \quad .
\end{aligned}
\tag{A.31}
$$

This completes our description of finite volume r=4 WENO reconstruction in 3D.

A.5) r=5 WENO Reconstruction in 3D

The reconstruction polynomial at fifth order is given by

$$
\begin{aligned}
u(x,y,z) = & u_{0,0,0} + u_x x + u_y y + u_z z + u_{xx}(x^2 - 1/12) + u_{yy}(y^2 - 1/12) + u_{zz}(z^2 - 1/12) \\
& + u_{xy}xy + u_{yz}yz + u_{xz}xz + u_{xxx}(x^3 - 3x/20) + u_{yyy}(y^3 - 3y/20) + u_{zzz}(z^3 - 3z/20) \\
& + u_{xxy}(x^2 - 1/12)y + u_{xxz}(x^2 - 1/12)z + u_{xyy}x(y^2 - 1/12) + u_{yyz}(y^2 - 1/12)z \\
& + u_{xzz}x(z^2 - 1/12) + u_{yzz}y(z^2 - 1/12) + u_{xyz}xyz \\
& + u_{xxxx}\left(x^4 - 3x^2/14 + 3/560\right) + u_{yyyy}\left(y^4 - 3y^2/14 + 3/560\right) + u_{zzzz}\left(z^4 - 3z^2/14 + 3/560\right) \\
& + u_{xxxy}(x^3 - 3x/20)y + u_{xxxz}(x^3 - 3x/20)z + u_{xyyy}x(y^3 - 3y/20) + u_{yyyz}(y^3 - 3y/20)z \\
& + u_{xzzz}x(z^3 - 3z/20) + u_{yzzz}y(z^3 - 3z/20) + u_{xxyy}(x^2 - 1/12)(y^2 - 1/12) \\
& + u_{yyzz}(y^2 - 1/12)(z^2 - 1/12) + u_{xxzz}(x^2 - 1/12)(z^2 - 1/12) + u_{xxyz}(x^2 - 1/12)yz \\
& + u_{xyyz}x(y^2 - 1/12)z + u_{xyzz}xy(z^2 - 1/12) \quad .
\end{aligned}
\tag{A.32}
$$

One-dimensional WENO-AO can be used to obtain the u_x, u_{xx}, u_{xxx} and u_{xxxx} terms and also the u_y, u_{yy}, u_{yyy} and u_{yyyy} terms and also the u_z, u_{zz}, u_{zzz} and u_{zzzz} terms. Furthermore, the two-dimensional WENO-AO can be used to obtain the $u_{xy}, u_{xxy}, u_{xyy}, u_{xxxy}, u_{xyyy}, u_{xxyy},$, $u_{yz}, u_{yyz}, u_{yzz}, u_{yyyz}, u_{yzzz}, u_{yyzz}$, $u_{xz}, u_{xxz}, u_{xzz}, u_{xxxz}, u_{xzzz}$ and u_{xxzz} cross-terms. There are nine possible stencils that can each be used to evaluate the three-dimensional cross terms $u_{xyz}, u_{xxyz}, u_{xyyz}$ and u_{xyyz}. The central stencil was added for stability reasons and has a linear weight that is 100 to 400 times larger than the linear weights of the other four directionally-biased stencils. For each of the nine stencils, we have nine possible evaluations of the cross-terms $u_{xyz}, u_{xxyz}, u_{xyyz}$ and u_{xyyz} which we catalogue below.

For the stencil S_1 we obtain

$$
\begin{aligned}
u_{xyz} = (&-350u_{0,0,0} + 560u_{1,1,1} - 70u_{1,1,2} - 70u_{1,2,1} - 70u_{2,1,1} - 280u_x - 140u_{xx} + 112u_{xxx} + 660u_{xxxx} \\
&+ 189u_{xxxy} + 189u_{xxxz} - 70u_{xxy} + 70u_{xxyy} - 70u_{xxz} + 70u_{xxzz} - 210u_{xy} - 70u_{xyy} + 189u_{xyyy} \\
&- 210u_{xz} - 70u_{xzz} + 189u_{xzzz} - 280u_y - 140u_{yy} + 112u_{yyy} + 660u_{yyyy} + 189u_{yyyz} - 70u_{yyz} \\
&+ 70u_{yyzz} - 210u_{yz} - 70u_{yzz} + 189u_{yzzz} - 280u_z - 140u_{zz} + 112u_{zzz} + 660u_{zzzz})/140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xxyz} = (&70u_{0,0,0} - 140u_{1,1,1} + 70u_{2,1,1} - 140u_{xx} - 420u_{xxx} - 1020u_{xxxx} - 420u_{xxxy} - 420u_{xxxz} - 140u_{xxy} \\
&- 140u_{xxyy} - 140u_{xxz} - 140u_{xxzz} + 70u_y + 70u_{yy} + 77u_{yyy} + 90u_{yyyy} \\
&+ 77u_{yyyz} + 70u_{yyz} + 70u_{yyzz} + 70u_{yz} + 70u_{yzz} + 77u_{yzzz} + 70u_z + 70u_{zz} + 77u_{zzz} + 90u_{zzzz})/140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyyz} = (&70u_{0,0,0} - 140u_{1,1,1} + 70u_{1,2,1} + 70u_x + 70u_{xx} + 77u_{xxx} + 90u_{xxxx} + 77u_{xxxz} - 140u_{xxyy} + 70u_{xxz} \\
&+ 70u_{xxzz} - 140u_{xyy} - 420u_{xyyy} + 70u_{xz} + 70u_{xzz} + 77u_{xzzz} - 140u_{yy} - 420u_{yyy} - 1020u_{yyyy} \\
&- 420u_{yyyz} - 140u_{yyz} - 140u_{yyzz} + 70u_z + 70u_{zz} + 77u_{zzz} + 90u_{zzzz})/140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyzz} = (&70u_{0,0,0} - 140u_{1,1,1} + 70u_{1,1,2} + 70u_x + 70u_{xx} + 77u_{xxx} + 90u_{xxxx} + 77u_{xxxy} + 70u_{xxy} + 70u_{xxyy} \\
&- 140u_{xxzz} + 70u_{xy} + 70u_{xyy} + 77u_{xyyy} - 140u_{xzz} - 420u_{xzzz} + 70u_y \\
&+ 70u_{yy} + 77u_{yyy} + 90u_{yyyy} - 140u_{yyzz} - 140u_{yzz} - 420u_{yzzz} - 140u_{zz} - 420u_{zzz} - 1020u_{zzzz})/140 \quad .
\end{aligned}
\tag{A.33}
$$

For the stencil S_2 we obtain

$$
\begin{aligned}
u_{xyz} = (&350u_{0,0,0} - 560u_{-1,1,1} + 70u_{-1,1,2} + 70u_{-1,2,1} + 70u_{-2,1,1} - 280u_x + 140u_{xx} + 112u_{xxx} \\
&- 660u_{xxxx} + 189u_{xxxy} + 189u_{xxxz} + 70u_{xxy} - 70u_{xxyy} + 70u_{xxz} - 70u_{xxzz} - 210u_{xy} - 70u_{xyy} \\
&+ 189u_{xyyy} - 210u_{xz} - 70u_{xzz} + 189u_{xzzz} + 280u_y + 140u_{yy} - 112u_{yyy} - 660u_{yyyy} - 189u_{yyyz} \\
&+ 70u_{yyz} - 70u_{yyzz} + 210u_{yz} + 70u_{yzz} - 189u_{yzzz} + 280u_z + 140u_{zz} - 112u_{zzz} - 660u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xxyz} = (&70u_{0,0,0} - 140u_{-1,1,1} + 70u_{-2,1,1} - 140u_{xx} + 420u_{xxx} - 1020u_{xxxx} + 420u_{xxxy} + 420u_{xxxz} \\
&- 140u_{xxy} - 140u_{xxyy} - 140u_{xxz} - 140u_{xxzz} + 70u_y + 70u_{yy} + 77u_{yyy} + 90u_{yyyy} \\
&+ 77u_{yyyz} + 70u_{yyz} + 70u_{yyzz} + 70u_{yz} + 70u_{yzz} + 77u_{yzzz} + 70u_z + 70u_{zz} + 77u_{zzz} + 90u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyyz} = (&-70u_{0,0,0} + 140u_{-1,1,1} - 70u_{-1,2,1} + 70u_x - 70u_{xx} + 77u_{xxx} - 90u_{xxxx} + 77u_{xxxz} + 140u_{xxyy} \\
&- 70u_{xxz} - 70u_{xxzz} - 140u_{xyy} - 420u_{xyyy} + 70u_{xz} + 70u_{xzz} + 77u_{xzzz} + 140u_{yy} + 420u_{yyy} \\
&+ 1020u_{yyyy} + 420u_{yyyz} + 140u_{yyz} + 140u_{yyzz} - 70u_z - 70u_{zz} - 77u_{zzz} - 90u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyzz} = (&-70u_{0,0,0} + 140u_{-1,1,1} - 70u_{-1,1,2} + 70u_x - 70u_{xx} + 77u_{xxx} - 90u_{xxxx} + 77u_{xxxy} - 70u_{xxy} \\
&- 70u_{xxyy} + 140u_{xxzz} + 70u_{xy} + 70u_{xyy} + 77u_{xyyy} - 140u_{xzz} - 420u_{xzzz} - 70u_y \\
&- 70u_{yy} - 77u_{yyy} - 90u_{yyyy} + 140u_{yyzz} + 140u_{yzz} + 420u_{yzzz} + 140u_{zz} + 420u_{zzz} + 1020u_{zzzz}) / 140 \quad .
\end{aligned}
\tag{A.34}
$$

For the stencil S_3 we obtain

$$
\begin{aligned}
u_{xyz} = (&350u_{0,0,0} - 560u_{1,-1,1} + 70u_{1,-1,2} + 70u_{1,-2,1} + 70u_{2,-1,1} + 280u_x + 140u_{xx} - 112u_{xxx} - 660u_{xxxx} \\
&+ 189u_{xxxy} - 189u_{xxxz} - 70u_{xxy} - 70u_{xxyy} + 70u_{xxz} - 70u_{xxzz} - 210u_{xy} + 70u_{xyy} + 189u_{xyyy} + 210u_{xz} \\
&+ 70u_{xzz} - 189u_{xzzz} - 280u_y + 140u_{yy} + 112u_{yyy} - 660u_{yyyy} + 189u_{yyyz} + 70u_{yyz} - 70u_{yyzz} - 210u_{yz} \\
&- 70u_{yzz} + 189u_{yzzz} + 280u_z + 140u_{zz} - 112u_{zzz} - 660u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xxyz} = (&-70u_{0,0,0} + 140u_{1,-1,1} - 70u_{2,-1,1} + 140u_{xx} + 420u_{xxx} + 1020u_{xxxx} - 420u_{xxxy} + 420u_{xxxz} \\
&- 140u_{xxy} + 140u_{xxyy} + 140u_{xxz} + 140u_{xxzz} + 70u_y - 70u_{yy} + 77u_{yyy} - 90u_{yyyy} \\
&+ 77u_{yyyz} - 70u_{yyz} - 70u_{yyzz} + 70u_{yz} + 70u_{yzz} + 77u_{yzzz} - 70u_z - 70u_{zz} - 77u_{zzz} - 90u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$u_{xyyz}=(70u_{0,0,0}-140u_{1,-1,1}+70u_{1,-2,1}+70u_{x}+70u_{xx}+77u_{xxx}+90u_{xxxx}+77u_{xxxz}-140u_{xxyy}$$
$$+70u_{xxz}+70u_{xxzz}-140u_{xyy}+420u_{xyyy}+70u_{xz}+70u_{xzz}+77u_{xzzz}-140u_{yy}+420u_{yyy}$$
$$-1020u_{yyyy}+420u_{yyyz}-140u_{yyz}-140u_{yyzz}+70u_{z}+70u_{zz}+77u_{zzz}+90u_{zzzz})/140 \;\;;$$

$$u_{xyzz}=(-70u_{0,0,0}+140u_{1,-1,1}-70u_{1,-1,2}-70u_{x}-70u_{xx}-77u_{xxx}-90u_{xxxx}+77u_{xxxy}+70u_{xxy}$$
$$-70u_{xxyy}+140u_{xxzz}+70u_{xy}-70u_{xyy}+77u_{xyyy}+140u_{xzz}+420u_{xzzz}+70u_{y}-70u_{yy}$$
$$+77u_{yyy}-90u_{yyyy}+140u_{yyzz}-140u_{yzz}-420u_{yzzz}+140u_{zz}+420u_{zzz}+1020u_{zzzz})/140 \;\;.$$

(A.35)

For the stencil S_4 we obtain

$$u_{xyz}=(-350u_{0,0,0}+560u_{-1,-1,1}-70u_{-1,-1,2}-70u_{-1,-2,1}-70u_{-2,-1,1}+280u_{x}-140u_{xx}-112u_{xxx}$$
$$+660u_{xxxx}+189u_{xxxy}-189u_{xxxz}+70u_{xxy}+70u_{xxyy}-70u_{xxz}+70u_{xxzz}-210u_{xy}+70u_{xyy}$$
$$+189u_{xyyy}+210u_{xz}+70u_{xzz}-189u_{xzzz}+280u_{y}-140u_{yy}-112u_{yyy}+660u_{yyyy}-189u_{yyyz}$$
$$-70u_{yyz}+70u_{yyzz}+210u_{yz}+70u_{yzz}-189u_{yzzz}-280u_{z}-140u_{zz}+112u_{zzz}+660u_{zzzz})/140 \;\;;$$

$$u_{xxyz}=(-70u_{0,0,0}+140u_{-1,-1,1}-70u_{-2,-1,1}+140u_{xx}-420u_{xxx}+1020u_{xxxx}+420u_{xxxy}-420u_{xxxz}$$
$$-140u_{xxy}+140u_{xxyy}+140u_{xxz}+140u_{xxzz}+70u_{y}-70u_{yy}+77u_{yyy}-90u_{yyyy}$$
$$+77u_{yyyz}-70u_{yyz}-70u_{yyzz}+70u_{yz}+70u_{yzz}+77u_{yzzz}-70u_{z}-70u_{zz}-77u_{zzz}-90u_{zzzz})/140 \;\;;$$

$$u_{xyyz}=(-70u_{0,0,0}+140u_{-1,-1,1}-70u_{-1,-2,1}+70u_{x}-70u_{xx}+77u_{xxx}-90u_{xxxx}+77u_{xxxz}+140u_{xxyy}$$
$$-70u_{xxz}-70u_{xxzz}-140u_{xyy}+420u_{xyyy}+70u_{xz}+70u_{xzz}+77u_{xzzz}+140u_{yy}-420u_{yyy}$$
$$+1020u_{yyyy}-420u_{yyyz}+140u_{yyz}+140u_{yyzz}-70u_{z}-70u_{zz}-77u_{zzz}-90u_{zzzz})/140 \;\;;$$

$$u_{xyzz}=(70u_{0,0,0}-140u_{-1,-1,1}+70u_{-1,-1,2}-70u_{x}+70u_{xx}-77u_{xxx}+90u_{xxxx}+77u_{xxxy}-70u_{xxy}$$
$$+70u_{xxyy}-140u_{xxzz}+70u_{xy}-70u_{xyy}+77u_{xyyy}+140u_{xzz}+420u_{xzzz}-70u_{y}+70u_{yy}$$
$$-77u_{yyy}+90u_{yyyy}-140u_{yyzz}+140u_{yzz}+420u_{yzzz}-140u_{zz}-420u_{zzz}-1020u_{zzzz})/140 \;\;.$$

(A.36)

For the stencil S_5 we obtain

$$
\begin{aligned}
u_{xyz} = (&350u_{0,0,0} - 560u_{1,1,-1} + 70u_{1,1,-2} + 70u_{1,2,-1} + 70u_{2,1,-1} + 280u_x + 140u_{xx} - 112u_{xxx} - 660u_{xxxx} \\
&- 189u_{xxxy} + 189u_{xxxz} + 70u_{xxy} - 70u_{xxyy} - 70u_{xxz} - 70u_{xxzz} + 210u_{xy} + 70u_{xyy} - 189u_{xyyy} \\
&- 210u_{xz} + 70u_{xzz} + 189u_{xzzz} + 280u_y + 140u_{yy} - 112u_{yyy} - 660u_{yyyy} + 189u_{yyyz} - 70u_{yyz} \\
&- 70u_{yyzz} - 210u_{yz} + 70u_{yzz} + 189u_{yzzz} - 280u_z + 140u_{zz} + 112u_{zzz} - 660u_{zzzz})/140 \ ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xxyz} = (&-70u_{0,0,0} + 140u_{1,1,-1} - 70u_{2,1,-1} + 140u_{xx} + 420u_{xxx} + 1020u_{xxxx} + 420u_{xxxy} - 420u_{xxxz} \\
&+ 140u_{xxy} + 140u_{xxyy} - 140u_{xxz} + 140u_{xxzz} - 70u_y - 70u_{yy} - 77u_{yyy} - 90u_{yyyy} + 77u_{yyyz} \\
&+ 70u_{yyz} - 70u_{yyzz} + 70u_{yz} - 70u_{yzz} + 77u_{yzzz} + 70u_z - 70u_{zz} + 77u_{zzz} - 90u_{zzzz})/140 \ ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyyz} = (&-70u_{0,0,0} + 140u_{1,1,-1} - 70u_{1,2,-1} - 70u_x - 70u_{xx} - 77u_{xxx} - 90u_{xxxx} + 77u_{xxxz} + 140u_{xxyy} \\
&+ 70u_{xxz} - 70u_{xxzz} + 140u_{xyy} + 420u_{xyyy} + 70u_{xz} - 70u_{xzz} + 77u_{xzzz} + 140u_{yy} + 420u_{yyy} \\
&+ 1020u_{yyyy} - 420u_{yyyz} - 140u_{yyz} + 140u_{yyzz} + 70u_z - 70u_{zz} + 77u_{zzz} - 90u_{zzzz})/140 \ ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyzz} = (&70u_{0,0,0} - 140u_{1,1,-1} + 70u_{1,1,-2} + 70u_x + 70u_{xx} + 77u_{xxx} + 90u_{xxxx} + 77u_{xxxy} + 70u_{xxy} \\
&+ 70u_{xxyy} - 140u_{xxzz} + 70u_{xy} + 70u_{xyy} + 77u_{xyyy} - 140u_{xzz} + 420u_{xzzz} + 70u_y + 70u_{yy} \\
&+ 77u_{yyy} + 90u_{yyyy} - 140u_{yyzz} - 140u_{yzz} + 420u_{yzzz} - 140u_{zz} + 420u_{zzz} - 1020u_{zzzz})/140 \ .
\end{aligned}
\tag{A.37}
$$

For the stencil S_6 we obtain

$$
\begin{aligned}
u_{xyz} = (&-350u_{0,0,0} + 560u_{-1,1,-1} - 70u_{-1,1,-2} - 70u_{-1,2,-1} - 70u_{-2,1,-1} + 280u_x - 140u_{xx} - 112u_{xxx} \\
&+ 660u_{xxxx} - 189u_{xxxy} + 189u_{xxxz} - 70u_{xxy} + 70u_{xxyy} + 70u_{xxz} + 70u_{xxzz} + 210u_{xy} + 70u_{xyy} \\
&- 189u_{xyyy} - 210u_{xz} + 70u_{xzz} + 189u_{xzzz} - 280u_y - 140u_{yy} + 112u_{yyy} + 660u_{yyyy} - 189u_{yyyz} \\
&+ 70u_{yyz} + 70u_{yyzz} + 210u_{yz} - 70u_{yzz} - 189u_{yzzz} + 280u_z - 140u_{zz} - 112u_{zzz} + 660u_{zzzz})/140 \ ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xxyz} = (&-70u_{0,0,0} + 140u_{-1,1,-1} - 70u_{-2,1,-1} + 140u_{xx} - 420u_{xxx} + 1020u_{xxxx} - 420u_{xxxy} + 420u_{xxxz} \\
&+ 140u_{xxy} + 140u_{xxyy} - 140u_{xxz} + 140u_{xxzz} - 70u_y - 70u_{yy} - 77u_{yyy} - 90u_{yyyy} + 77u_{yyyz} \\
&+ 70u_{yyz} - 70u_{yyzz} + 70u_{yz} - 70u_{yzz} + 77u_{yzzz} + 70u_z - 70u_{zz} + 77u_{zzz} - 90u_{zzzz})/140 \ ;
\end{aligned}
$$

$$u_{xyyz}=(70u_{0,0,0}-140u_{-1,1,-1}+70u_{-1,2,-1}-70u_{x}+70u_{xx}-77u_{xxx}+90u_{xxxx}+77u_{xxxz}-140u_{xxyy}$$
$$-70u_{xxz}+70u_{xxzz}+140u_{xyy}+420u_{xyyy}+70u_{xz}-70u_{xzz}+77u_{xzzz}-140u_{yy}-420u_{yyy}$$
$$-1020u_{yyyy}+420u_{yyyz}+140u_{yyz}-140u_{yyzz}-70u_{z}+70u_{zz}-77u_{zzz}+90u_{zzzz})/140 \;\;;$$

$$u_{xyzz}=(-70u_{0,0,0}+140u_{-1,1,-1}-70u_{-1,1,-2}+70u_{x}-70u_{xx}+77u_{xxx}-90u_{xxxx}+77u_{xxxy}-70u_{xxy}$$
$$-70u_{xxyy}+140u_{xxzz}+70u_{xy}+70u_{xyy}+77u_{xyyy}-140u_{xzz}+420u_{xzzz}-70u_{y}-70u_{yy}$$
$$-77u_{yyy}-90u_{yyyy}+140u_{yyzz}+140u_{yzz}-420u_{yzzz}+140u_{zz}-420u_{zzz}+1020u_{zzzz})/140 \;\;.$$

(A.38)

For the stencil S_7 we obtain

$$u_{xyz}=(-350u_{0,0,0}+560u_{1,-1,-1}-70u_{1,-1,-2}-70u_{1,-2,-1}-70u_{2,-1,-1}-280u_{x}-140u_{xx}+112u_{xxx}$$
$$+660u_{xxxx}-189u_{xxxy}-189u_{xxxz}+70u_{xxy}+70u_{xxyy}+70u_{xxz}+70u_{xxzz}+210u_{xy}-70u_{xyy}$$
$$-189u_{xyyy}+210u_{xz}-70u_{xzz}-189u_{xzzz}+280u_{y}-140u_{yy}-112u_{yyy}+660u_{yyyy}+189u_{yyyz}$$
$$+70u_{yyz}+70u_{yyzz}-210u_{yz}+70u_{yzz}+189u_{yzzz}+280u_{z}-140u_{zz}-112u_{zzz}+660u_{zzzz})/140 \;\;;$$

$$u_{xxyz}=(70u_{0,0,0}-140u_{1,-1,-1}+70u_{2,-1,-1}-140u_{xx}-420u_{xxx}-1020u_{xxxx}+420u_{xxxy}+420u_{xxxz}$$
$$+140u_{xxy}-140u_{xxyy}+140u_{xxz}-140u_{xxzz}-70u_{y}+70u_{yy}-77u_{yyy}+90u_{yyyy}+77u_{yyyz}$$
$$-70u_{yyz}+70u_{yyzz}+70u_{yz}-70u_{yzz}+77u_{yzzz}-70u_{z}+70u_{zz}-77u_{zzz}+90u_{zzzz})/140 \;\;;$$

$$u_{xyyz}=(-70u_{0,0,0}+140u_{1,-1,-1}-70u_{1,-2,-1}-70u_{x}-70u_{xx}-77u_{xxx}-90u_{xxxx}+77u_{xxxz}+140u_{xxyy}$$
$$+70u_{xxz}-70u_{xxzz}+140u_{xyy}-420u_{xyyy}+70u_{xz}-70u_{xzz}+77u_{xzzz}+140u_{yy}-420u_{yyy}$$
$$+1020u_{yyyy}+420u_{yyyz}-140u_{yyz}+140u_{yyzz}+70u_{z}-70u_{zz}+77u_{zzz}-90u_{zzzz})/140 \;\;;$$

$$u_{xyzz}=(-70u_{0,0,0}+140u_{1,-1,-1}-70u_{1,-1,-2}-70u_{x}-70u_{xx}-77u_{xxx}-90u_{xxxx}+77u_{xxxy}+70u_{xxy}$$
$$-70u_{xxyy}+140u_{xxzz}+70u_{xy}-70u_{xyy}+77u_{xyyy}+140u_{xzz}-420u_{xzzz}+70u_{y}-70u_{yy}$$
$$+77u_{yyy}-90u_{yyyy}+140u_{yyzz}-140u_{yzz}+420u_{yzzz}+140u_{zz}-420u_{zzz}+1020u_{zzzz})/140 \;\;.$$

(A.39)

For the stencil S_8 we obtain

$$
\begin{aligned}
u_{xyz} = (&350u_{0,0,0} - 560u_{-1,-1,-1} + 70u_{-1,-1,-2} + 70u_{-1,-2,-1} + 70u_{-2,-1,-1} - 280u_x + 140u_{xx} + 112u_{xxx} \\
&- 660u_{xxxx} - 189u_{xxxy} - 189u_{xxxz} - 70u_{xxy} - 70u_{xxyy} - 70u_{xxz} - 70u_{xxzz} + 210u_{xy} - 70u_{xyy} \\
&- 189u_{xyyy} + 210u_{xz} - 70u_{xzz} - 189u_{xzzz} - 280u_y + 140u_{yy} + 112u_{yyy} - 660u_{yyyy} - 189u_{yyyz} \\
&- 70u_{yyz} - 70u_{yyzz} + 210u_{yz} - 70u_{yzz} - 189u_{yzzz} - 280u_z + 140u_{zz} + 112u_{zzz} - 660u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xxyz} = (&70u_{0,0,0} - 140u_{-1,-1,-1} + 70u_{-2,-1,-1} - 140u_{xx} + 420u_{xxx} - 1020u_{xxxx} - 420u_{xxxy} - 420u_{xxxz} \\
&+ 140u_{xxy} - 140u_{xxyy} + 140u_{xxz} - 140u_{xxzz} - 70u_y + 70u_{yy} - 77u_{yyy} + 90u_{yyyy} + 77u_{yyyz} \\
&- 70u_{yyz} + 70u_{yyzz} + 70u_{yz} - 70u_{yzz} + 77u_{yzzz} - 70u_z + 70u_{zz} - 77u_{zzz} + 90u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyyz} = (&70u_{0,0,0} - 140u_{-1,-1,-1} + 70u_{-1,-2,-1} - 70u_x + 70u_{xx} - 77u_{xxx} + 90u_{xxxx} + 77u_{xxxz} - 140u_{xxyy} \\
&- 70u_{xxz} + 70u_{xxzz} + 140u_{xyy} - 420u_{xyyy} + 70u_{xz} - 70u_{xzz} + 77u_{xzzz} - 140u_{yy} + 420u_{yyy} \\
&- 1020u_{yyyy} - 420u_{yyyz} + 140u_{yyz} - 140u_{yyzz} - 70u_z + 70u_{zz} - 77u_{zzz} + 90u_{zzzz}) / 140 \quad ;
\end{aligned}
$$

$$
\begin{aligned}
u_{xyzz} = (&70u_{0,0,0} - 140u_{-1,-1,-1} + 70u_{-1,-1,-2} - 70u_x + 70u_{xx} - 77u_{xxx} + 90u_{xxxx} + 77u_{xxxy} - 70u_{xxy} \\
&+ 70u_{xxyy} - 140u_{xxzz} + 70u_{xy} - 70u_{xyy} + 77u_{xyyy} + 140u_{xzz} - 420u_{xzzz} - 70u_y + 70u_{yy} \\
&- 77u_{yyy} + 90u_{yyyy} - 140u_{yyzz} + 140u_{yzz} - 420u_{yzzz} - 140u_{zz} + 420u_{zzz} - 1020u_{zzzz}) / 140 \quad .
\end{aligned}
\tag{A.40}
$$

For the central stencil S_9 we obtain

$$
\begin{aligned}
u_{xyz} &= (-u_{-1,-1,-1} + u_{-1,-1,1} + u_{-1,1,-1} - u_{-1,1,1} + u_{1,-1,-1} - u_{1,-1,1} - u_{1,1,-1} + u_{1,1,1}) / 8 \quad ; \\
u_{xxyz} &= (u_{-1,-1,-1} - u_{-1,-1,1} - u_{-1,1,-1} + u_{-1,1,1} + u_{1,-1,-1} - u_{1,-1,1} - u_{1,1,-1} + u_{1,1,1}) / 8 \\
&\quad - u_{yz} - 11u_{yyyz} / 10 - 11u_{yzzz} / 10 \quad ; \\
u_{xyyz} &= (u_{-1,-1,-1} - u_{-1,-1,1} + u_{-1,1,-1} - u_{-1,1,1} - u_{1,-1,-1} + u_{1,-1,1} - u_{1,1,-1} + u_{1,1,1}) / 8 \\
&\quad - u_{xz} - 11u_{xxxz} / 10 - 11u_{xzzz} / 10 \quad ; \\
u_{xyzz} &= (u_{-1,-1,-1} + u_{-1,-1,1} - u_{-1,1,-1} - u_{-1,1,1} - u_{1,-1,-1} - u_{1,-1,1} + u_{1,1,-1} + u_{1,1,1}) / 8 \\
&\quad - u_{xy} - 11u_{xxxy} / 10 - 11u_{xyyy} / 10 \quad .
\end{aligned}
\tag{A.41}
$$

The smoothness indicator for eqn. (A.32), written as a sum of perfect squares, is given by

$$
\begin{aligned}
IS &= (u_x + u_{xxx}/10)^2 + 13(u_{xx} + 123u_{xxxx}/455)^2/3 + 781u_{xxx}^2/20 + 1421461u_{xxxx}^2/2275 + (u_y + u_{yyy}/10)^2 \\
&+ 13(u_{yy} + 123u_{yyyy}/455)^2/3 + 781u_{yyy}^2/20 + 1421461u_{yyyy}^2/2275 + (u_z + u_{zzz}/10)^2 \\
&+ 13(u_{zz} + 123u_{zzzz}/455)^2/3 + 781u_{zzz}^2/20 + 1421461u_{zzzz}^2/2275 + 47u_{xxy}^2/10 + 47u_{xyy}^2/10 \\
&+ 47u_{xxz}^2/10 + 47u_{xzz}^2/10 + 47u_{yyz}^2/10 + 47u_{yzz}^2/10 + 61u_{xyz}^2/48 \\
&+ 7(u_{xy} + 13u_{xxxy}/140 + 13u_{xyyy}/140)^2/6 + 7(u_{yz} + 13u_{yyyz}/140 + 13u_{yzzz}/140)^2/6 \\
&+ 7(u_{xz} + 13u_{xxxz}/140 + 13u_{xzzz}/140)^2/6 + 88841(u_{xxxy}^2 + u_{xyyy}^2)/2100 + (u_{xxxy} - u_{xyyy})^2/16800 \\
&+ 5083u_{xxyy}^2/270 + 88841(u_{yyyz}^2 + u_{yzzz}^2)/2100 + (u_{yyyz} - u_{yzzz})^2/16800 + 5083u_{yyzz}^2/270 \\
&+ 88841(u_{xzzz}^2 + u_{xxxz}^2)/2100 + (u_{xzzz} - u_{xxxz})^2/16800 + 5083u_{xxzz}^2/270 + 10999u_{xxyz}^2/2160 \\
&+ 10999u_{xyyz}^2/2160 + 10999u_{xyzz}^2/2160 \quad .
\end{aligned}
\tag{A.42}
$$

This completes our description of finite volume r=5 WENO reconstruction in 3D.

Supplement B

We now present a strategy for satisfying all the coefficients in eqns. (2.8), (2.9) and (2.10) of Section II in a format that also satisfies all the constraints. We do this in such a way that the person seeking to make a computer implementation simply has to sequentially follow all the steps that are given below.

Let us denote D_y^{x++} and D_z^{x++} to be the facial values of the *y*- and *z*-gradients in the right *x*-face of the right neighbor to the zone that we are considering. Similarly, let D_y^{x--} and D_z^{x--} to be the facial values of the *y*- and *z*-gradients in the left *x*-face of the left neighbor to the zone that we are considering. We can define three possible values of a_{xxy} , denote them by a_{xxy}^1 and a_{xxy}^2 for each of the two smaller stencils and a_{xxy}^3 for the larger stencil. We can write

$$a_{xxy}^1 = \left(D_y^{x++} - 2D_y^{x+} + D_y^{x-}\right)/2 \quad ; \quad a_{xxy}^2 = \left(D_y^{x+} - 2D_y^{x-} + D_y^{x--}\right)/2 \quad ; \quad a_{xxy}^3 = \left(a_{xxy}^1 + a_{xxy}^2\right)/2 \quad . \tag{B.1}$$

We can achieve a CWENO-style non-linear hybridization of the three stencils as follows

$$\begin{gathered} w^1 = 1\Big/\left(\left(a_{xxy}^1\right)^2 + \varepsilon\right) \quad ; \quad w^2 = 1\Big/\left(\left(a_{xxy}^2\right)^2 + \varepsilon\right) \quad ; \quad w^3 = 100\Big/\left(\left(a_{xxy}^3\right)^2 + \varepsilon\right) \quad ; \\ \overline{w}^1 = w^1\Big/\left(w^1 + w^2 + w^3\right) \quad ; \quad \overline{w}^2 = w^2\Big/\left(w^1 + w^2 + w^3\right) \quad ; \quad \overline{w}^3 = w^3\Big/\left(w^1 + w^2 + w^3\right) \quad ; \\ a_{xxy} = \overline{w}^1 a_{xxy}^1 + \overline{w}^2 a_{xxy}^2 + \overline{w}^3 a_{xxy}^3 \quad . \end{gathered} \tag{B.2}$$

Here ε is any very tiny positive number. A similar strategy can be applied to obtain a_{xxz} by using D_z^{x++} , D_z^{x+} , D_z^{x-} and D_z^{x--} . We can also use a similar line of reasoning to obtain b_{xyy} and b_{yyz} ; an analogous line of reasoning gives us c_{xzz} and c_{yzz} . By matching the quadratic cross terms in the faces we also obtain

$$a_{xyz} = D_{yz}^{x+} - D_{yz}^{x-} \quad ; \quad b_{xyz} = D_{xz}^{y+} - D_{xz}^{y-} \quad ; \quad c_{xyz} = D_{xy}^{z+} - D_{xy}^{z-} \quad . \tag{B.3}$$

By matching the fourth order facial values at the right and left *x*-faces we obtain

$$\begin{gathered} a_{yyyz} = \left(D_{yyyz}^{x+} + D_{yyyz}^{x-}\right)/2 \quad ; \quad a_{xyyyz} = D_{yyyz}^{x+} - D_{yyyz}^{x-} \quad ; \\ a_{yzzz} = \left(D_{yzzz}^{x+} + D_{yzzz}^{x-}\right)/2 \quad ; \quad a_{xyzzz} = D_{yzzz}^{x+} - D_{yzzz}^{x-} \quad . \end{gathered} \tag{B.4}$$

Doing the same at the upper and lower y-faces gives

$$b_{xxxz}=\left(D_{xxxz}^{y+}+D_{xxxz}^{y-}\right)/2 \quad ; \quad b_{xxxyz}=D_{xxxz}^{y+}-D_{xxxz}^{y-} \quad ;$$
$$b_{xzzz}=\left(D_{xzzz}^{y+}+D_{xzzz}^{y-}\right)/2 \quad ; \quad b_{xyzzz}=D_{xzzz}^{y+}-D_{xzzz}^{y-} \quad . \tag{B.5}$$

Doing the same at the top and bottom z-faces gives

$$c_{xxxy}=\left(D_{xxxy}^{z+}+D_{xxxy}^{z-}\right)/2 \quad ; \quad c_{xxxyz}=D_{xxxy}^{z+}-D_{xxxy}^{z-} \quad ;$$
$$c_{xyyy}=\left(D_{xyyy}^{z+}+D_{xyyy}^{z-}\right)/2 \quad ; \quad c_{xyyyz}=D_{xyyy}^{z+}-D_{xyyy}^{z-} \quad . \tag{B.6}$$

We now wish to satisfy all the fourth order constraints and some of the second order constraints. This is accomplished by first defining three auxiliary variables, α , β and γ . We obtain

$$\alpha=35\left(q_{xy}-2a_{xxy}-2b_{xyy}-c_{xyz}\right)/12 \quad ; \quad \beta=35\left(q_{yz}-a_{xyz}-2b_{yyz}-2c_{yzz}\right)/12 \quad ;$$
$$\gamma=35\left(q_{xz}-2a_{xxz}-b_{xyz}-2c_{xzz}\right)/12 \quad . \tag{B.7}$$

The above equation also ensures that the second order constraints involving the cross terms are satisfied. The six fourth order constraints are also satisfied in an energy-minimizing fashion by making the following choices

$$a_{xxxxy}=b_{xyyyy}=\alpha \quad ; \quad b_{xxxyy}=-2\alpha-c_{xxxyz}/2 \quad ; \quad a_{xxyyy}=-2\alpha-c_{xyyyz}/2 \quad ;$$
$$b_{yyyyz}=c_{yzzzz}=\beta \quad ; \quad b_{yyzzz}=-2\beta-a_{xyzzz}/2 \quad ; \quad c_{yyyzz}=-2\beta-a_{xyyyz}/2 \quad ; \tag{B.8}$$
$$a_{xxxxz}=c_{xzzzz}=\gamma \quad ; \quad a_{xxzzz}=-2\gamma-b_{xyzzz}/2 \quad ; \quad c_{xxxzz}=-2\gamma-b_{xxxyz}/2 \quad .$$

By matching the third order facial values in the right and left x-faces we obtain

$$a_{yyy}=\left(D_{yyy}^{x+}+D_{yyy}^{x-}\right)/2-a_{xxyyy}/6 \quad ; \quad a_{xyyy}=D_{yyy}^{x+}-D_{yyy}^{x-} \quad ;$$
$$a_{zzz}=\left(D_{zzz}^{x+}+D_{zzz}^{x-}\right)/2-a_{xxzzz}/6 \quad ; \quad a_{xzzz}=D_{zzz}^{x+}-D_{zzz}^{x-} \quad ; \tag{B.9}$$
$$a_{yyz}=\left(D_{yyz}^{x+}+D_{yyz}^{x-}\right)/2 \quad ; \quad a_{xyyz}=D_{yyz}^{x+}-D_{yyz}^{x-} \quad ; \quad a_{yzz}=\left(D_{yzz}^{x+}+D_{yzz}^{x-}\right)/2 \quad ; \quad a_{xyzz}=D_{yzz}^{x+}-D_{yzz}^{x-} \quad .$$

Similarly, matching the third order facial values in the upper and lower y-faces gives

$$b_{xxx}=\left(D_{xxx}^{y+}+D_{xxx}^{y-}\right)/2-b_{xxxyy}/6 \quad ; \quad b_{xxxy}=D_{xxx}^{y+}-D_{xxx}^{y-} \quad ;$$
$$b_{zzz}=\left(D_{zzz}^{y+}+D_{zzz}^{y-}\right)/2-b_{yyzzz}/6 \quad ; \quad b_{yzzz}=D_{zzz}^{y+}-D_{zzz}^{y-} \quad ; \tag{B.10}$$
$$b_{xxz}=\left(D_{xxz}^{y+}+D_{xxz}^{y-}\right)/2 \quad ; \quad b_{xxyz}=D_{xxz}^{y+}-D_{xxz}^{y-} \quad ; \quad b_{xzz}=\left(D_{xzz}^{y+}+D_{xzz}^{y-}\right)/2 \quad ; \quad b_{xyzz}=D_{xzz}^{y+}-D_{xzz}^{y-} \quad .$$

Likewise, matching the third order facial values in the top and bottom z-faces gives

$$c_{xxx} = \left(D_{xxx}^{z+} + D_{xxx}^{z-}\right)/2 - c_{xxxzz}/6 \quad ; \quad c_{xxxz} = D_{xxx}^{z+} - D_{xxx}^{z-} \quad ;$$
$$c_{yyy} = \left(D_{yyy}^{z+} + D_{yyy}^{z-}\right)/2 - c_{yyyzz}/6 \quad ; \quad c_{yyyz} = D_{yyy}^{z+} - D_{yyy}^{z-} \quad ; \tag{B.11}$$
$$c_{xxy} = \left(D_{xxy}^{z+} + D_{xxy}^{z-}\right)/2 \quad ; \quad c_{xxyz} = D_{xxy}^{z+} - D_{xxy}^{z-} \quad ; \quad c_{xyy} = \left(D_{xyy}^{z+} + D_{xyy}^{z-}\right)/2 \quad ; \quad c_{xyyz} = D_{xyy}^{z+} - D_{xyy}^{z-} \quad .$$

Applying the third order constraints while minimizing the energy in the field gives

$$a_{xxxx} = \left(q_{xxx} - b_{xxxy} - c_{xxxz}\right)/4 \quad ; \quad b_{yyyy} = \left(q_{yyy} - a_{xyyy} - c_{yyyz}\right)/4 \quad ; \quad c_{zzzz} = \left(q_{zzz} - a_{xzzz} - b_{yzzz}\right)/4 \quad ;$$
$$b_{xxyy} = 3\left(q_{xxy} - c_{xxyz}\right)/20 \quad ; \quad a_{xxxy} = 14\, b_{xxyy}/9 \quad ; \quad c_{xxzz} = 3\left(q_{xxz} - b_{xxyz}\right)/20 \quad ; \quad a_{xxxz} = 14\, c_{xxzz}/9 \quad ;$$
$$a_{xxyy} = 3\left(q_{xyy} - c_{xyyz}\right)/20 \quad ; \quad b_{xyyy} = 14\, a_{xxyy}/9 \quad ; \quad c_{yyzz} = 3\left(q_{yyz} - a_{xyyz}\right)/20 \quad ; \quad b_{yyyz} = 14\, c_{yyzz}/9 \quad ;$$
$$a_{xxzz} = 3\left(q_{xzz} - b_{xyzz}\right)/20 \quad ; \quad c_{xzzz} = 14\, a_{xxzz}/9 \quad ; \quad b_{yyzz} = 3\left(q_{yzz} - a_{xyzz}\right)/20 \quad ; \quad c_{yzzz} = 14\, b_{yyzz}/9 \quad ;$$
$$a_{xxyz} = b_{xyyz} = c_{xyzz} = q_{xyz}/6 \quad .$$

(B.12)

Matching the second order facial values in the right and left x-faces gives

$$a_{yy} = \left(D_{yy}^{x+} + D_{yy}^{x-}\right)/2 - a_{xxyy}/6 \quad ; \quad a_{xyy} = D_{yy}^{x+} - D_{yy}^{x-} \quad ;$$
$$a_{zz} = \left(D_{zz}^{x+} + D_{zz}^{x-}\right)/2 - a_{xxzz}/6 \quad ; \quad a_{xzz} = D_{zz}^{x+} - D_{zz}^{x-} \quad ; \quad a_{yz} = \left(D_{yz}^{x+} + D_{yz}^{x-}\right)/2 - a_{xxyz}/6 \quad . \tag{B.13}$$

Similarly, matching the second order facial values in the upper and lower y-faces gives

$$b_{xx} = \left(D_{xx}^{y+} + D_{xx}^{y-}\right)/2 - b_{xxyy}/6 \quad ; \quad b_{xxy} = D_{xx}^{y+} - D_{xx}^{y-} \quad ;$$
$$b_{zz} = \left(D_{zz}^{y+} + D_{zz}^{y-}\right)/2 - b_{yyzz}/6 \quad ; \quad b_{yzz} = D_{zz}^{y+} - D_{zz}^{y-} \quad ; \quad b_{xz} = \left(D_{xz}^{y+} + D_{xz}^{y-}\right)/2 - b_{xyyz}/6 \quad . \tag{B.14}$$

Likewise, matching the second order facial values in the top and bottom z-faces gives

$$c_{xx} = \left(D_{xx}^{z+} + D_{xx}^{z-}\right)/2 - c_{xxzz}/6 \quad ; \quad c_{xxz} = D_{xx}^{z+} - D_{xx}^{z-} \quad ;$$
$$c_{yy} = \left(D_{yy}^{z+} + D_{yy}^{z-}\right)/2 - c_{yyzz}/6 \quad ; \quad c_{yyz} = D_{yy}^{z+} - D_{yy}^{z-} \quad ; \quad c_{xy} = \left(D_{xy}^{z+} + D_{xy}^{z-}\right)/2 - c_{xyzz}/6 \quad . \tag{B.15}$$

Some of the second order constraints were resolved in eqn. (B.7). But we still have three more second order constraints which can now be resolved as

$$a_{xxx} = \left(q_{xx} - b_{xxy} - c_{xxz}\right)/3 \quad ; \quad b_{yyy} = \left(q_{yy} - a_{xyy} - c_{yyz}\right)/3 \quad ; \quad c_{zzz} = \left(q_{zz} - a_{xzz} - b_{yzz}\right)/3 \quad . \tag{B.16}$$

Matching the first order facial values at the right and left x-faces gives

$$a_y = \left(D_y^{x+} + D_y^{x-}\right)/2 - a_{xxy}/6 - a_{xxxxy}/70 \quad ; \quad a_{xy} = \left(D_y^{x+} - D_y^{x-}\right) - a_{xxxy}/10 \quad ;$$
$$a_z = \left(D_z^{x+} + D_z^{x-}\right)/2 - a_{xxz}/6 - a_{xxxxz}/70 \quad ; \quad a_{xz} = \left(D_z^{x+} - D_z^{x-}\right) - a_{xxxz}/10 \quad . \tag{B.17}$$

Similarly, matching the first order facial values at the upper and lower *y*-faces gives

$$b_x = \left(D_x^{y+} + D_x^{y-}\right)/2 - b_{xyy}/6 - b_{xyyyy}/70 \quad ; \quad b_{xy} = \left(D_x^{y+} - D_x^{y-}\right) - b_{xyyy}/10 \quad ;$$
$$b_z = \left(D_z^{y+} + D_z^{y-}\right)/2 - b_{yyz}/6 - b_{yyyyz}/70 \quad ; \quad b_{yz} = \left(D_z^{y+} - D_z^{y-}\right) - b_{yyyz}/10 \quad . \tag{B.18}$$

Likewise, matching the first order facial values at the top and bottom *z*-faces gives

$$c_x = \left(D_x^{z+} + D_x^{z-}\right)/2 - c_{xzz}/6 - c_{xzzzz}/70 \quad ; \quad c_{xz} = \left(D_x^{z+} - D_x^{z-}\right) - c_{xzzz}/10 \quad ;$$
$$c_y = \left(D_y^{z+} + D_y^{z-}\right)/2 - c_{yzz}/6 - c_{yzzzz}/70 \quad ; \quad c_{yz} = \left(D_y^{z+} - D_y^{z-}\right) - c_{yzzz}/10 \quad . \tag{B.19}$$

The three first order constraints then give us

$$\begin{aligned}
a_{xx} &= \left(q_x - b_{xy} - c_{xz}\right)/2 - 3\,q_{xxx}/40 - \left(q_{xyy} + q_{xzz}\right)/24 + 3\,a_{xxxx}/14 + \left(a_{xxyy} + a_{xxzz}\right)/12 \\
&\quad + \left(b_{xyzz} + c_{xyyz}\right)/24 + 3\left(b_{xxxy} + b_{xyyy} + c_{xxxz} + c_{xzzz}\right)/40 \quad ; \\
b_{yy} &= \left(q_y - a_{xy} - c_{yz}\right)/2 - 3\,q_{yyy}/40 - \left(q_{xxy} + q_{yzz}\right)/24 + 3\,b_{yyyy}/14 + \left(b_{xxyy} + b_{yyzz}\right)/12 \\
&\quad + \left(a_{xyzz} + c_{xxyz}\right)/24 + 3\left(a_{xxxy} + a_{xyyy} + c_{yyyz} + c_{yzzz}\right)/40 \quad ; \\
c_{zz} &= \left(q_z - a_{xz} - b_{yz}\right)/2 - 3\,q_{zzz}/40 - \left(q_{xxz} + q_{yyz}\right)/24 + 3\,c_{zzzz}/14 + \left(c_{xxzz} + c_{yyzz}\right)/12 \\
&\quad + \left(a_{xyyz} + b_{xxyz}\right)/24 + 3\left(a_{xxxz} + a_{xzzz} + b_{yyyz} + b_{yzzz}\right)/40 \quad .
\end{aligned} \tag{B.20}$$

Matching the zeroth order facial values at the right and left *x*-faces give

$$a_0 = \left(D_0^{x+} + D_0^{x-}\right)/2 - a_{xx}/6 - a_{xxxx}/70 \quad ; \quad a_x = \left(D_0^{x+} - D_0^{x-}\right) - a_{xxx}/10 \quad . \tag{B.21}$$

Similarly, matching the zeroth order facial values at the upper and lower *y*-faces gives

$$b_0 = \left(D_0^{y+} + D_0^{y-}\right)/2 - b_{yy}/6 - b_{yyyy}/70 \quad ; \quad b_y = \left(D_0^{y+} - D_0^{y-}\right) - b_{yyy}/10 \quad . \tag{B.22}$$

Likewise, matching the zeroth order facial values at the top and bottom *z*-faces gives

$$c_0 = \left(D_0^{z+} + D_0^{z-}\right)/2 - c_{zz}/6 - c_{zzzz}/70 \quad ; \quad c_z = \left(D_0^{z+} - D_0^{z-}\right) - c_{zzz}/10 \quad . \tag{B.23}$$

The constant part of the constraint is actually a consistency condition and gives us consistency with the equation given in eqn. (2.5). It is given by

$$
\begin{aligned}
a_x + b_y + c_z - 3\left(a_{xxx} + b_{yyy} + c_{zzz}\right)\Big/20 - \left(a_{xyy} + a_{xzz} + b_{xxy} + b_{yzz} + c_{xxz} + c_{yyz}\right)\Big/12 \\
= q_0 - \left(q_{xx} + q_{yy} + q_{zz}\right)\Big/12 \quad .
\end{aligned}
\tag{B.24}
$$

The equations in this Supplement tell us how to start with the facial moments in eqns. (2.2), (2.3) and (2.4) and obtain the volumetric modes in eqns. (2.8), (2.9) and (2.10).

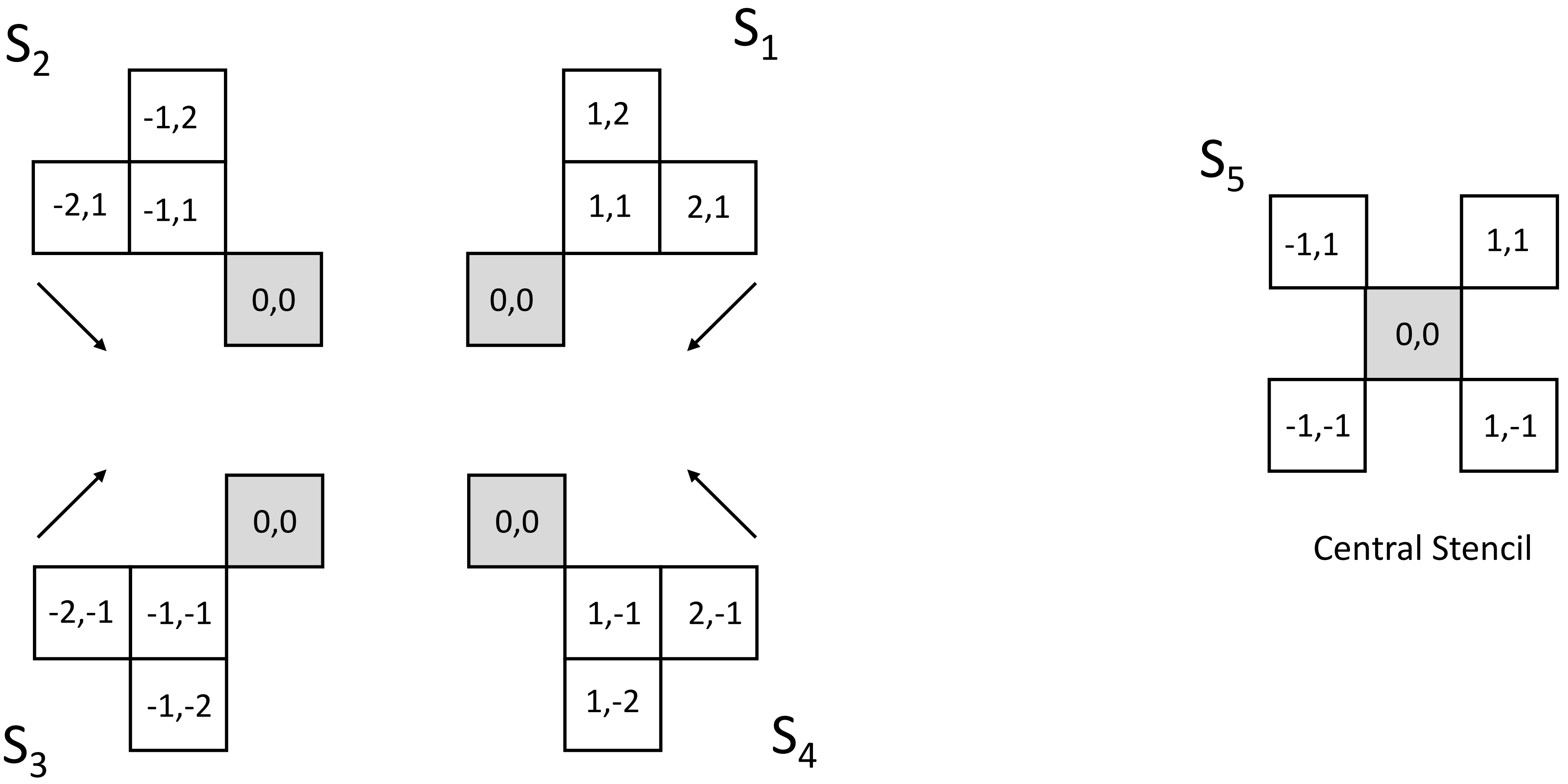
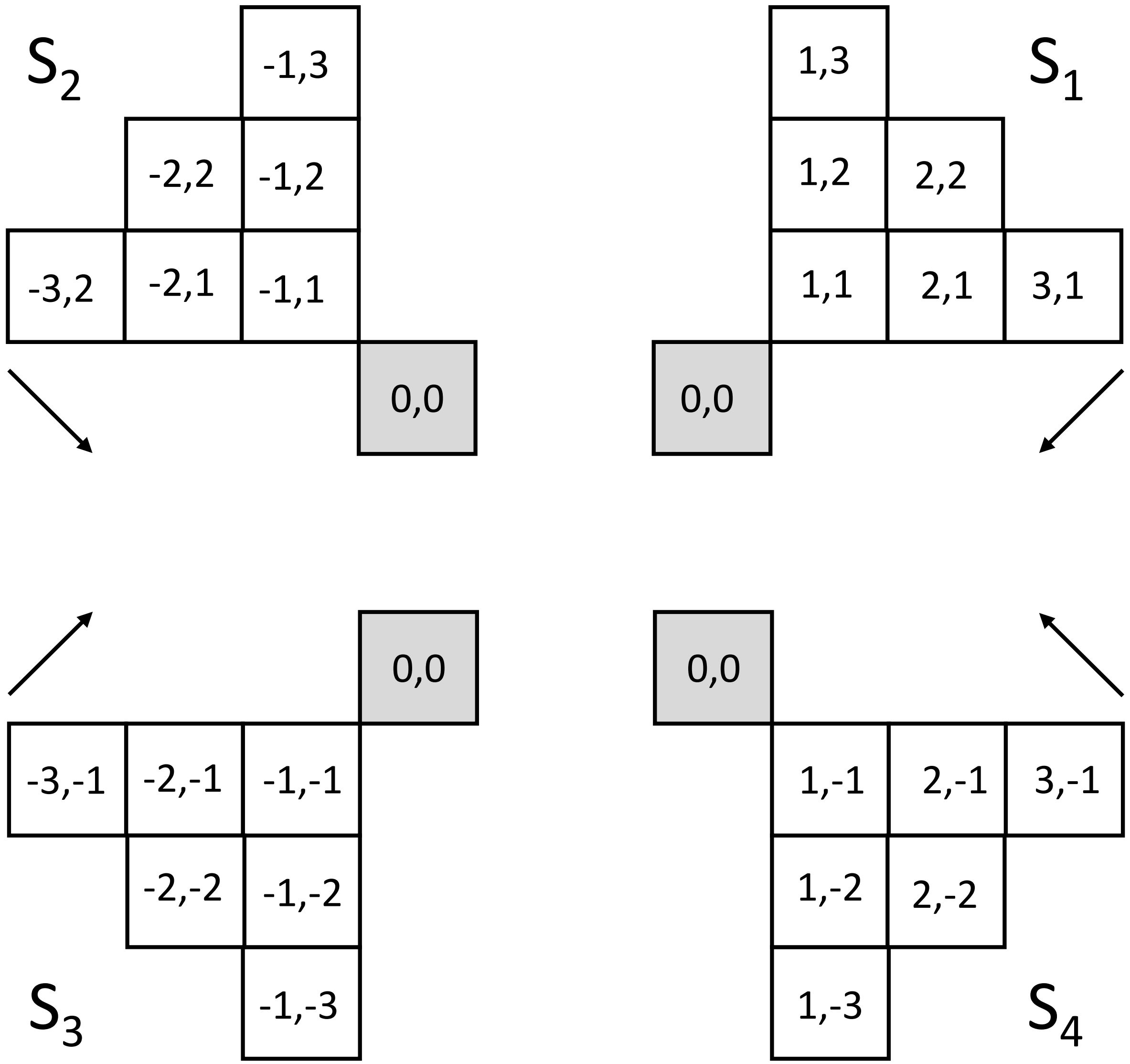
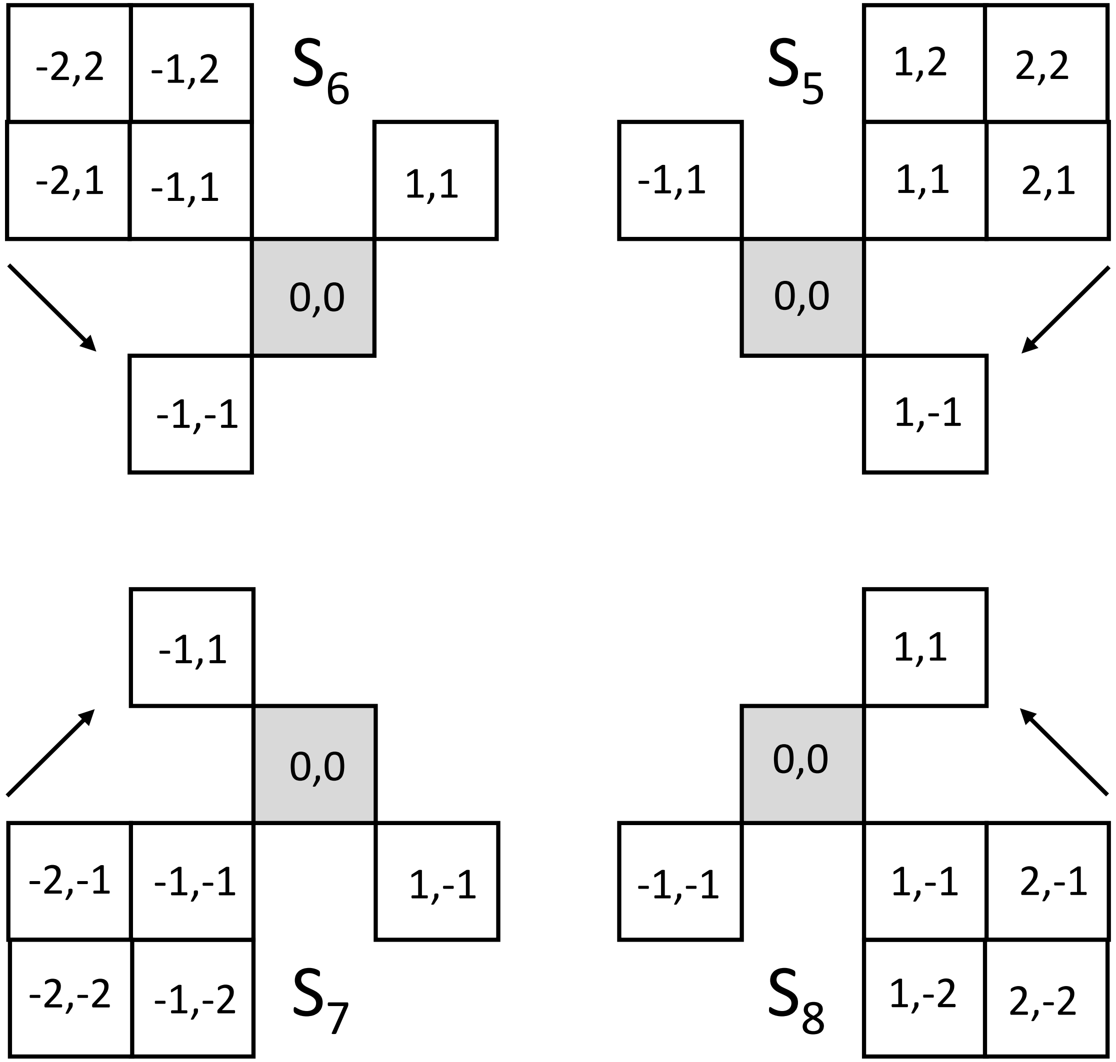


Fig. A.1 shows the four upwinded stencils for obtaining the cross terms in 2D at fourth order. They are labeled S_1 to S_4 and the direction of the wind is also shown with an arrow. The stencil S_5 is a central stencil at fourth order. The zone of interest, where the reconstruction has to be carried out, is shaded and labeled "0,0". The other zones are labeled relative to that zone.



Arrows show upwind direction



-2,2
-1,2
S6
-2,1
-1,1
1,1
0,0
-1,-1
S5
1,2
2,2
-1,1
1,1
2,1
0,0
1,-1
-1,1
0,0
-2,-1
-1,-1
1,-1
-2,-2
-1,-2
S7
1,1
0,0
-1,-1
1,-1
2,-1
S8
1,-2
2,-2

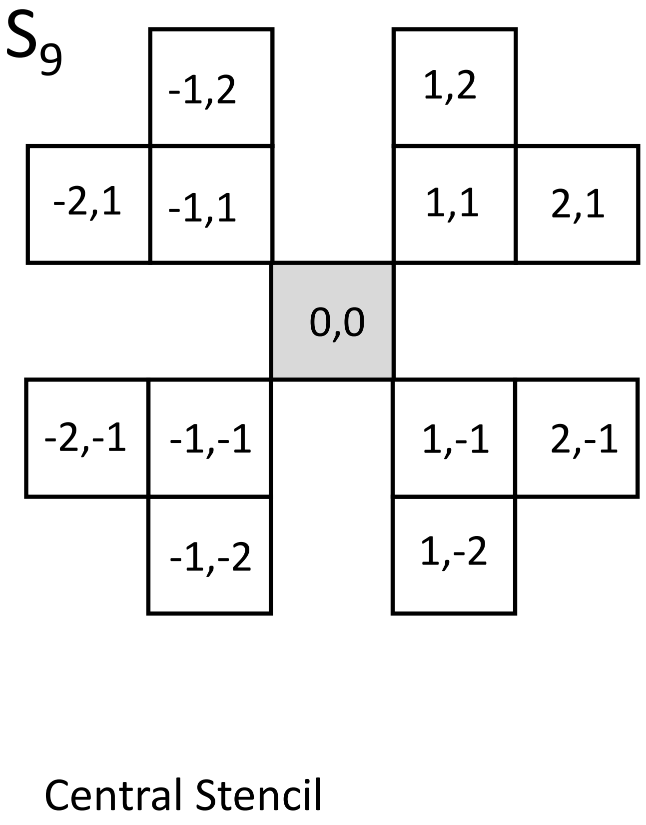


Fig. A.2 shows the eight upwinded stencils for obtaining the cross terms in 2D at fifth order. They are labeled S_1 to S_8 and the direction of the wind is also shown with an arrow. The stencil S_9 is a central stencil at fifth order. The zone of interest, where the reconstruction has to be carried out, is shaded and labeled "0,0". The other zones are labeled relative to that zone.