Online Appendix for: Discretizing Earnings Dynamics: Implications of Gaussian-Mixture Shocks for Life-Cycle Models

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Abstract

This document contains the Online Appendices to the paper 'Discretizing Earnings Dynamics: Implications of Gaussian-Mixture Shocks for Life-Cycle Models'. Appendix A describes the life-cycle model in full, how it is solved, the parameter values used for each of the six calibrations, and how welfare is calculated. Appendix B describes each of the six 'Models' of earnings from Guvenen, Karahan, Ozkan, and Song (2021), and provides the parameter values for these. Appendix C provides details for the discretization of 'Model 4' (with shocks that depend on previous period value), and for the discretization of the non-employment shocks.

Matlab codes implementing the discretization method (and many other discretization methods) can be found at:
https://github.com/vfitoolkit/VFIToolkit-matlab/tree/master/DiscretizationMethods

Matlab codes implementing the models and generating all the results in this paper are available at:
https://github.com/robertdkirkby/DiscretizedEarningsDynamics

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A Life-Cycle Model

As explained in Section 3, the household problem is,

$$
\begin{aligned}
V(a,z,\upsilon,\epsilon,j;i) \quad = \quad & \max_{c,a'} \frac{c^{1-\sigma}}{1-\sigma} + (1-s_j)\beta \mathbb{I}_{(j\geq Jbeq)}\phi(a') \\
& \qquad + s_j \beta E[V(a',z',\upsilon',\epsilon',j+1;i)|z,\upsilon] \\
& \text{if } j < Jr: \; c + a' = (1+r)a + w(1-\upsilon)y(z,\upsilon,\epsilon,i,j) + \theta\upsilon \\
& \text{if } j \geq Jr: \; c + a' = (1+r)a + b \\
& a' \geq 0
\end{aligned}
$$

for $j = 1,\dots,J$ and $i = 1,\dots,I$ which index age and permanent type, respectively. Earnings here is either y or $(1-\upsilon)y$, and the process for (log) $y(z,\upsilon,\epsilon,i,j)$ is as specified in equation (1); in this general setting models without υ (models 1,3,4) are nested as having $\upsilon = 0$. The parameter values are given in below. b is a pension, θ is an income floor (income in non-employment). $\phi(a')$ is a warm-glow of bequest, which is received at once older than age $Jbeq$. We set $\phi(a') = \phi_1 \frac{(1+a'/\phi_2)^{1-\phi_3}}{1-\phi_3}$ as the warm-glow function, following De Nardi (2004). The terminal period is $V(\cdot, J+1) = 0$ (the warm-glow of bequests is caputred seperately by $\phi(a')$). The permanent type $i = 1,\dots,I$ determines α_i, and in model 6 only it also determines $\kappa_{j,\beta,i}$.

The variables are: c is consumption, a is assets, a' is next period assets, j is model age (actual age is $j + ageshifterj$; as reported in table of parameter values), s_j is the age-conditional survival probability, z is exogenous earnings, w is wage, r is the interest rate (rate of return on assets), *pension* is the pension, J is total number of periods, Jr is retirement age, $\phi(a')$ is the warm-glow of bequests. $V(a,z,j)$ is the value function of a household with assets a, earnings z, and age j. Households choose c and a'; a consumption-savings decision.

Note that when solving the models, z is multidimensional, e.g., it is often a markov process on persistent earnings, another markov process on non-employment, and an i.i.d. on transitory earnings.

We solve the model using VFI Toolkit, which involves pure discretization. We solve the value function problem GPU parallelizing over all states except the i.i.d. shocks which we loop over. We then iterate the agent distribution using the Tan improvement. Individual annual earnings are up to \$250,000 (in Model 6) and we use an asset grid from \$0 to \$5,000,000.[1] We use 21 points to discretize z and 9 points to discretize ϵ. The fixed-effect, α_i, (and $\kappa_{\beta,i,j}$) is discretized with 5 points (each, so Model 6 has 25 permanent types of agent). Matlab codes implementing the models and generating all the results in this paper are available at:

[1] The asset grid contains 2501 points and the spacing increases with assets. The maximum space between the two highest grid points is \$6,000. If they were evenly spaced, there would be \$2000 between grid points, as it is the first few points are less than a dollar apart and this spacing gradually increases to the aforementioned max of \$6,000.

https://github.com/robertdkirkby/DiscretizedEarningsDynamics

When performing the welfare analysis in Section 3.4 we calculate welfare as behind-the-veil expected lifetime utility at birth, that is

$$W = E_0[V(\cdot, j=1)] \tag{1}$$

where the expectation is taken with respect to the initial age $j = 1$ distribution of households. The expectation is over z, υ, ϵ in period 1, but also over permanent types α (and HIP $\kappa_{j,\beta,i}$ in model 6).

That covers is how we compute the welfare in a given environment, e.g., in model 2, or in model 2 with no shocks. The next step is to compute the consumption-equivalent variation (CEV) between two such environments, between environment A, e.g., model 2 without idiosyncratic shocks, and environment B, e.g., model 2. Let W_B be the welfare in environment B, calculated by equation (1). The CEV is calculated as the percentage increase in consumption in environment A that makes behind-the-veil welfare equal to welfare in environment B, that is we are looking for CEV that solves $W_A(CEV) = W_B$.

This just leaves us to explain how to calculate $W_A(CEV)$; the behind-the-veil welfare in environment A when consumption is increased by CEV percent. First ignore CEV for a moment (when it is zero), and for the model associated with environment A we are solving,

$$\begin{aligned}
V(a, z, \upsilon, \epsilon, j; i) \quad = \quad & \max_{c,a'} \frac{c^{1-\sigma}}{1-\sigma} + (1-s_j)\beta \mathbb{I}_{(j \geq Jbeq)}\phi(a') \\
& \qquad + s_j \beta E[V(a', z', \upsilon', \epsilon', j+1; i)|z, \upsilon] \\
& \text{if } j < Jr: \; c + a' = (1+r)a + w(1-\upsilon)y(z, \upsilon, \epsilon, i, j) + \theta\upsilon \\
& \text{if } j \geq Jr: \; c + a' = (1+r)a + b \\
& a' \geq 0
\end{aligned}$$

to get the (optimal) policy function $Policy_{c,A}$ and the value function V_A (this V_A is what we would use to calculate W_A in equation (1) if we were interested in it). Note that once we have the policy function, $Policy_{c,A}$, the value function V_A can be calculated by iteratively using the formula,

$$V_A(a, z, j) = \frac{Policy_{c,A}^{1-\sigma}}{1-\sigma} + (1-s_j)\beta \mathbb{I}_{(j \geq Jbeq)}\phi(a'(Policy_{c,A})) + s_j \beta E[V_A(a'(Policy_{c,A}), j+1)|z]$$

We are interested in the value function associated with this consumption policy, but consumption being increased by CEV percent, call it $V_{A,CEV}$. $V_{A,CEV}$ is calculated from CEV and $Policy_{c,A}$ by iteratively using the formula,

$$V_A(a, z, j) = \frac{((1+CEV) * Policy_{c,A})^{1-\sigma}}{1-\sigma} + (1-s_j)\beta \mathbb{I}_{(j \geq Jbeq)}\phi(a'(Policy_{c,A})) + s_j \beta E[V_A(a'(Policy_{c,A}), j+1)|z]$$

Once we have $V_{A,CEV}$, then we calculate $W_A(CEV) = E_0[V_{A,CEV}(\cdot, j=1)]$; that is we just apply the usual welfare function, equation (1) to $V_{A,CEV}$. To find the consumption-equivalent variation we are looking for CEV that satisfies $W_A(CEV) = W_B$ (the percentage increase in consumption in every state that gives us the same welfare in environment A as we get in environment B).

When solving for the welfare CEV, we try values of CEV from 0 to 300, in increments of 1 (percentage point) and calculate $W_A(CEV)$, we then report as our result the value (out of these 301) of CEV that gives the closest value of $W_A(CEV)$ to the value of W_B.

The parameters of the life-cycle model differ across the six models (they are all calibrated to the same targets) and are given in Tables A, except the age-conditional survival probabilities which are provided in Table A.

Parameters of the Life-Cycle Model: Model 1		
Total periods	J	76
Retirement period	Jr	42
Leave bequests after age	$Jbeq$	56
Age in years at j=1		25
Preferences		
Discount factor	β	1.06
CRRA	σ	2.00
Prices and policies		
Wage	w	4272.19
Interest rate	r	0.05
Pension	b	15400.00
Income floor	θ	6400.00
Warm glow of bequests		
relative importance of bequests	ϕ_1	415.3
bequests as a luxury good	ϕ_2	0.0
curvature for bequests	ϕ_3	2.0

Parameters of the Life-Cycle Model: Model 2		
Preferences		
Discount factor	β	1.01
CRRA	σ	2.00
Prices and policies		
Wage	w	939.06
Interest rate	r	0.05
Pension	b	15400.00
Income floor	θ	6400.00
Warm glow of bequests		
relative importance of bequests	ϕ_1	326.0
bequests as a luxury good	ϕ_2	0.0
curvature for bequests	ϕ_3	2.0

Parameters of the Life-Cycle Model: Model 3		
Preferences		
Discount factor	β	0.99
CRRA	σ	2.00
Prices and policies		
Wage	w	2588.19
Interest rate	r	0.05
Pension	b	15400.00
Income floor	θ	6400.00
Warm glow of bequests		
relative importance of bequests	ϕ_1	51.7
bequests as a luxury good	ϕ_2	1.1
curvature for bequests	ϕ_3	2.0

Parameters of the Life-Cycle Model: Model 4		
Preferences		
Discount factor	β	1.91
CRRA	σ	2.00
Prices and policies		
Wage	w	3400.29
Interest rate	r	0.05
Pension	b	15400.00
Income floor	θ	6400.00
Warm glow of bequests		
relative importance of bequests	ϕ_1	0.0
bequests as a luxury good	ϕ_2	0.0
curvature for bequests	ϕ_3	2.0

Parameters of the Life-Cycle Model: Model 5		
Preferences		
Discount factor	β	0.98
CRRA	σ	2.00
Prices and policies		
Wage	w	731.03
Interest rate	r	0.05
Pension	b	15400.00
Income floor	θ	6400.00
Warm glow of bequests		
relative importance of bequests	ϕ_1	4.0
bequests as a luxury good	ϕ_2	6.8
curvature for bequests	ϕ_3	2.0

Parameters of the Life-Cycle Model: Model 6		
Preferences		
Discount factor	β	1.00
CRRA	σ	2.00
Prices and policies		
Wage	w	793.71
Interest rate	r	0.05
Pension	b	15400.00
Income floor	θ	6400.00
Warm glow of bequests		
relative importance of bequests	ϕ_1	8500.0
bequests as a luxury good	ϕ_2	0.0
curvature for bequests	ϕ_3	2.0

Age-Conditional Survival/Death Probabilities for the Life-Cycle Model

Model age	j	1	2	3	4	5	6	7	8	9	10
Age in Years		25	26	27	28	29	30	31	32	33	34
Death	d_j	0.000983	0.000967	0.000960	0.000970	0.000994	0.001027	0.001065	0.001115	0.001154	0.001209
Survival	$s_j = 1 - d_j$	0.999017	0.999033	0.999040	0.999030	0.999006	0.998973	0.998935	0.998885	0.998846	0.998791
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.999017	0.998051	0.997093	0.996126	0.995135	0.994113	0.993055	0.991948	0.990803	0.989605
Model age	j	11	12	13	14	15	16	17	18	19	20
Age in Years		35	36	37	38	39	40	41	42	43	44
Death	d_j	0.001271	0.001351	0.001460	0.001603	0.001769	0.001943	0.002120	0.002311	0.002520	0.002747
Survival	$s_j = 1 - d_j$	0.998729	0.998649	0.998540	0.998397	0.998231	0.998057	0.997880	0.997689	0.997480	0.997253
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.988347	0.987012	0.985571	0.983991	0.982250	0.980342	0.978263	0.976003	0.973543	0.970869
Model age	j	21	22	23	24	25	26	27	28	29	30
Age in Years		45	46	47	48	49	50	51	52	53	54
Death	d_j	0.002989	0.003242	0.003512	0.003803	0.004118	0.004464	0.004837	0.005217	0.005591	0.005963
Survival	$s_j = 1 - d_j$	0.997011	0.996758	0.996488	0.996197	0.995882	0.995536	0.995163	0.994783	0.994409	0.994037
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.967967	0.964829	0.961440	0.957784	0.953840	0.949582	0.944989	0.940059	0.934803	0.929229
Model age	j	31	32	33	34	35	36	37	38	39	40
Age in Years		55	56	57	58	59	60	61	62	63	64
Death	d_j	0.006346	0.006768	0.007261	0.007866	0.008596	0.009473	0.010450	0.011456	0.012407	0.013320
Survival	$s_j = 1 - d_j$	0.993654	0.993232	0.992739	0.992134	0.991404	0.990527	0.989550	0.988544	0.987593	0.986680
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.923332	0.917083	0.910424	0.903262	0.895498	0.887015	0.877745	0.867690	0.856925	0.845510
Model age	j	41	42	43	44	45	46	47	48	49	50
Age in Years		65	66	67	68	69	70	71	72	73	74
Death	d_j	0.014299	0.015323	0.016558	0.018029	0.019723	0.021607	0.023723	0.026143	0.028892	0.031988
Survival	$s_j = 1 - d_j$	0.985701	0.984677	0.983442	0.981971	0.980277	0.978393	0.976277	0.973857	0.971108	0.968012
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.833420	0.820650	0.807062	0.792511	0.776880	0.760094	0.742063	0.722663	0.701784	0.679335
Model age	j	51	52	53	54	55	56	57	58	59	60
Age in Years		75	76	77	78	79	80	81	82	83	84
Death	d_j	0.035476	0.039238	0.043382	0.047941	0.052953	0.058457	0.064494	0.071107	0.078342	0.086244
Survival	$s_j = 1 - d_j$	0.964524	0.960762	0.956618	0.952059	0.947047	0.941543	0.935506	0.928893	0.921658	0.913756
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.655235	0.629525	0.602215	0.573344	0.542984	0.511243	0.478270	0.444262	0.409458	0.374144
Model age	j	61	62	63	64	65	66	67	68	69	70
Age in Years		85	86	87	88	89	90	91	92	93	94
Death	d_j	0.094861	0.104242	0.114432	0.125479	0.137427	0.150317	0.164187	0.179066	0.194979	0.211941
Survival	$s_j = 1 - d_j$	0.905139	0.895758	0.885568	0.874521	0.862573	0.849683	0.835813	0.820934	0.805021	0.788059
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.338653	0.303351	0.268638	0.234929	0.202644	0.172183	0.143913	0.118143	0.095107	0.074950
Model age	j	71	72	73	74	75	76				
Age in Years		95	96	97	98	99	100				
Death	d_j	0.229957	0.249020	0.269112	0.290198	0.312231	1.000000				
Survival	$s_j = 1 - d_j$	0.770043	0.750980	0.730888	0.709802	0.687769	0.000000				
Cumulative survival	$\Pi_{j=1}^{J} s_j$	0.057715	0.043343	0.031679	0.022486	0.015465	0.000000				

Source: s_j is the probability of surviving to be age $j+1$, given alive at age j. Age-conditional death probabilites for the US are taken from "National Vital Statistics Report, volume 58, number 10, March 2010.", first column (qx) of Table 1 (Total Population). Conditional survival probabilities are calculated as then $s_j = 1 - d_j$.

B Six processes for Earnings Dynamics

The six earnings dynamics processes we consider are Models 1, 2, 3, 4, 5 & 6 of Guvenen, Karahan, Ozkan, and Song (2021). This appendix describes all six models in full. Whenever gaussian-mixture distributions are used the mixture is of two normal distributions.

We can think of the main differences between the models as: Model 1 is a life-cycle AR(1) with gaussian innovations. Model 2 is a standard life-cycle AR(1) with non-employment shocks. Model 3 is a life-cycle AR(1) with gaussian-mixture innovations. Model 5 is a life-cycle AR(1) with gaussian-mixture AR(1) with gaussian-mixture innovations and with non-employment shocks. Model 6 is a life-cycle AR(1) with gaussian-mixture AR(1) with gaussian-mixture innovations and with non-employment shocks, which also has heterogenous income profiles. Model 4 is slightly more complex, it is Model 3, but where the parameters of the AR(1) are also allowed to depend on both age and income (that is, on the period and on the lagged value of the shock itself).

B.1 GKOS Earnings Dynamics Model 1: Gaussian (without non-employment shock)

Model 1 of Guvenen, Karahan, Ozkan, and Song (2021) models (log) earnings as,

$$
\begin{aligned}
ln(y_j) &= \kappa_j + \alpha_i + z_j + \epsilon_j \\
\kappa_j &= \kappa_0 + \kappa_1(j/10) + \kappa_2(j/10)^2 \\
z_j &= \rho z_{j-1} + \eta_j, \quad \eta_j \sim N(0, \sigma_\eta) \\
z_0 &\sim N(0, \sigma_{z,0}) \\
\epsilon_j &\sim N(0, \sigma_\epsilon) \\
\alpha_i &\sim N(0, \sigma_\alpha)
\end{aligned}
$$

where κ_j is a deterministic life-cycle profile (modelled as a quadratic polynomial), α_i is a fixed-effect, z_j is a persistent AR(1) with with gaussian innovations, ϵ_j is a transitory iid gaussian distribution. This model does not contain a non-employment shock.

B.2 GKOS Earnings Dynamics Model 2: Gaussian

Model 2 of Guvenen, Karahan, Ozkan, and Song (2021) models earnings as $(1 - \upsilon_j)y_j$, where υ_j is a non-employment shock; see Section C.2 for explanation of non-employment shocks. It models (log) earnings conditional on employment as,

$$\begin{aligned}
ln(y_j) &= \kappa_j + \alpha_i + z_j + \epsilon_j \\
\kappa_j &= \kappa_0 + \kappa_1(j/10) + \kappa_2(j/10)^2 \\
z_j &= \rho z_{j-1} + \eta_j, \quad \eta_j \sim N(0, \sigma_\eta) \\
z_0 &\sim N(0, \sigma_{z,0}) \\
\epsilon_j &\sim N(0, \sigma_\epsilon) \\
\alpha_i &\sim N(0, \sigma_\alpha)
\end{aligned}$$

where κ_j is a deterministic life-cycle profile (modelled as a quadratic polynomial), α_i is a fixed-effect, z_j is a persistent AR(1) with with gaussian innovations, ϵ_j is a transitory iid gaussian distribution.

B.3 GKOS Earnings Dynamics Model 3: Gaussian-Mixtures (without non-employment shock)

Model 3 of Guvenen, Karahan, Ozkan, and Song (2021) models (log) earnings as,

$$\begin{aligned}
ln(y_j) &= \kappa_j + \alpha_i + z_j + \epsilon_j \\
\kappa_j &= \kappa_0 + \kappa_1(j/10) + \kappa_2(j/10)^2 \\
z_j &= \rho z_{j-1} + \eta_j, \quad \eta_j \sim F_\eta \\
z_0 &\sim N(0, \sigma_{z,0}) \\
\epsilon_j &\sim F_\epsilon \\
\alpha_i &\sim N(0, \sigma_\alpha)
\end{aligned}$$

where the important differences from Model 1 are that both the innovations to the the persistent shocks, namely η_j, and the transitory shocks themselves, ϵ_j, are modelled as gaussian-mixtures rather than normal distributions. Both are modelled as a mixture of two normal distributions. This model does not contain a non-employment shock.

B.4 GKOS Earnings Dynamics Model 4: Gaussian-Mixture depending on age and lagged value (without non-employment shock)

Model 4 of Guvenen, Karahan, Ozkan, and Song (2021) models (log) earnings as,

$$\begin{aligned}
ln(y_j) &= \kappa_j + \alpha_i + z_j + \epsilon_j \\
\kappa_j &= \kappa_0 + \kappa_1(j/10) + \kappa_2(j/10)^2 \\
z_j &= \rho_j z_{j-1} + \eta_j, \quad \eta_j \sim F_{\eta,j,z_{j-1}} \\
z_0 &\sim N(0, \sigma_{z,0}) \\
\epsilon_j &\sim F_{\epsilon,j} \\
\alpha_i &\sim N(0, \sigma_\alpha)
\end{aligned}$$

Like Model 3 both ϵ_j and the innovations to z_j are gaussian-mixtures, and this model does not have non-employment shocks. The difference is that the gaussian-mixture distribution of the innovations to z_j, namely $F_{\eta,j,z_{j-1}}$ has parameters that depend on both j and the lag of the AR(1) itself, z_{j-1}.

B.5 GKOS Earnings Dynamics Model 5: Gaussian-Mixtures

Model 5 of Guvenen, Karahan, Ozkan, and Song (2021) models earnings as $(1-\upsilon_j)y_j$, where υ_j is a non-employment shock; see Section C.2 for explanation of non-employment shocks. It models (log) earnings conditional on employment as,

$$\begin{aligned}
ln(y_j) &= \kappa_j + \alpha_i + z_j + \epsilon_j \\
\kappa_j &= \kappa_0 + \kappa_1(j/10) + \kappa_2(j/10)^2 \\
z_j &= \rho z_{j-1} + \eta_j, \quad \eta_j \sim F_\eta \\
z_0 &\sim N(0, \sigma_{z,0}) \\
\epsilon_j &\sim F_\epsilon \\
\alpha_i &\sim N(0, \sigma_\alpha)
\end{aligned}$$

where the important differences from Model 2 are that both the innovations to the the persistent shocks, namely η_j, and the transitory shocks themselves, ϵ_j, are modelled as gaussian-mixtures rather than normal distributions. Both are modelled as a mixture of two normal distributions.

B.6 GKOS Earnings Dynamics Model 6: Gaussian-Mixtures and Heterogenous Income Profiles

Model 6 of Guvenen, Karahan, Ozkan, and Song (2021) models earnings as $(1-\upsilon_j)y_j$, where υ_j is a non-employment shock; see Section C.2 for explanation of non-employment shocks. It models (log) earnings conditional on employment as,

$$\begin{aligned}
ln(y_j) &= (\kappa_j + \kappa_{\beta,i,j}) + \alpha_i + z_j + \epsilon_j \\
\kappa_j &= \kappa_0 + \kappa_1(j/10) + \kappa_2(j/10)^2 \\
\kappa_{\beta,i,j} &= \kappa_{\beta,i}(j/10) \\
z_j &= \rho z_{j-1} + \eta_j, \quad \eta_j \sim F_\eta \\
z_0 &\sim N(0, \sigma_{z,0}) \\
\epsilon_j &\sim F_\epsilon \\
(\alpha_i, \kappa_{\beta,i}) &\sim N(0, \Sigma_{\alpha,\beta})
\end{aligned}$$

where the important difference from Model 5 is the inclusion of heterogenous income profiles, the $\kappa_{\beta,i,j}$ terms, which act as a permanent type that modify the slope (linear) term of κ_j (they leave the constant and quadratic terms of κ_j unaffected).[2] Notice also that the two permanent types, α_i and $\kappa_{\beta,i,j}$, are *not* independent; they are modelled as joint normally distributed. As in model 5 both the innovations to the the persistent shocks, namely η_j, and the transitory shocks themselves, ϵ_j, are modelled as gaussian-mixtures rather than normal distributions.

B.7 Parameters for Earnings Dynamics

We report the parameters of the earnings dynamics processes in Table B.7. Note that this is essentially just a copy-paste of numbers from Guvenen, Karahan, Ozkan, and Song (2021) and is done purely for convenience/clarity.

G indicates gaussian innovations, mix indicates guassian-mixture innovations. HIP=heterogenous income profiles. Note that in the model, $j = 1$ corresponds to age 25. Note that the deterministic life-cycle uses $j/10$, rather than j. Guvenen, Karahan, Ozkan, and Song (2021) also report the standard errors of the parameters for Model 6: because they have a few hundred million observations the standard errors are two to three orders of magnitude smaller than the point estimates as are unimportant for our present purposes. *: -1 was the lower bound on $\mu_{\eta,1}$ during estimation, and point estimate is up against this bound.

$mixprob_{\eta,1}$ is the probability of the first of the two mixture distributions. The other is given by $mixprob_{\eta,2} = 1 - mixprob_{\eta,1}$. The mean of the second of the mixture distributions is determined as the mean of the mixture is zero (by assumption/definition), so $\mu_{\eta,2} = -(mixprob_{\eta,1}\mu_{\eta,1})/mixprob_{\eta_2}$ (simple rearrangement of sum(weights*means)=0). Same applies to mixture probabilites and means for ϵ.

[2] The awkward notation of $\kappa_{\beta,i,j}$ is because I wish to use β for the discount factor in the life-cycle models, but also want to not depart too far from Guvenen, Karahan, Ozkan, and Song (2021) who call it β_i.

Parameters for Earnings Processes

Model	1	2	3	4	5	6
Functional forms						
AR(1) component	G	G	mix	mix	mix	mix
$\hookrightarrow$ Probability age/inc.	-	-	no/no	yes/yes	no/no	no/no
Non-employment shocks	no	yes	no	no	yes	yes
$\hookrightarrow$ Probability age/inc.	-	yes/yes	-	-	yes/yes	yes/yes
Transitory shocks	G	G	mix	mix	mix	mix
HIP	no	no	no	no	no	yes
Parameters						
ρ	1.005	0.967	1.010	0.992	0.991	0.959
$mixprob_{\eta,1}$			0.05	(below)	0.176	0.407
$\mu_{\eta,1}$			-1.0*	-1.0*	-0.0524	-0.085
$\sigma_{\eta,1}$	0.134	0.197	1.421	1.070	0.113	0.364
$\sigma_{\eta,2}$			0.010	0.032	0.046	0.069
$\sigma_{z,0}$	0.343	0.563	0.213	0.446	0.450	0.714
λ		0.030			0.016	0.0001
$mixprob_{\epsilon}$			0.118	0.088	0.044	0.130
$\mu_{\epsilon,1}$			-0.826	0.311	0.134	0.271
$\sigma_{\epsilon,1}$	0.696	0.163	1.549	0.795	0.762	0.285
$\sigma_{\epsilon,2}$			0.020	0.020	0.055	0.037
σ_{α}	1.182	0.655	0.273	0.473	0.472	0.300
σ_{β}						0.196
$corr_{\alpha\beta}$						0.768
Deterministic Life-Cycle Parameters (κ_j)						
$a_0 \times x1$	0.740	2.569	2.547	2.176	2.746	2.581
$a_1 \times xj/10$	0.337	0.766	-0.144	0.169	0.624	0.812
$a_2 \times x(j/10)^2$	0.070	-0.152	-0.059	-0.100	-0.167	-0.185
Nonemployment Shock Probability Function Parameters (υ_j)						
$a_{\upsilon} \times 1$		-3.036			-2.495	-3.353
$b_{\upsilon} \times j$		-0.917			-1.037	-0.859
$c_{\upsilon} \times x_j$		-5.397			-5.051	-5.034
$d_{\upsilon} \times j \times x_j$		-4.442			-1.087	-2.895
Normal Mixture Probability Function Parameters (η_j)						
$a_{z_1} \times 1$			0.05	-0.474	0.176	0.407
$b_{z_1} \times j$				1.961		
$c_{z_1} \times x_{j-1}$				-3.183		
$d_{z_1} \times j \times x_{j-1}$				-0.187		

C Discretization

This section just covers two further details relating the to discretization of the earnings processes. They are minor relative to the content of the main body of the paper and are included here for completeness.

C.1 Parameters depend on lagged-value

We now turn to discretizing z_j in Model 4, which is the same as in Section 2 except that now the gaussian-mixture innovation parameters are allowed to depend on the lagged value of z_j itself; actually we will only allow the mixture probabilities to depend on the lagged value of z_j, the means and standard deviations of the normal distributions being mixed can only depend on j. To be precise, we will discretize z_j given by,

$$
\begin{aligned}
& z_j = \rho z_{j-1} + \eta_j, \quad \eta_j \sim F_{\eta,j,z_{j-1}} \\
& F_{\eta,j,z_{j-1}} = \sum_{i=1}^{q} w_{i,z_{j-1}} N(\mu_i, \sigma^2_{\eta,i})
\end{aligned}
\tag{2}
$$

This creates a challenge for our previous approach as we do not know the standard deviation of $F_{\eta,j,z_{j-1}}$ prior to creating the quadrature grid as it depends on z_{j-1}. Our approach is simply to evaluate the standard deviation given $z_{j-1} = 0$, which we know to be relevant as by construction z_j is zero mean (it is an AR(1) with zero-mean innovations and no constant term). We then construct the grid based on this in the same was as before.[3]

C.2 Non-employment shocks

The actual processes we discretize following Guvenen, Karahan, Ozkan, and Song (2021) also include a *non-employment* shock, υ_j, which takes a value of 1 for non-employment and 0 for employment. Actual earnings are then $(1 - \upsilon_j)y_j$, where $\ln(y_j)$ follows an earnings process.[4] This shock is important to capture that younger people often have some periods of zero earnings; Kaplan (2012) uses a similar shock in a model with endogenous labor and justifies it as a reduced-form way to capture the concepts of search-matching labor market models.

[3] We originally used an iterative approach, in which this was the first step, and we then did a second step: Based on the quadrature grids for each age we can recalculate the parameters using values for z_{j-1} at each (lagged) grid point, then the calculate the standard deviation at each grid point, and then take the maximum of these standard deviations and use that to recreate the grid. But this second step did not perform well in practice and just the single step seems to work fine.

[4] Guvenen, Karahan, Ozkan, and Song (2021) actually allow a slightly more general process during estimation, namely that υ_j takes the values 0 and $min(1, exp(\lambda))$, but since their estimates of λ are always greater than zero this simplifies to 0 and 1 for the actual estimated processes.

The probability that υ_j equals 1 is given by $p_\upsilon(j, z_j)$; so $\upsilon_j = 0$ with probability $1 - p_\upsilon(j, z_j)$. This probability $p_\upsilon(j, z_j)$ depends both on age, and on the current value of the persistent shock z_j. Guvenen, Karahan, Ozkan, and Song (2021) model as $p_\upsilon(j, z_j) = \frac{exp(\xi_{j,z_j})}{1+exp(\xi_{j,z_j})}$, where $\xi_{j,z_j} = \xi_0 + \xi_1 j + \xi_2 z_j + \xi_3 j z_j$.

After z_j has been discretized this is simply a matter of creating a Markov process on υ_j that takes the values 0 and 1 and with the transition probabilities depending on j and z_j. Creating this as a joint Markov on (z_j, υ_j) is 'trivial' once z_j has already been created. 'Trivial' is of course conditional on a solid understanding of discrete Markov processes, but we will not attempt to explain it here. Interested readers can see the codes[5] where it occupies just a few lines as a for-loop over the states.

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[5]Specifically it is in 'DiscretizeEarningsDynamics.m'