A Conditional Probabilities

We have to show that, $\lim_{z^* \to -\infty} P(A(\theta, z^*) - \theta > 0|z^*) = 0$ and $\lim_{z^* \to \infty} P(A(\theta, z^*) - \theta > 0|z^*) = 1$. To do so, we recall that $z_i = A - \theta + \sigma_z \epsilon_i$ with $\epsilon_i \sim \mathcal{N}(0, 1)$, and examine the conditional probability:

$$P(A(\theta, z^*) - \theta > 0|z^*) = c. \tag{20}$$

We begin by defining $y(\theta, z^*) := A(\theta, z^*) - \theta$, and we recall Bayes's formula:

$$f(y|z^*) = \frac{h(z^*|y)f(y)}{\int_{-\infty}^{\infty} h(z^*|y)f(y)dy}, \tag{21}$$

Where $h(z^*|y)$ is a normal distribution. Moreover, we note that:

$$f(y(\theta)) = g(\theta(y))\frac{d\theta}{dy}. \tag{22}$$

Where $\frac{d\theta}{dy} = \frac{1}{A_\theta(\theta, z^*) - 1}$. Recalling (11), $A = 1 - \Phi(\sqrt{\alpha_z}(z^* - A + \theta))$, we have $A_\theta(\theta, z^*) = \frac{-\sqrt{\alpha_z}\phi(\sqrt{\alpha_z}(z^* - A + \theta))}{1 - \sqrt{\alpha_z}\phi(\sqrt{\alpha_z}(z^* - A + \theta))} < 0$, for $A_j, j = 1, 3$. Agents hold a uniform uninformative prior over θ, such that $g(\theta)$ is a constant $\bar{g}$. Moreover, we have $\lim_{z^* \to \infty} A_\theta = 0$ and thus $\lim_{z^* \to \infty} \frac{d\theta}{dy} = -1$ and $f(y) = -1\bar{g}$. Substituting into (21) yields:

$$\lim_{z^* \to \infty} f(y|z^*) = \lim_{z^* \to \infty} \frac{h(z^*|y)}{\int_{-\infty}^{\infty} h(z^*|y)dy}, \tag{23}$$

where $h(z^*|y) = \phi(\sqrt{\alpha_z}(z^* - y))$. Finally, we have:

$$\lim_{z^* \to \infty} P(y > 0|z^*) = \lim_{z^* \to \infty} \frac{\int_0^{\infty} h(z^*|y)dy}{\int_{-\infty}^{\infty} h(z^*|y)dy} = \lim_{z^* \to \infty} \Phi(\alpha_z z^*) = 1. \tag{24}$$

The same argument can be made to show that $\lim_{z^* \to -\infty} P(y > 0|z^*) = 0$.

B Proof of Proposition 4

The proof involves two steps:

1. In Section B.1, we prove Lemma 2, which helps us to collect signal combinations for which agents do/don't attack. In turn, we compute bounds for the mass of attacking agents.

2. In Section B.2 we start with a guess, i.e. equation (28), of agents' aggregate attack. In turn, we use Lemma 2, together with the theorem of Arzelà–Ascoli, and show that there exists a continuum of equilibria $A_t(\theta)$. Regarding notation $A_t^n(t^-)$ and $A_t^n(t^+)$ denote left and right limits of the function $A_t(\theta)$ shown in Figure 7. Moreover, index n accounts for the n-th iteration over the agents' action A_t, and will be used in Section B.2.

B.1 Signals, Cutoffs, and the Mass of Attacking Agents

Lemma 2. *Suppose that for a given t $A(\theta) \in [1-\delta, 1]$ if $\theta < t$, and $A(\theta) \in [0, \delta]$ otherwise. Then the following holds:*

1. *If $y_i \geq 1-\delta-\gamma$ and $x_i \leq t+\xi$, then $P[\theta < t | A(\cdot), (x_i, y_i)] \geq c$.*
2. *If $y_i \leq \delta+\gamma$ and $x \geq t-\xi$, then $P[\theta \geq t | A(\cdot), (x_i, y_i)] \geq c$.*
3. *$P[y_i \geq 1-\delta-\gamma \cap x \leq t+\xi | A(\cdot), \theta < t] \geq 1-\delta$*
4. *$P[y_i \leq \delta+\gamma \cap x > t-\xi | A(\cdot), \theta \geq t] \geq 1-\delta$*

Proof. 1. To prove the first statement, we define

$$\kappa_N := \frac{\int_t^{t+N} f_x(x_i-\theta) f_y(y_i - A(\theta))\, d\theta}{\int_{t-N}^{t} f_x(x_i-\theta) f_y(y_i - A(\theta))\, d\theta},$$

and show that the first two conditions can be reduced to statements about κ_N. We denote by $\mathcal{U}_N^t$ the uniform distribution of θ over the interval $[t-N, t+N]$.

$$\begin{aligned} P[\theta < t | A(\cdot), (x_i, y_i)] &= \lim_{N\to\infty} \frac{P[\theta < t \wedge (x_i, y_i) | A(\cdot), \mathcal{U}_N^t]}{P[(x_i, y_i) | A(\cdot), \mathcal{U}_N^t]} \\ &= \lim_{N\to\infty} \frac{\int_{t-N}^{t} \frac{1}{2N} f_x(x_i-\theta) f_y(y_i - A(\theta))\, d\theta}{\int_{t-N}^{t} \frac{1}{2N} f_x(x_i-\theta) f_y(y_i - A(\theta))\, d\theta + \int_t^{t+N} \frac{1}{2N} f_x(x_i-\theta) f_y(y_i - A(\theta))\, d\theta} \\ &= \lim_{N\to\infty} \frac{1}{1+\kappa_N} \geq c \Leftrightarrow \lim_{N\to\infty} \kappa_N \leq \frac{1-c}{c} \end{aligned}$$

Using appropriate variable transformations ($\tau = \theta - t$ in the nominator and $\tau = t - \theta$ in the denominator) and symmetry of f_x we get

$$\kappa_N = \frac{\int_0^N f_x(x_i - (t+\tau)) f_y(y_i - A(t+\tau))\, d\tau}{\int_0^N f_x(x_i - (t-\tau)) f_y(y_i - A(t-\tau))\, d\tau}.$$

Recall, that $y_i \geq 1-\delta-\gamma$ and $x_i \leq t+\xi$ holds in this case.

$$\begin{aligned} \kappa_N &\leq \frac{\int_0^N f_x(x_i - (t+\tau))\, d\tau}{\int_0^N f_x(x_i - (t-\tau))\, d\tau} \frac{\sup_{a\in[0,\delta]} f_y(y_i - a)}{\inf_{a\in[1-\delta,1]} f_y(y_i - a)} \\ &= \frac{F_x(x_i - t) - F_x(x_i - t - N)}{F_x(x_i - t + N) - F_x(x_i - t)} \frac{\sup_{a\in[0,\delta]} f_y(y_i - a)}{\inf_{a\in[1-\delta,1]} f_y(y_i - a)} \\ &\underset{N\to\infty}{\longrightarrow} \frac{F_x(x_i - t)}{1 - F_x(x_i - t)} \frac{\sup_{a\in[0,\delta]} f_y(y_i - a)}{\inf_{a\in[1-\delta,1]} f_y(y_i - a)} \\ &\leq \frac{F_x(\xi)}{1 - F_x(\xi)} \frac{\sup_{a\in[0,\delta]} f_y(y_i - a)}{\inf_{a\in[1-\delta,1]} f_y(y_i - a)} \leq \frac{1-c}{c} \end{aligned}$$

The last but one inequality uses condition (18).

2. The proof of the second statement relies on the same arguments used in 1. First, observe that

$$P[\theta \geq t | A(\cdot), (x_i, y_i)] = \lim_{N\to\infty} \frac{1}{1+\frac{1}{\kappa_N}} \geq c \Leftrightarrow \lim_{N\to\infty} \frac{1}{\kappa_N} \leq \frac{1-c}{c}.$$

Note that $y_i \leq \delta + \gamma$ and $x \geq t - \xi$ holds. Again, we can obtain a statement:

$$\frac{1}{\kappa_N} \leq \frac{1-F_x(-\xi)}{F_x(-\xi)} \frac{\sup_{a\in[1-\delta,1]} f_y(y_i - a)}{\inf_{a\in[0,\delta]} f_y(y_i - a)}$$
$$= \frac{F_x(\xi)}{1-F_x(\xi)} \frac{\sup_{a\in[0,\delta]} f_y((1-y_i)-a)}{\inf_{a\in[1-\delta,1]} f_y((1-y_i)-a)} \leq \frac{1-c}{c}.$$

The equality uses the symmetry of F_x and f_y. The last inequality exploits condition (18) (note that $(1-y_i) > 1-\delta-\gamma$).

3. The third statement follows from condition (19). Regarding notation, we use $A_t^n(t^-)$ to denote left limits of the function A_t depicted in Figure 6:

$$\begin{aligned} &P[y_i \geq 1-\delta-\gamma \wedge x \leq t+\xi | A(\cdot), \theta < t] \\ &\qquad \geq P[y_i \geq 1-\delta-\gamma \wedge x \leq t+\xi | A(\theta) = 1-\delta, \theta = t^-] \\ &\qquad \geq F_x(t+\xi-t)(1-F_y(1-\delta-\gamma-(1-\delta))) = F_x(\xi)F_y(\gamma) \geq 1-\delta \end{aligned} \tag{25}$$

4. The fourth statement follows from condition (19) and the following inequalities:

$$\begin{aligned} &P[y_i \leq \delta+\gamma \wedge x > t-\xi | A(\cdot), \theta \geq t] \\ &\qquad \geq P[y_i \leq \delta+\gamma \wedge x > t-\xi | A(\theta) = \delta, \theta = t] \\ &\qquad \geq F_y(\gamma)(1-F_x(-\xi)) = F_x(\xi)F_y(\gamma) \geq 1-\delta \end{aligned} \tag{26}$$

□

B.2 Iteration

We now use an iteration, in which agents' action A^{n+1} is a function of agents' previous action A^n, to prove the existence of an equilibrium for a given[14]

$$t \in [\delta+\gamma, 1-\delta-\gamma]. \tag{27}$$

[14]The requirement $1-\delta \geq \gamma$, $1 > 3\delta + 2\gamma$ in Proposition 4 implies that the interval is non-degenerated.

We start from a hypothetical situation in which player i faces an aggregate attack A_t^0 defined by (see green line below)

$$A_t^0(\theta) := \begin{cases} 1-\delta & \theta < t \\ \delta & \text{otherwise.} \end{cases} \tag{28}$$

We define the set of signals for which a player who would find it optimal to attack given a conjecture about other players' behavior A_t^n by Γ^n. Using this definition we have

$$A_t^{n+1}(\theta) := \iint_{\Gamma^n} f_x(x-\theta) f_y(y - A_t^n(\theta))\, dx\, dy \tag{29}$$

By induction it follows from the definition of A_t^{n+1} and the inequalities in Lemma 2 that $A_t^n \geq 1-\delta$ ($\leq \delta$) for $\theta \leq t$ (otherwise) for all $n \geq 0$.

To complete the proof of Proposition 4 we use the theorem of Arzelà–Ascoli to show that the sequence A_t^n has a convergent subsequence. That is, we note that A_t^n are continuous, uniformly bounded (by zero and one), and have a uniformly bounded derivative for all $\theta \neq t$, which implies equicontinuity. Hence, the preconditions of the Arzelà–Ascoli theorem are met[15] on each interval $[-k, t]$ (define A_t^n at t by the right limit) and $[t, k+1]$ for $k \in \mathbf{N}$ (they are also met for subsequences). Hence, for $k = 1$ there is a convergent subsequence on $[-1, t]$ denoted by $A_t^{n_1}$, from which we can select yet another convergent subsequence on $[t, 2]$ (and thus on $[-1, 2]$) denoted by $A_t^{n_2}$. We can carry out the same procedure for each $k > 1$, and receive a (sub)sequence $A_t^{n_{2k}}$ that converges on $[-k, k+1]$. Last but not least, select the k-th element of sequence $A_t^{n_{2k}}$ to create a new sequence $A_t^{n_0}$. This sequence is a subsequence of a converging sequence, and hence it converges.

We denote the limit of this sequence by A_t. It constitutes an equilibrium due to the continuity of the best-response operator.

C One Private Signal with General Error Terms

In this section we extend our results from Section 2.2, and show that "precise private information" ensures multiple equilibria for more general distribution functions.

Suppose agents receive signals:

$$z_i = A - \theta + \rho_i, \tag{30}$$

which inform about the attack's net size.

[15] See, e.g., Shilov (2013, p. 32).

We denote by G the cumulative distribution function of the error ρ_i, and by $g = G'$ the respective density function[16]. We simplify our proof and assume that g is supposed to be symmetric around 0.

Proposition 5. *Suppose that there exist $\delta > 0$ and $\gamma > 0$ such that the following conditions hold:*

$$\frac{1 - G(\xi - \alpha)}{G(\xi - \beta)} \geq \frac{1 - c}{c}, \quad \textit{for all } \xi \geq 1 - \delta - \gamma,\ \alpha \in [0, \delta],\ \textit{ and } \beta \in [1 - \delta, 1] \tag{31}$$

$$G(\gamma) \geq 1 - \delta, \textit{ and} \tag{32}$$

$$g(\delta - \gamma) < 1 \tag{33}$$

Then, there exists a continuum of equilibria.

Proof. See Appendix D. □

The proof's strategy is to fix a t, and to start with a guess of the equilibrium aggregate attack. In turn, we compute the best response to that attack. This yields another guess for the aggregate attack, to which we compute the best response.... . We use this argument to construct a converging sequence of aggregate attacks. The limit aggregate attack is an equilibrium. This process can be repeated for an open interval of t values, which establishes that there exists a continuum of equilibria.

D Proof of Proposition 5

The proof is an adapted version of the proof of Proposition 4. We begin by proving a supporting lemma:

Lemma 3. *Suppose that for a given t $A(\theta) \in [1 - \delta, 1]$ if $\theta < t$, and $A(\theta) \in [0, \delta]$ otherwise. Then the following holds:*

1. *If $z_i \geq 1 - \delta - t - \gamma$ then $P[\theta < t | A(\cdot), z_i] \geq c$.*
2. *If $z_i \leq \delta - t + \gamma$, then $P[\theta \geq t | A(\cdot), z_i] \geq c$.*
3. *$P[z_i \geq 1 - \delta - t - \gamma | A(\cdot), \theta < t] \geq 1 - \delta$*

[16]We assume that G is differentiable, i.e., we rule out atoms

4. $P[z_i \leq \delta - t + \gamma | A(\cdot), \theta \geq t] \geq 1 - \delta$

Proof. 1. To prove the first inequality in 1. we define

$$\kappa_N := \frac{\int_t^{t+N} g(z_i - A(\theta) + \theta)\, d\theta}{\int_{t-N}^{t} g(z_i - A(\theta) + \theta)\, d\theta},$$

and show that both inequalities in 1. can be reduced to statements about κ_N.

$$\begin{aligned} P[\theta < t | A(\cdot), z_i] &= \lim_{N\to\infty} \frac{P[\theta < t \wedge z_i | A(\cdot), \mathcal{U}_N^t]}{P[z_i | A(\cdot), \mathcal{U}_N^t]} \\ &= \lim_{N\to\infty} \frac{\int_t^{t+N} g(z_i - A(\theta) + \theta)\, d\theta}{\int_t^{t+N} g(z_i - A(\theta) + \theta)\, d\theta + \int_t^{t+N} g(z_i - A(\theta) + \theta)\, d\theta} \\ &= \lim_{N\to\infty} \frac{1}{1 + \kappa_N} \geq c \Leftrightarrow \lim_{N\to\infty} \kappa_N \leq \frac{1-c}{c} \end{aligned}$$

Due to continuity of g there exist $\alpha_N \in [0, \delta]$ and $\beta_N \in [1 - \delta, 1]$ such that

$$\begin{aligned} \kappa_N = \frac{\int_t^{t+N} g(z_i - \alpha_N + \theta)\, d\theta}{\int_{t-N}^{t} g(z_i - \beta_N + \theta)\, d\theta} &= \frac{G(z - \alpha_N + \theta + N) - G(z - \alpha_N + \theta)}{G(z - \beta_N + \theta) - G(z - \beta_N + \theta - N)} \\ &\xrightarrow[N\to\infty]{} \frac{1 - G(z - \alpha + \theta)}{G(z - \beta + \theta)} \leq \frac{1-c}{c}, \end{aligned}$$

where $\alpha \in [0, \delta]$ and $\beta \in [1 - \delta, 1]$ are the limits of the respective sequence. The last inequality exploits condition (31) and implies that $P[\theta < t | A(\cdot), z_i] \geq c$.

2. The proof of property 2. works along the lines of step 1., or alternatively like step 2. in the proof of Lemma 2.

3. To prove the third property, we use the notation $A_t^n(t^-)$ to denote left limits of the function A_t depicted in Figure 6:

$$\begin{aligned} P[z_i \geq 1 - \delta - t - \gamma | A(\cdot), \theta < t] &\geq P[z_i \geq 1 - \delta - t - \gamma | A(\theta) = 1 - \delta, \theta = t^-] \\ &= P[\rho_i \geq -\gamma] = 1 - G(-\gamma) = G(\gamma) \geq 1 - \delta \end{aligned}$$

The last but one equality holds due G being symmetric, the last equality due to condition (32).

4. Part 4. follows from:

$$\begin{aligned} P[z_i \leq \delta - t + \gamma | A(\cdot), \theta \geq t] &\geq P[z_i \leq \delta - t + \gamma | A(\theta) = \delta, \theta = t] \\ &= P[\rho_i \leq \gamma] = G(\gamma) \geq 1 - \delta \end{aligned}$$

Again, the last equality is due to condition (32).

□

The remaining part of the proof of Proposition 4 applies directly with two exceptions:

1. For given A^n there exists a cutoff z_n such that a player attacks iff $z_i < z_n$. This allows to adjust the definition of A_t^{n+1} given A_t^n;.

$$A_t^{n+1}(\theta) := \int_{-\infty}^{z_n} g(z - A_t^n(\theta) + \theta)) \, dz \, .$$

2. To establish equicontinuity of A_t^n, we start with another supporting Lemma:

Lemma 4. *Suppose agents consider the aggregate attack (function) to be equal to A_t^n and that $A_t^n(\theta) > t + \delta$ for $\theta < t$, and $A_t^n(\theta) < t - \delta$ otherwise. Then, the agent's attack cutoff is $z^n \in (-\gamma, \gamma)$.*

Proof. An agent receiving a signal $z_i \geq \gamma$ would attack, where we use the notation $A_t^n(t^-)$ to denote left limits and $A_t^n(t^+)$ to denote right limits:

$$\begin{aligned} P[\theta < t | z_i = z, A_t^n] &= {}^{17} \frac{G(z - A_t^n(t^-) + t)}{G(z - A_t^n(t^-) + t) + 1 - G(z - A_t^n(t^+) + t)} \\ &\geq {}^{18} \frac{1 - G(z + \delta)}{G(z - \delta) + 1 - G(z + \delta)} = P[\theta < t | z_i = z, A_t^0] > P^* \end{aligned}$$

Hence, for the cutoff $z^n < \gamma$ has to hold. On the other hand, $z^n > -\gamma$ holds, as an agent receiving a signal $z_i \leq -\gamma$ would not attack:

$$\begin{aligned} P[\theta \geq t | z_i = z, A_t^n] &= \frac{1 - G(z - A_t^n(t^+) + t)}{G(z - A_t^n(t^-) + t) + 1 - G(z - A_t^n(t^+) + t)} \\ &\geq \frac{1 - G(z + \delta)}{G(z - \delta) + 1 - G(z + \delta)} = P[\theta \geq t | z_i = z, A_t^0] > 1 - P^* \end{aligned}$$

□

[17] Use analogous computations as in step 2.

[18] The inequality can be shown by rearranging the following inequality:

$$\frac{G(z - A_t^n(t^-) + t)}{G(z - \delta)} \geq 1 \geq \frac{1 - G(z - A_t^n(t^+) + t)}{1 - G(z + \delta)}$$

Now, note that A_t^n are continuous, uniformly bounded and have a uniformly bounded derivative for all $\theta \neq t$:

$$\begin{aligned}
\frac{dA_t^{n+1}}{d\theta}(\theta) &= -g(z^n - A_t^n(\theta) + \theta)\left(\frac{dA_t^n}{d\theta}(\theta) - 1\right) \\
&= -g(z^n - A_t^n(\theta) + \theta)\,\frac{1 - g(z^n - A_t^n(\theta) + \theta)^n}{1 - g(z^n - A_t^n(\theta) + \theta)} \\
&\geq -\max\left\{g(\gamma - \delta)\,\frac{1 - g(\gamma - \delta)^n}{1 - g(\gamma - \delta)}, g(-\gamma + \delta)\,\frac{1 - g(-\gamma + \delta)^n}{1 - g(-\gamma + \delta)}\right\} \\
&> \frac{-g(\gamma - \delta)}{1 - g(\gamma - \delta)}
\end{aligned}$$

The second equality can be shown by induction using $\frac{dA_t^0}{d\theta}(\theta) = 0$. We use the notation to $A_t^n(t^-)$ for the right limit of A_t^n for $\theta \to t$. The first inequality holds because g is increasing (decreasing) on $\theta < t$ (otherwise), $z^n \in (-\gamma, \gamma)$ (Lemma 4), and $A_t^n(\theta) \geq t + \delta$ on $\theta < t$ and $A_t^n(\theta) < t - \delta$ otherwise. The second inequality holds due to the symmetry of g and condition (33). Also note, that the second equality implies that the derivative of A_t^n is non-positive. Thus, we have established uniform upper and lower bounds.