

Supplementary information for “Flat-band Fulde-Ferrell-Larkin-Ovchinnikov State from Quantum Geometric Discrepancy”

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I. EFFECTIVE ACTION IN THE BAND-PROJECTION FORMALISM

To study the core physics of the superconducting state in a multi-band system, it is convenient to project the attractive interaction onto the target flat band. In this section, we derive the effective action for the single-band description, with the full numerical results presented in the next section.

A. Linearized gap and number equations

To begin with, we consider a general local attractive Hubbard interaction:

$$\hat{H}_{\text{int}} = -U \sum_{i\alpha} \hat{c}_{i\alpha\uparrow}^\dagger \hat{c}_{i\alpha\downarrow}^\dagger \hat{c}_{i\alpha\downarrow} \hat{c}_{i\alpha\uparrow}. \quad (\text{S1})$$

Here U denotes the attractive interaction strength driving the superconducting order. In real space, i and α are the site and orbital index, respectively. The fermion operator $\hat{c}_{\alpha,\mathbf{k}\sigma}^\dagger$ in the momentum space can be expressed by Fourier transform via

$$\hat{c}_{\alpha,\mathbf{k}\sigma}^\dagger = \frac{1}{\sqrt{N_c}} \sum_i e^{-i\mathbf{k}\cdot\mathbf{r}_{i\alpha}} \hat{c}_{i\alpha\sigma}^\dagger, \quad \hat{c}_{i\alpha\sigma}^\dagger = \frac{1}{\sqrt{N_c}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_{i\alpha}} \hat{c}_{\alpha,\mathbf{k}\sigma}^\dagger. \quad (\text{S2})$$

Therefore, we can rewrite the interacting Hamiltonian in the momentum space as

$$\hat{H}_{\text{int}} = -\frac{U}{N_c} \sum_{\alpha\mathbf{k}\mathbf{k}'\mathbf{q}} \hat{c}_{\alpha,\mathbf{k}+\mathbf{q}\uparrow}^\dagger \hat{c}_{\alpha,-\mathbf{k}\downarrow}^\dagger \hat{c}_{\alpha,-\mathbf{k}'\downarrow} \hat{c}_{\alpha,\mathbf{k}'+\mathbf{q}\uparrow}. \quad (\text{S3})$$

For the convenience of the analytical treatment, we approximate the interaction term [1–3] as

$$\hat{H}_{\text{int}} \approx -\frac{U}{N_{\text{orb}}N_c} \sum_{\alpha\beta\mathbf{k}\mathbf{k}'\mathbf{q}} \hat{c}_{\alpha,\mathbf{k}+\mathbf{q}\uparrow}^\dagger \hat{c}_{\alpha,-\mathbf{k}\downarrow}^\dagger \hat{c}_{\beta,-\mathbf{k}'\downarrow} \hat{c}_{\beta,\mathbf{k}'+\mathbf{q}\uparrow}. \quad (\text{S4})$$

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Here we have used $g = U/(N_{orb}N_c)$ to characterize the effective interaction. We then project the interaction term onto the target flat band by $\hat{c}_{\alpha,\mathbf{k}\sigma} \rightarrow u_{\mathbf{k}\sigma}^*(\alpha)\hat{a}_{\mathbf{k}\sigma}$, where $\hat{a}_{\mathbf{k}\sigma}$ denotes the band-basis electron annihilation operator. Then from $\hat{H}_{\text{int}}$, the effective action S after the band projection can be expressed as

$$S = \sum_{\mathbf{k},\sigma} \bar{a}_{\mathbf{k}\sigma} (-i\omega_n + \xi_{\mathbf{k}\sigma}) a_{\mathbf{k}\sigma} - g \sum_q \bar{\theta}_q \theta_q, \quad (\text{S5})$$

where $\theta_q = \sum_{\mathbf{k}} \Lambda^*(\mathbf{k}, \mathbf{q}) a_{-\mathbf{k}\downarrow} a_{\mathbf{k}+\mathbf{q}\uparrow}$. Here $\Lambda(\mathbf{k}, \mathbf{q}) = \sum_{\alpha} u_{\mathbf{k}+\mathbf{q}\uparrow}(\alpha) u_{-\mathbf{k}\downarrow}(\alpha) = \langle u_{\mathbf{k}+\mathbf{q}\uparrow} | u_{-\mathbf{k}\downarrow}^* \rangle$ is introduced as the form factor. Here we have used the notations $\mathbf{k} = (\mathbf{k}, \omega_n)$ and $\mathbf{q} = (\mathbf{q}, \Omega_m)$ with ω_n (Ω_m) the fermionic (bosonic) Matsubara frequencies.

After a standard Hubbard-Stratonovich decoupling, one can find the action in the Nambu basis $\psi_{\mathbf{k},\mathbf{q}} = (a_{\mathbf{k}+\mathbf{q}\uparrow}, \bar{a}_{-\mathbf{k}\downarrow})^T$ takes the form as

$$S = \sum_q \frac{|\Delta_q|^2}{g} + \sum_{\mathbf{k},\mathbf{q}} \bar{\psi}_{\mathbf{k},\mathbf{q}} \hat{\mathbb{G}}_{\mathbf{k},\mathbf{q}}^{-1} \psi_{\mathbf{k},\mathbf{q}}, \quad (\text{S6})$$

where

$$\hat{\mathbb{G}}_{\mathbf{k},\mathbf{q}}^{-1} = \begin{pmatrix} -i\omega_n - i\Omega_m + \xi_{\mathbf{k}+\mathbf{q}\uparrow} & -\Lambda(\mathbf{k}, \mathbf{q})\Delta_q \\ -\Lambda^*(\mathbf{k}, \mathbf{q})\Delta_q^* & -i\omega_n - \xi_{-\mathbf{k}\downarrow} \end{pmatrix}. \quad (\text{S7})$$

After integrating out the fermionic field, we can get

$$S[\Delta_q] = \sum_q \frac{|\Delta_q|^2}{g} - \text{Tr} \ln \hat{\mathbb{G}}_{\mathbf{k},\mathbf{q}}^{-1}, \quad (\text{S8})$$

and expanding it up to the second order in Δ_q , the Gaussian action around the trivial saddle point $\Delta_q = 0$ reads

$$S_G[\Delta_q] = S[\Delta_q = 0] + \sum_q \Gamma_q^{-1} |\Delta_q|^2, \quad (\text{S9})$$

in which the coefficient

$$\Gamma_q^{-1} = \left. \frac{\delta^2 S[\Delta_q]}{\delta \Delta_q^* \delta \Delta_q} \right|_{\Delta_q = \Delta_q^* = 0} = g^{-1} - \chi_q^c \quad (\text{S10})$$

is the pairing propagator, and the pairing susceptibility reads

$$\chi_q^c = \frac{1}{N_c} \sum_{\mathbf{k}} \frac{1 - n_F(\xi_{\mathbf{k}+\mathbf{q}\uparrow}) - n_F(\xi_{-\mathbf{k}\downarrow})}{\xi_{\mathbf{k}+\mathbf{q}\uparrow} + \xi_{-\mathbf{k}\downarrow} - i\Omega_m} |\Lambda(\mathbf{k}, \mathbf{q})|^2. \quad (\text{S11})$$

At the static limit ($\Omega_m = 0$), $\chi_q^c \equiv \chi_{q=(\mathbf{q},0)}^c$, and the linearized gap equation is given by $g\chi_q^c = 1$.

Near the critical temperature, following the Nosières and Schmitt-Rink (NSR) method [4], we consider the Gaussian fluctuations contribution to the thermodynamic potential as $\Omega = \Omega_0 - \beta^{-1} \sum_q \ln \Gamma_q$, where $\Omega_0 = S[\Delta_q = 0]/\beta$, and the number equation says $N_{\sigma} = -\partial\Omega/\partial\mu_{\sigma}$. We can rewrite Ω in terms of a phase shift defined by $\Gamma_{q=(\mathbf{q}, i\Omega_m \rightarrow \omega \pm i0)} = |\Gamma_{q=(\mathbf{q}, i\Omega_m \rightarrow \omega)}| \exp[\pm i\Psi(\mathbf{q}, \omega)]$. So the number equation incorporating the effects of Gaussian fluctuations is given by $N_{\sigma} = N_{0\sigma} + N_{G\sigma}$, where

$$N_{0\sigma} = \frac{1}{2} \sum_{\mathbf{k}} \left[1 - \tanh \left(\frac{\beta \xi_{\mathbf{k}\sigma}}{2} \right) \right], \quad (\text{S12})$$

$$N_{G\sigma} = \sum_{\mathbf{q}} \int_{-\infty}^{+\infty} \frac{d\omega}{\pi} n_B(\omega) \frac{\partial \Psi(\mathbf{q}, \omega)}{\partial \mu_{\sigma}}. \quad (\text{S13})$$

We can introduce the filling factor $\nu_{\sigma} = N_{\sigma}/N_c$. In the ideal flat-band limit, we have $\xi_{\mathbf{k}\sigma} = -\mu_{\sigma}$. When $\nu_{\uparrow} = \nu_{\downarrow} = \nu$ and $\mu_{\uparrow} = \mu_{\downarrow} = \mu$ (no spin-population imbalance), we have $\chi_q^c = \tanh \left(\frac{\beta\mu}{2} \right) \frac{1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})}{2\mu}$. Then we consider the number equation, the free part is $2\nu_0 = N_0/N_c = 1 + \tanh \left(\frac{\beta\mu}{2} \right)$. If we only consider the contribution from N_0 , i.e., neglecting the fluctuations, we can recover Eq. (6) in the main text regardless of the filling factor. However, if we include N_G , it is not the case for general filling. But worth mention that, Eq. (6) can approximately stand even with fluctuations when $\beta\mu \ll 1$, where $\chi_q^c \rightarrow \beta(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q}))/4$. When neglecting N_G , this condition corresponds to the neighborhood of half filling.

B. Corrections from Gaussian fluctuations

Then we seriously consider the corrections brought by N_G , the dynamical pairing propagator at the ideal flat-band limit is

$$\Gamma_q^{-1} = g^{-1} - \tanh\left(\frac{\beta\mu}{2}\right) \frac{1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})}{2\mu + i\Omega_m}. \quad (\text{S14})$$

For the convenience of calculation, we notice that

$$\frac{\partial \Psi}{\partial \mu} = \cos \Psi \frac{\partial \sin \Psi}{\partial \mu} - \sin \Psi \frac{\partial \cos \Psi}{\partial \mu}, \quad (\text{S15})$$

where $\cos \Psi = \text{Re} \Gamma_{q=(\mathbf{q}, i\Omega_m \rightarrow \omega \pm i0)} / |\Gamma_{q=(\mathbf{q}, i\Omega_m \rightarrow \omega)}|$ and $\sin \Psi = \text{Im} \Gamma_{q=(\mathbf{q}, i\Omega_m \rightarrow \omega \pm i0)} / |\Gamma_{q=(\mathbf{q}, i\Omega_m \rightarrow \omega)}|$.

If we replace $i\Omega_m$ by $\omega + ic$, c is an infinitesimal positive number, we can obtain

$$\cos \Psi = \frac{(2\mu + \omega)^2 + c^2 - g(2\mu + \omega)(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)}{\sqrt{\left[(2\mu + \omega)^2 + c^2\right] \left[\left(2\mu + \omega - g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)\right)^2 + c^2\right]}}, \quad (\text{S16})$$

$$\sin \Psi = \frac{-gc(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)}{\sqrt{\left[(2\mu + \omega)^2 + c^2\right] \left[\left(2\mu + \omega - g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)\right)^2 + c^2\right]}}. \quad (\text{S17})$$

Then from Eq. S15 we have

$$\frac{\partial \Psi}{\partial \mu} = \frac{-gc(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \beta}{2 \cosh^2\left(\frac{\beta\mu}{2}\right) \left[\left(2\mu + \omega - g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)\right)^2 + c^2\right]} \quad (\text{S18})$$

$$+ \frac{2gc(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right) \left(4\mu + 2\omega - g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)\right)}{\left[(2\mu + \omega)^2 + c^2\right] \left[\left(2\mu + \omega - g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)\right)^2 + c^2\right]}, \quad (\text{S19})$$

To evaluate N_G , we use $\delta(x + a) = \lim_{c \rightarrow 0+} \frac{1}{\pi} \frac{c}{(a+x)^2 + c^2}$ to

$$\frac{\partial \Psi}{\partial \mu} = \left[2\pi - \frac{g\pi(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q}))\beta}{2 \cosh^2\left(\frac{\beta\mu}{2}\right)} \right] \delta\left(2\mu + \omega - g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)\right) - 2\pi\delta(2\mu + \omega), \quad (\text{S20})$$

so that we arrive at

$$N_G = -2N_c n_B(-2\mu) + \sum_{\mathbf{q}} \left[2 - \frac{g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q}))\beta}{2 \cosh^2\left(\frac{\beta\mu}{2}\right)} \right] n_B\left(-2\mu + g(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q})) \tanh\left(\frac{\beta\mu}{2}\right)\right). \quad (\text{S21})$$

Then we want to show that, even in the presence of Gaussian fluctuations, the FFLO state with the highest critical temperature can still be determined solely by $\bar{d}_{\mathcal{A}}^2(\mathbf{Q})$. We dope the half filling with $\delta\nu$, then the chemical potential becomes μ . Besides, we use the linearized gap equation concerning the FFLO state with finite momentum $\mathbf{Q}$, namely, $\beta g = 4x / [(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{Q})) \tanh(x)]$ to replace the corresponding βg in the number equation, in which we use $x = \beta\mu/2$ for convenience. Then we reduce the number equation to

$$\delta\nu = \frac{1}{2} \tanh x - \frac{1}{e^{-4x} - 1} + \frac{1}{N_c} \sum_{\mathbf{q}} \left[1 - \frac{2x(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{q}))}{\sinh 2x(1 - \bar{d}_{\mathcal{A}}^2(\mathbf{Q}))} \right] / \left[\exp\left(\frac{4x(\bar{d}_{\mathcal{A}}^2(\mathbf{Q}) - \bar{d}_{\mathcal{A}}^2(\mathbf{q}))}{1 - \bar{d}_{\mathcal{A}}^2(\mathbf{Q})}\right) - 1 \right]. \quad (\text{S22})$$

We observe that the solution of x is independent of g , which implies the critical temperature is always proportional to g . When fixing $\delta\nu$, the solution of x is only determined by the value of $\bar{d}_{\mathcal{A}}^2(\mathbf{Q})$, so does the critical temperature.

However, unlike the free case, $\beta_{c\mathbf{Q}}$ may not be a monotonic function of $\bar{d}_{\mathcal{A}}^2(\mathbf{Q})$ now. We can plot $\beta_{c\mathbf{Q}} - \bar{d}_{\mathcal{A}}^2(\mathbf{Q})$ curve to look for the global minimum point of $\beta_{c\mathbf{Q}}$ (highest critical temperature), then we can find out the corresponding FFLO instability with $\bar{d}_{\mathcal{A}}^2(\mathbf{Q})$. We calculate the critical temperatures corrected by Gaussian fluctuations in Fig. S1 as a comparison of Fig. 3(b) in the main text. We can see that the emergence of FFLO instability in the presence of Gaussian fluctuations on the whole aligns with the predictions from QGD and AQD.

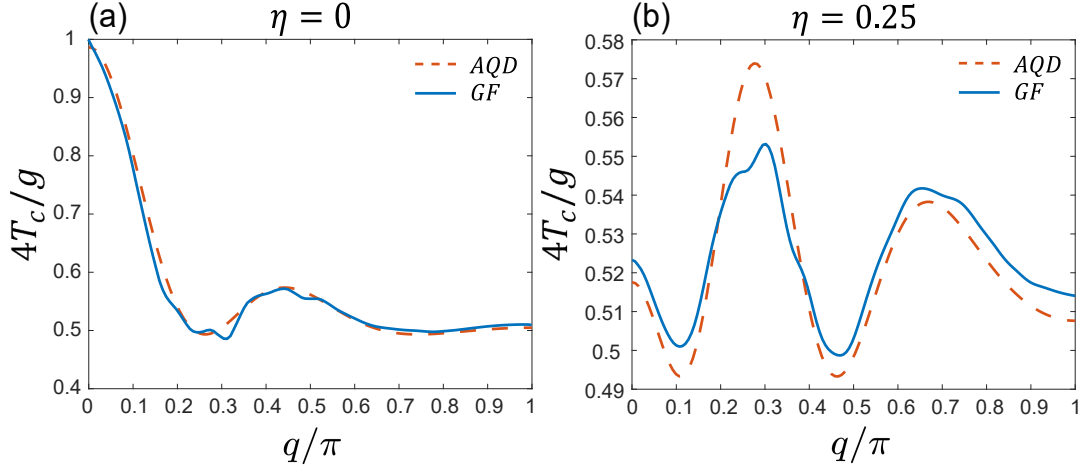


FIG. S1. (a) $T_{c\mathbf{q}}(\eta)$ from Eq. (6) in the main text (dashed orange line) and from Gaussian fluctuations correction (solid blue line) for $\mathbf{q} = q(\hat{x} + \hat{y})$, $q \in [0, \pi]$, at $\eta = 0$. (b) Same calculations for $\eta = 0.25$. In our calculations, we set $\zeta_{\downarrow} = 3$ and $\zeta_{\uparrow} = \zeta_{\downarrow}(1 + \eta)/(1 - \eta)$, doping factor $\delta\nu = 0.1$.

II. SELF-CONSISTENT MULTIORBITAL MEAN-FIELD THEORY

In this section, we provide the detailed methodology for the mean-field numerical simulations discussed in the main text. To be specific, we perform the decoupling of the on-site attractive Hubbard interaction within the mean-field approximation:

$$\begin{aligned} \hat{H}_{\text{int}} &= U \sum_{i\alpha} \hat{c}_{i\alpha\uparrow}^{\dagger} \hat{c}_{i\alpha\downarrow}^{\dagger} \hat{c}_{i\alpha\downarrow} \hat{c}_{i\alpha\uparrow} \\ &\approx \sum_{\alpha\mathbf{k}} \left(\Delta_{\alpha} \hat{c}_{\alpha,\mathbf{k}+\mathbf{q}\uparrow}^{\dagger} \hat{c}_{\alpha,-\mathbf{k}\downarrow}^{\dagger} + h.c. \right) + \frac{N_c}{U} \sum_{\alpha} |\Delta_{\alpha}|^2, \end{aligned} \quad (\text{S23})$$

where we have used $\Delta_{\alpha} = -U \langle \hat{c}_{\alpha,-\mathbf{k}\downarrow} \hat{c}_{\alpha,\mathbf{k}+\mathbf{q}\uparrow} \rangle$ and $\Delta_{\alpha} \equiv \Delta_{\alpha}(\mathbf{q})$ can be determined by the self-consistent gap equation for a given Cooper pair momentum $\mathbf{q}$. The full mean-field Hamiltonian can be rewritten in the Nambu basis $\psi(\mathbf{k}, \mathbf{q}) = \left(\hat{c}_{A,\mathbf{k}+\mathbf{q}\uparrow}, \hat{c}_{B,\mathbf{k}+\mathbf{q}\uparrow}, \hat{c}_{A,-\mathbf{k}\downarrow}^{\dagger}, \hat{c}_{B,-\mathbf{k}\downarrow}^{\dagger} \right)^T$ as

$$\hat{\mathcal{H}}_{\text{MF}} = \sum_{\mathbf{k}, \mathbf{q}} \psi^{\dagger}(\mathbf{k}, \mathbf{q}) H_{\text{BdG}}(\mathbf{k}, \mathbf{q}) \psi(\mathbf{k}, \mathbf{q}) + \sum_{\mathbf{k}, \mathbf{q}} \text{Tr}[H_{-\mathbf{k}\downarrow} - \mu(\mathbf{q}) \mathbb{I}_{2 \times 2}] + \frac{N_c}{U} \sum_{\alpha} |\Delta_{\alpha}|^2, \quad (\text{S24})$$

in which

$$H_{\text{BdG}}(\mathbf{k}, \mathbf{q}) = \begin{pmatrix} H_{\mathbf{k}+\mathbf{q}\uparrow} - \mu(\mathbf{q}) \mathbb{I}_{2 \times 2} & \text{diag}(\Delta_{\alpha}) \\ \text{diag}(\Delta_{\alpha}^{\dagger}) & -H_{-\mathbf{k}\downarrow}^T + \mu(\mathbf{q}) \mathbb{I}_{2 \times 2} \end{pmatrix}, \quad (\text{S25})$$

where $\text{diag}(\Delta_{\alpha}) = \text{diag}(\Delta_A, \Delta_B)$ and the momentum $\mathbf{q} = q_x \hat{x} + q_y \hat{y}$. The grand potential is defined as

$$\begin{aligned} \Omega[\mu(\mathbf{q}), \Delta_{\alpha}(\mathbf{q}), \mathbf{q}] &= -\frac{1}{\beta} \log \mathcal{Z} = -\frac{1}{\beta} \ln \text{Tr}[e^{-\beta \hat{\mathcal{H}}_{\text{MF}}}] \\ &= \sum_{\mathbf{k}} \text{Tr}(H_{-\mathbf{k}\downarrow} - \mu(\mathbf{q}) \mathbb{I}_{2 \times 2}) + \frac{N_c}{U} \sum_{\alpha} |\Delta_{\alpha}|^2 - \frac{1}{\beta} \sum_{\mathbf{k}} \text{Tr} \left[\ln \left(1 + e^{-\beta H_{\text{BdG}}(\mathbf{k}, \mathbf{q})} \right) \right] \\ &= \sum_{\mathbf{k}} \text{Tr}(H_{-\mathbf{k}\downarrow} - \mu(\mathbf{q}) \mathbb{I}_{2 \times 2}) + \frac{N_c}{U} \sum_{\alpha} |\Delta_{\alpha}|^2 - \frac{1}{\beta} \sum_{n\mathbf{k}} \ln \left(1 + e^{-\beta E_n(\mathbf{k}, \mathbf{q})} \right), \end{aligned} \quad (\text{S26})$$

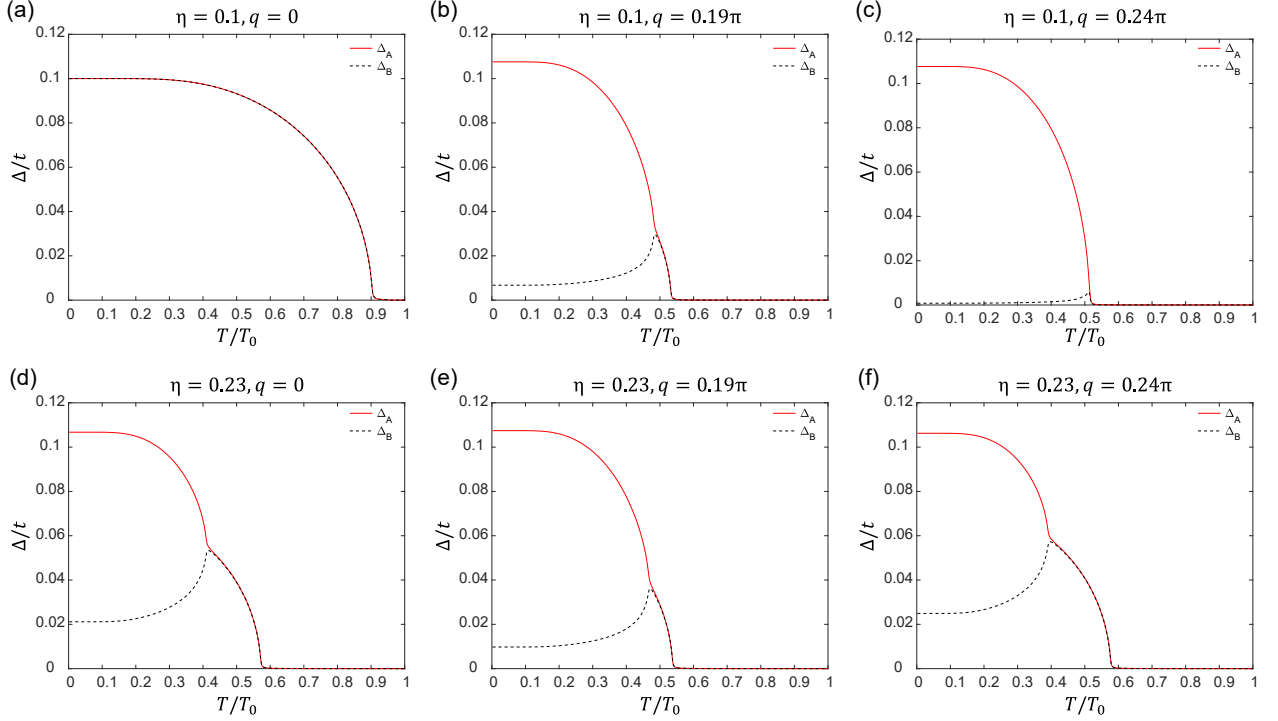


FIG. S2. (a)-(c): The $\Delta - T$ relation for the $\eta = 0.1$ (in BCS region) states paired with the finite momenta of $\mathbf{q} = q(\hat{x} + \hat{y})$, and $q = 0, 0.19\pi, 0.24\pi$ respectively. The favoured finite-momentum $Q = 0$. (d)-(f): The $\Delta - T$ relation for the $\eta = 0.23$ (in FFLO region) states paired with the same q in (a)-(c), and the favoured finite-momentum $Q = 0.24\pi$. In (a)-(f), Δ_A is donated by the red solid curve and Δ_B is donated by the black dashed curve. In these figures, when $\Delta_A \neq \Delta_B$, we only show the $\Delta_A > \Delta_B$ solution. Of course, there is another solution at $\Delta_B > \Delta_A$ due to the equivalence of A, B orbital.

where $E_n(\mathbf{k}, \mathbf{q})$ is the n -th eigenvalue of the BdG Hamiltonian $H_{\text{BdG}}(\mathbf{k}, \mathbf{q})$. By minimizing the grand potential with respect to the pairing potential Δ_α we can get the self-consistent equation:

$$\begin{aligned} \frac{\partial \Omega}{\partial \Delta_\alpha^*} &= \frac{N_c}{U} \Delta_\alpha(\mathbf{q}) + \sum_{\mathbf{k}} \langle \hat{c}_{\alpha, -\mathbf{k}\downarrow} \hat{c}_{\alpha, \mathbf{k}+\mathbf{q}\uparrow} \rangle = 0, \\ \Delta_\alpha(\mathbf{q}) &= -\frac{U}{N_c} \sum_{\mathbf{k}} \langle \hat{c}_{\alpha, -\mathbf{k}\downarrow} \hat{c}_{\alpha, \mathbf{k}+\mathbf{q}\uparrow} \rangle = -\frac{U}{N_c} \sum_{\mathbf{k}} \text{Tr} \left[\mathcal{U}_{\mathbf{k}, \mathbf{q}}^\dagger \mathcal{M}_\alpha \mathcal{U}_{\mathbf{k}, \mathbf{q}} \hat{\rho} \right]. \end{aligned} \quad (\text{S27})$$

And the filling factor ν can be obtained from the thermodynamic relation:

$$\begin{aligned} N &= 2N_c \nu = -\frac{\partial \Omega}{\partial \mu} = \sum_{\mathbf{k}} \langle \hat{c}_{a, \mathbf{k}\uparrow}^\dagger \hat{c}_{a, \mathbf{k}\uparrow} + \hat{c}_{a, \mathbf{k}\downarrow}^\dagger \hat{c}_{a, \mathbf{k}\downarrow} + \hat{c}_{b, \mathbf{k}\uparrow}^\dagger \hat{c}_{b, \mathbf{k}\uparrow} + \hat{c}_{b, \mathbf{k}\downarrow}^\dagger \hat{c}_{b, \mathbf{k}\downarrow} \rangle, \\ 2\nu &= 1 + \sum_{\mathbf{k}} \text{Tr} \left[\mathcal{U}_{\mathbf{k}, \mathbf{q}}^\dagger \tau_z \otimes \mathbb{I}_{2 \times 2} \mathcal{U}_{\mathbf{k}, \mathbf{q}} \hat{\rho} \right], \end{aligned} \quad (\text{S28})$$

where $\mathcal{U}_{\mathbf{k}, \mathbf{q}}$ is the unitary operator that diagonalizes the BdG Hamiltonian as $\mathcal{U}_{\mathbf{k}, \mathbf{q}}^\dagger H_{\text{BdG}}(\mathbf{k}, \mathbf{q}) \mathcal{U}_{\mathbf{k}, \mathbf{q}} = \text{diag}(E_n)$, $\mathcal{M}_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\mathcal{M}_B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, and the density matrix $\hat{\rho} = \frac{1}{2} \text{diag} \left(\frac{1}{1 + e^{\beta E_n(\mathbf{k}, \mathbf{q})}} \right)$. In turn, if the particle number (filling factor ν) is fixed, the chemical potential $\mu(\mathbf{q})$ can be determined from this equation. Finally, the free energy density (up to a constant given by the summation $\frac{1}{N_c} \sum_{\mathbf{k}} \text{Tr}(H_{-\mathbf{k}\downarrow})$) is:

$$F(\mathbf{q}) = \frac{1}{N_c} (\Omega[\mu(\mathbf{q}), \Delta_\alpha(\mathbf{q}), \mathbf{q}] + \mu(\mathbf{q})N) = \frac{1}{U} \sum_{\alpha} |\Delta_\alpha|^2 - \frac{1}{\beta N_c} \sum_{n\mathbf{k}} \ln \left(1 + e^{-\beta E_n(\mathbf{k}, \mathbf{q})} \right). \quad (\text{S29})$$

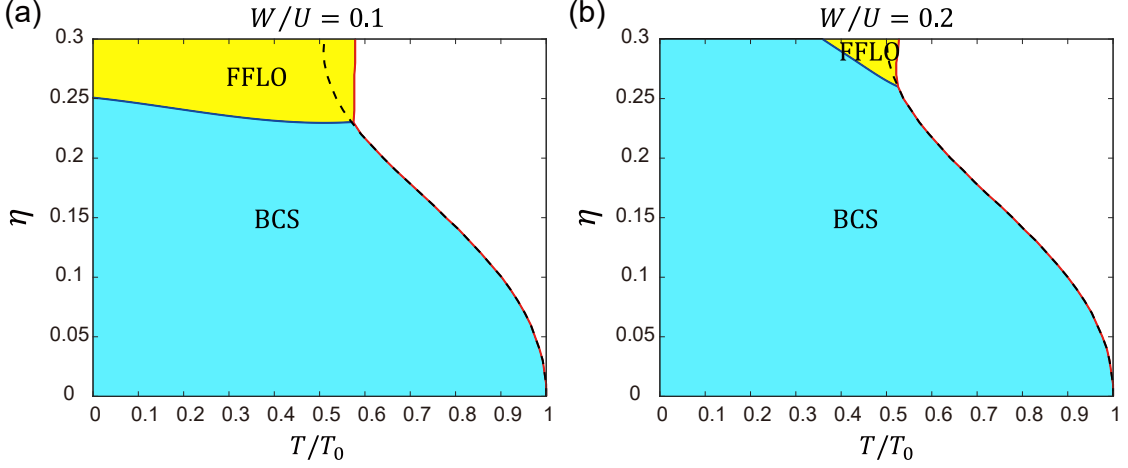


FIG. S3. The superconducting phase diagram of the ζ -lattice model. The red solid line labels $T_{c0}(\eta)$, the black dashed line labels $T_{cQ}(\eta)$, where $\mathbf{Q} = Q(\hat{x} + \hat{y})$. The BCS regime and the FFLO regime are labeled in light blue and yellow, respectively. Parameters used here: $U = 0.4t$, $\zeta_{\downarrow} = 3$, $W/U = 0.1$ in (a) and $W/U = 0.2$ in (b).

We note that in the previous section, the band-projected interaction Hamiltonian is approximated under the uniform pairing condition, where we only consider the trivial irreducible representation as the intra-orbital pairing channel, such that $\Delta_{\alpha\beta} \propto \delta_{\alpha\beta}$ is selected as the only relevant pairing channel, while other pairing channels are neglected. To demonstrate that this approximation is valid near the critical temperature in this model, we plot the $\Delta - T$ relation for different values of the QGD η and finite momentum $\mathbf{q}$ in Fig. S2. At low temperatures, Δ_A and Δ_B can differ depending on η and $\mathbf{q}$. However, near the critical temperature, we observe that $\Delta_A = \Delta_B$, which satisfies the uniform pairing condition. This observation explains why our critical temperature analysis, based on the uniform pairing condition, aligns well with the numerical results.

We also provide additional numerical results for the case of finite band dispersion. To ensure a narrow bandwidth compared to the band gap and the interaction strength, we maintain the condition $W \ll U \ll 2t$. In Fig. S3 we show the phase diagram for $W/U = 0.1$ and $W/U = 0.2$. Compared to Fig. 4 in the main text, it is evident that the region corresponding to the FFLO phase shrinks as W/U increases. In particular, Fig. S3(b) shows that the FFLO states are confined to a very narrow region. From this trend, it is reasonable to infer that further increasing W/U could lead to the complete disappearance of the FFLO states. This observation supports the claim made in the main text.

III. ADDITIONAL MODELS FOR FFLO PHASE WITH QGD

A. ζ -lattice model on triangular lattice

In the main text, we investigate a ζ -lattice model on a square lattice, which preserves C_4 symmetry. To further demonstrate the generality of our approach, we extend our analysis in this section to a triangular lattice featuring two orbitals (A and B) per site. Analogously, we can express the Bloch Hamiltonian as follows

$$\begin{aligned} \mathcal{H}_{\mathbf{k}\sigma} &= -t(\lambda_x \sin \zeta_{\sigma} p_{\mathbf{k}} + m_{\sigma} \lambda_y \cos \zeta_{\sigma} p_{\mathbf{k}}) - \mu_{\sigma} \lambda_0, \\ p_{\mathbf{k}} &= \sum_{j=1}^3 \cos(\mathbf{k} \cdot \mathbf{R}_j). \end{aligned} \quad (\text{S30})$$

where $\mathbf{R}_1 = a(1, 0)$, $\mathbf{R}_2 = a(-1/2, \sqrt{3}/2)$, $\mathbf{R}_3 = a(-1/2, -\sqrt{3}/2)$. We calculate the averaged anomalous quantum distance $\bar{d}_{\eta}^2(\mathbf{q})$ for this model in the absence and presence of QGD, as shown in Fig. S4(c) and (d), respectively. For comparison, we also present the corresponding plots for the square lattice model from the main text in Fig. S4(a) and (b). In all panels, the dark blue regions indicate the global minimum of $\bar{d}_{\eta}^2(\mathbf{q})$. We observe that in both (a) and (c), when $\eta = 0$ the global minimum of $\bar{d}_{\eta}^2(\mathbf{q})$ occurs at $\mathbf{q} = 0$. However, in (b) and (d), the global minima shift away from $\mathbf{q}$ implying the FFLO instability, as discussed in the main text. Notice that in Fig. S4(b), the FFLO states exhibit four-fold degeneracy, whereas in Fig. S4(d), they display six-fold degeneracy. This difference arises from the distinct

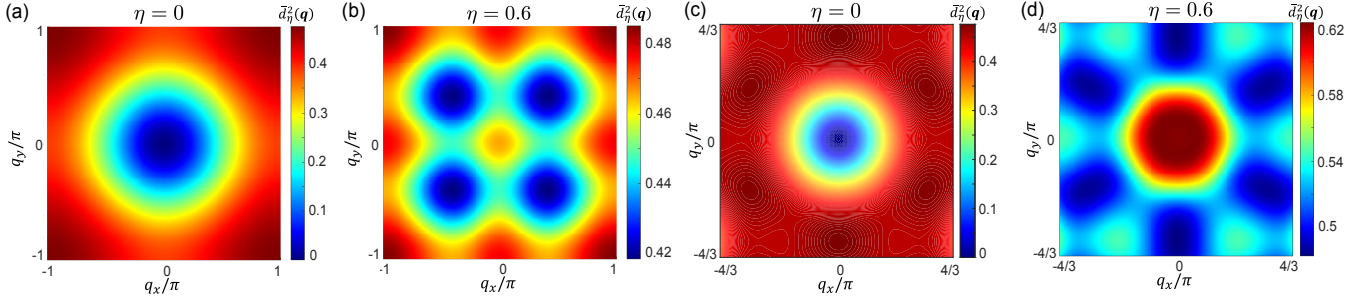


FIG. S4. Averaged anomalous quantum distance $\bar{d}_\eta^2(\mathbf{q})$ for the square and triangular ζ -lattice model, in the absence (presence) of QGD. (a) Square lattice for $\eta = 0$. (b) Square lattice for $\eta = 0.6$. (c) Triangular lattice for $\eta = 0$. (d) Triangular lattice for $\eta = 0.6$. Other parameters used here: $\zeta_\uparrow = 1$, $t = 1$.

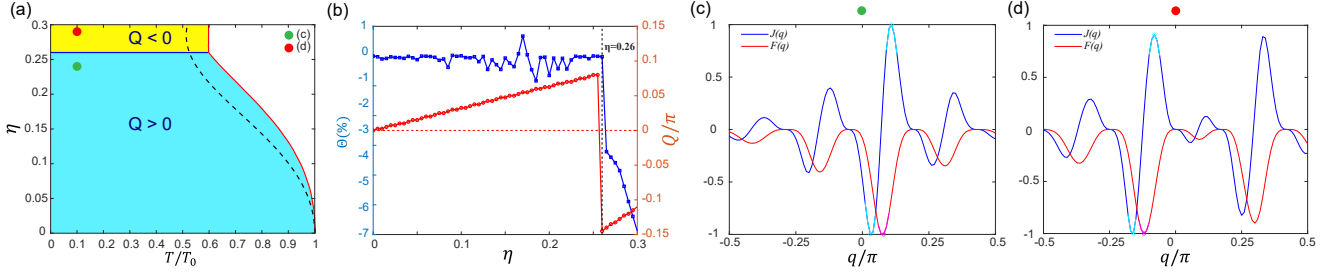


FIG. S5. (a) The superconducting phase diagram of ζ -lattice model with an inversion breaking term γ . The $Q > 0$ (weak helical superconducting state) and $Q < 0$ (strong helical superconducting state) regimes are colored in blue and yellow, respectively. Parameters used here: $U = 0.4t$, $\zeta_\uparrow = 3$, $\gamma = 3$. (b) The red curve shows the finite-momentum $Q = Q(1, 1)$ versus QGD (η). Meanwhile, the blue curve is the corresponding diode efficiency Θ . When increasing η to across the phase boundary $\eta = 0.26$, the diode efficiency Θ is enhanced dramatically, and the sign of Q is reversed. (c) and (d) The free energy (red curve) and supercurrent (blue curve) of 2 highlighted points in (a) with the temperature $T = 0.1T_0$ but different η [$\eta = 0.24$ in (c) and 0.29 in (d)]. Both the free energy and the supercurrent are normalized to $[-1, 1]$. The green point is in the $Q > 0$ phase, and the red point is in the $Q < 0$ phase.

crystalline symmetries of the Bloch functions. These results underscore the universality of our theory across various model Hamiltonians.

B. Flat-band helical pairing state and superconducting diode effect

Helical superconductivity can be realized in noncentrosymmetric superconductors with the aid of an in-plane magnetic field [5]. To emulate an analogous helical pairing state in a flat-band system with QGD, we introduce an additional inversion-breaking term manually into the ζ -lattice model. This is done by modifying (recall that the periodic function is $\alpha_{\mathbf{k},\sigma} = \zeta_\sigma p_{\mathbf{k}}$)

$$\alpha_{\mathbf{k},\sigma} = \zeta_\sigma p_{\mathbf{k}} + m_\sigma \gamma (\sin k_x + \sin k_y). \quad (\text{S31})$$

Here the γ term breaks the inversion symmetry in the Bloch wave function such that $\alpha_{-\mathbf{k},\sigma} \neq \alpha_{\mathbf{k},\sigma}$, while still preserving the time-reversal symmetry as $\alpha_{\mathbf{k},\uparrow} = \alpha_{-\mathbf{k},\downarrow}$ when $\eta = 0$. However, when $\eta \neq 0$, this term lifts $\pm Q$ degeneracy [see free energy plot in Fig. S5(d)], resulting in FF-type pairing state. Here the Cooper pair momentum is along the $\Gamma - M$ line as $\mathbf{q} = q(\hat{x}, \hat{y})$.

The mean-field numerical results are summarized in Fig. S5. In this case, the finite momentum pairing state is favored for any nonzero η . Interestingly, $Q(\eta)$ shows a sharp change near a critical η_c , manifested as the sign change [see Fig. S5 (b)], which suggests a first-order phase transition. In Fig. S5(a), we color the $Q > 0$ and $Q < 0$ regions in blue and yellow, respectively. This is an analog to the weak and strong helical superconducting states [5].

Recently, the superconducting diode effect (SDE), in which the magnitudes of the critical supercurrents differ in opposite directions, has been studied in helical superconductors [5, 6]. Inspired by this, here we also calculate the diode efficiency $\Theta = (J_{c+} - |J_{c-}|)/(J_{c+} + |J_{c-}|)$ in Fig. S5(b), where $J_{c\pm}$ is the upper and lower critical supercurrent

in the (1, 1)-direction. It shows that a dramatic enhancement of Θ appears near η_c , which is similar to what has been proposed in Ref. [5].

The supercurrent can be evaluated by $J(q) = 2[\partial F(q)]/\partial q$, and the minimized free energy can be calculated from Eq. (S29). To give a clearer picture of the phase transition, in Fig. S5(c) and (d), we show the free energy $F(q)$ (red curves) as well as the supercurrent $J(q)$ (blue curves) for the two points (green and red dots) in Fig. S5(a). In the blue region [Fig. S5 (a)], the global minimum of the free energy $F(q)$ is at $Q > 0$ [Fig. S5 (c)], but $|Q|$ is small, associated with the small diode efficiency. When $\eta > \eta_c$ [Fig. S5 (d)], the pairing state with $Q < 0$ becomes the global minimum. In this case, an enhanced diode efficiency can be observed in Fig. S5(b).

C. Inter-flavor FFLO phase within pseudo-Landau levels

In this section, we explore the formation of the FFLO phase in pseudo-Landau level systems. As schematically illustrated in Fig. S6, we consider a strained spinless Dirac material with a series of pseudo-Landau levels under a uniform pseudomagnetic field. The K and K' valleys are time-reversal partners, which naturally leads to the inter-flavor Cooper pairings in the presence of attractive interactions. Under this scenario, we consider Cooper pairs form within Landau level n_1 from K valley and n_2 from K' valley. Here, an imbalance in the populations of the two flavors (such as valley polarization), i.e., $\mu_K \neq \mu_{K'}$ may lead to $n_1 \neq n_2$, where QGD comes into being as the quantum metrics of the different pseudo-Landau levels are distinct, and the FFLO phase can manifest. This scenario is also proposed in Skyrmion crystals [7].

To evaluate the critical temperature T_{cq} , we first use the form factor of Landau levels on the torus geometry to obtain the anomalous quantum distance [8]:

$$\langle u_{n_2}^K(\mathbf{k} + \mathbf{q}) | u_{n_1}^{*K'}(-\mathbf{k}) \rangle = \langle u_{n_2}^K(\mathbf{k} + \mathbf{q}) | u_{n_1}^K(\mathbf{k}) \rangle = e^{\frac{i}{2}(\mathbf{k} + \mathbf{q}) \times \mathbf{k}} \sqrt{\frac{n_1!}{n_2!}} L_{n_1}^{n_2-n_1}(qq^*) (iq)^{n_2-n_1} e^{-\frac{1}{2}qq^*}. \quad (\text{S32})$$

Here $q = (q_x + q_y)/\sqrt{2}$ and $L_n^m(x)$ is the generalized Laguerre polynomial. And we have used $|u_{n_1}^{*K'}(-\mathbf{k})\rangle = |u_{n_1}^K(\mathbf{k})\rangle$. Then we use the formula in main text $T_{cq}/T_0 = 1 - \bar{d}_A^2(\mathbf{q})$ to calculate T_{cq} as summarized in Fig. S6.

Firstly, we notice that the zero-momentum pairing is favored when $n_1 = n_2 = n$ as shown in Fig. S6(b). Notice that for Landau levels the quantum metric satisfies $\int \text{tr}[g_n(\mathbf{k})] = 2n + 1$. Thus $T_{cq}/T_0 = 1 - \frac{2n+1}{2}q^2$ for small q . In Fig. S6(c), when $n_1 \neq n_2$, it can be seen that a global maximum occurs at $q = Q$ corresponding to the FFLO phase.

We can make an rough estimation of the finite momentum Q by evaluating $\partial[L_{n_1}^{n_2-n_1}(q^2/2)(q/\sqrt{2})^{n_2-n_1}]/\partial q = 0$ at $q = Q$, leading to

$$-Q^2 L_{n_1-1}^{n_2-n_1+1}(Q^2/2) + (n_2 - n_1) L_{n_1}^{n_2-n_1}(Q^2/2) = 0. \quad (\text{S33})$$

Expanding it to the lowest order of Q and at large n_1, n_2 , we can make an approximate result as

$$Q \approx \sqrt{\frac{2(n_2 - n_1) \binom{n_2}{n_1}}{(n_2 - n_1 + 2) \binom{n_2}{n_1 - 1}}} = \sqrt{\frac{2(n_2 - n_1)(n_2 - n_1 + 1)}{n_1(n_2 - n_1 + 2)}}. \quad (\text{S34})$$

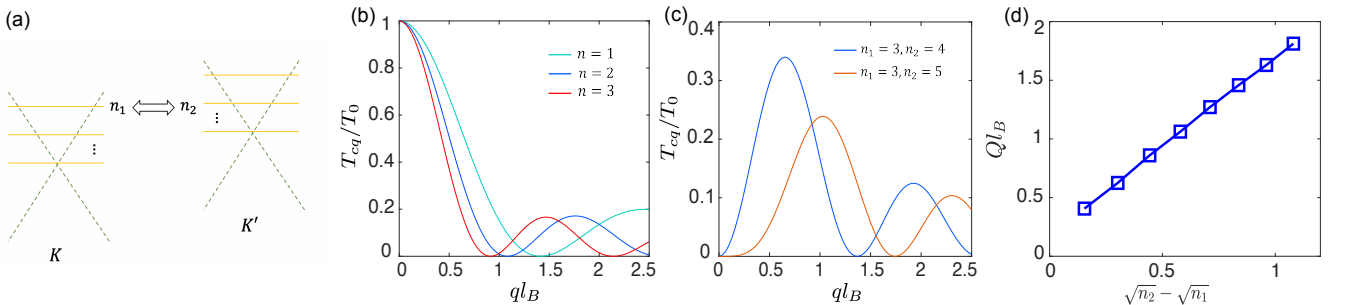


FIG. S6. (a) The inter-flavor (valley) pairing involves the electrons on Landau level n_1 from K valley and n_2 from K' valley. (b) The T_{cq} for the BCS-type pairing ($Q = 0$) at $n_1 = n_2 = n$. (c) Calculated T_{cq}/T_0 for the FFLO pairing at $n_1 \neq n_2$. (d) The stabilized finite momentum Q scales as $\sqrt{n_2} - \sqrt{n_1}$ at $n_1 = 10$.

For n_2 close to n_1 , $Q \sim \sqrt{n_2} - \sqrt{n_1}$ as demonstrate in Fig. S6(d). We can also observe a direct relationship between Q and the QGD, as the quantum metrics of the different Landau levels are distinct.

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