Supplementary information for: The role of master clock stability in quantum information processing

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In this supplementary material we provide the reader with full theoretical details required in order to understand and quantitatively apply the concepts reviewed in the main text to an experimental system. As noted in the main text the concepts we introduce are well known in various fields, but here present a single, comprehensive, and notationally consistent exposition of the relevant content, drawing from sources in the precision oscillator/clock community, signal processing, and quantum information.

I. MATHEMATICAL BACKGROUND AND DEFINITIONS

In this section we provide the mathematical definition of the power spectral density (PSD) of a time-dependent stochastic field, and derive some familiar properties. This derivation follows the treatment presented in Ref. [1]. Let $y(t)$ denote a real-valued stochastic process defined continuously on the time domain. For a set $\mathcal{E}$ of individual realizations of this process, let $y_k(t)$, $k \in \mathcal{E}$, denote the kth realization. Further, let $Y_{k,t} \equiv y_k(t)$ denote the (scalar) random variable taking the value of the kth realization at fixed time t. We introduce the following notation to distinguish between the *ensemble* average and the *time* average:

$$\langle y(t)\rangle \equiv \lim_{|\mathcal{E}|\to\infty} \frac{1}{|\mathcal{E}|}\sum_{k\in\mathcal{E}} Y_{k,t} \qquad (\textit{ensemble average}), \quad (1)$$

$$\overline{y_k(t)} \equiv \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} y_k(t)dt \quad (\textit{time average}), \qquad (2)$$

where $|\mathcal{E}|$ is the number of elements in the set $\mathcal{E}$, and T is the duration of the measurement time spanned by the time window $[-\frac{T}{2}, \frac{T}{2}]$. We explicitly include the subscript k in the expression for the time average to emphasize that this is computed over a *single* realization of $y(t)$. We assume the following properties of $y(t)$:

1. $y(t)$ is an ergodic process.
2. $y(t)$ is a wide-sense stationary (w.s.s.) process.
3. $y(t)$ is a zero-mean process: $\langle y(t)\rangle = \overline{y_k(t)} = 0$ for all t.

The assumption of ergodicity implies that, in the limit of infinite ensemble sizes, the ensemble mean of $y(t)$ – or the ensemble mean of a function of $y(t)$ – approaches the corresponding *time-average* over any given realization $y_k(t)$. That is,

$$\langle f(y(t))\rangle \longleftrightarrow \overline{f(y_k(t))}. \qquad (3)$$

The assumption of wide-sense stationarity carries with it two implications. First, the ensemble average is invariant with respect to translation in time, namely $\langle y(t)\rangle = \langle y(t+\tau)\rangle$ for all τ. Second, the autocorrelation function $C_y(t_1,t_2) \equiv \langle y(t_1)y(t_2)\rangle$ describing correlations between $y(t)$ at times t_1 and t_2 depends only on the time *difference*, and consequently may be parameterized by the single variable $\tau = t_2 - t_1$. Thus,

$$C_y(\tau) = \langle y(t_1)y(t_1+\tau)\rangle = \overline{y_k(t_1)y_k(t_1+\tau)} \qquad (4)$$

where the dependence on t_1 is averaged out, this being an arbitrary reference point, and the equivalence between the ensemble-averaged and time-averaged formulation in the second equality is permitted by ergodicity.

Next we define the PSD of $y(t)$ by transition from the time-domain description to a Fourier-domain description. In this paper we use the nonunitary angular frequency notation for the Fourier transform, specifically defining the Fourier transform pair

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt \qquad (5)$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}d\omega \qquad (6)$$

where $\omega = 2\pi\nu$ is the angular frequency in radians per second, and ν is the frequency in Hz. We define the time-gated function

$$y_T(t) = \begin{cases} y(t) & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & |T| > \frac{T}{2} \end{cases} \qquad (7)$$

associated with sampling the signal over the finite time window $[-\frac{T}{2}, \frac{T}{2}]$ as done in an actual measurement. The Fourier transform of this time-gated function is therefore

$$Y_T(\omega) = \int_{-\infty}^{\infty} y_T(t)e^{-i\omega t}dt = \int_{-T/2}^{T/2} y(t)e^{-i\omega t}dt \qquad (8)$$

where the last equality yields the equivalent interpretation of $Y_T(\omega)$ as the truncated Fourier transform of the un-gated signal. This definition avoids the difficulty that the full Fourier transform may not be well defined if the integral over $y(t)$ fails to converge in the limit $T \to \infty$. The power spectral density (PSD) of $y(t)$ may then be defined by

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle |Y_T(\omega)|^2 \right\rangle. \tag{9}$$

Writing this out explicitly we obtain

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle \int_{-T/2}^{T/2} y^*(t_1) e^{i\omega t_1} dt_1 \int_{-T/2}^{T/2} y(t_2) e^{-i\omega t_2} dt_2 \right\rangle \tag{10}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \langle y(t_1) y(t_2) \rangle \, e^{i\omega(t_1-t_2)} dt_1 dt_2 \tag{11}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} C_y(\tau) e^{-i\omega\tau} dt_1 dt_2, \tag{12}$$

where we have used the fact that $y(t)$ is real, and invoked Eq. 4 with the time difference variable $\tau = t_2 - t_1$. Transforming the integral domain we get

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} (T - |\tau|) C_y(\tau) e^{-i\omega\tau} d\tau \tag{13}$$

$$= \lim_{T\to\infty} \int_{-T}^{T} C_y(\tau) e^{-i\omega\tau} d\tau \tag{14}$$

$$+ \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} |\tau| \, C_y(\tau) e^{-i\omega\tau} d\tau. \tag{15}$$

Assuming $y(t)$ has *finite* temporal correlations, appropriate for realistic physical processes, the autocorrelation function $C_y(\tau)$ vanishes in the limit $\tau \to \infty$. Consequently the second term above vanishes in the limit $T \to \infty$, yielding

$$S_y(\omega) = \int_{-\infty}^{\infty} C_y(\tau) e^{-i\omega\tau} d\tau, \tag{16}$$

$$C_y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) e^{i\omega\tau} d\omega. \tag{17}$$

That is, $S_y(\omega)$ and $C_y(\tau)$ form a Fourier transform pair, which is a statement of the Wiener-Khintchine theorem [1–3].

The power spectral density formulated in Eq. 9 is by definition a non-negative even function on the real domain $\omega \in (-\infty, \infty)$. This is known as the *bilateral* PSD, with total power (or mean-square fluctuation) given by integrating over both positive and negative frequency domain:

$$P_{\text{tot}} = C_y(0) = \left\langle y(t)^2 \right\rangle \tag{18}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega \tag{19}$$

$$= \int_{-\infty}^{\infty} S_y(\nu) d\nu, \quad \omega = 2\pi\nu. \tag{20}$$

The bilateral PSD naturally arises in mathematical analysis involving Fourier transformations, satisfying the Wiener-Khinchine theorem describing its relationship with the autocorrelation function as in Eqs. 16 and 17.

It is also useful when considering systems subject to quantum noise, for which the corresponding quantum operator $\hat{y}(t)$ in general does not commute at different times and is in principle a complex number, i.e., $\langle \hat{y}(t) \hat{y}(t+\tau) \rangle = a + ib$. The spectral density of a complex number is no longer an even function in ω so must be specified for both positive and negative frequencies. Such non-symmetric spectra arise, for example, in the context of qubit stimulated and spontaneous emission and absorption to/from the environment.

For real-valued $y(t)$, however, the bilateral PSD is symmetric and may be replaced by the *unilateral* PSD, more typically used in engineering and metrological applications dealing with manifestly classical signals. Environmental fluctuations responsible for qubit dephasing fall into this category. For clarity, in this paper we use the following notation to keep the usage of bilateral (two-sided) and unilateral (one-sided) PSDs distinct:

$$S_y^{(2)}(\omega) \qquad (\textit{bilateral}), \tag{21}$$

$$S_y^{(1)}(\omega) \qquad (\textit{unilateral}) \tag{22}$$

where the unilateral PSD is defined by

$$S_y^{(1)}(\omega) = \begin{cases} 2S_y^{(2)}(\omega) & \text{for } \omega \geq 0 \\ 0 & \text{for } \omega < 0 \end{cases}. \tag{23}$$

Alternatively, we may explicitly incorporate this distinction by returning to Eq. 9, and defining from the outset

$$S_y^{(2)}(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle |Y_T(\omega)|^2 \right\rangle, \qquad \omega \in (-\infty, \infty) \tag{24}$$

$$S_y^{(1)}(\omega) = \lim_{T\to\infty} \frac{2}{T} \left\langle |Y_T(\omega)|^2 \right\rangle, \qquad \omega \in [0, \infty). \tag{25}$$

With these definitions the total power in the signal is given by

$$P_{\text{tot}} = \int_{-\infty}^{\infty} S_y^{(2)}(\nu) d\nu \tag{26}$$

$$= 2 \int_{0}^{\infty} S_y^{(2)}(\nu) d\nu \tag{27}$$

$$= \int_{0}^{\infty} S_y^{(1)}(\nu) d\nu. \tag{28}$$

II. RELATIONSHIP BETWEEN $S_{\delta\omega_{\mathrm{LO}}}(\omega)$ AND $S_{\phi_N}(\omega)$

A. PSDs for phase and frequency noise

The instantaneous frequency of a local oscillator is given by the time derivative of the instantaneous phase. Fluctuations in the phase consequently gives rise to fluctuations in frequency, implying a relationship between the PSDs of the phase and frequency noise fields. This relationship is well-known, as set out in Ref. [4]. In this section we reproduce the relevant results in a manner consistent with the notation introduced above. Consider a local oscillator

$$\Omega(t)\cos[\omega_{\mathrm{LO}}t+\phi_N(t)] \tag{29}$$

with instantaneous frequency

$$\omega(t)\equiv\frac{d}{dt}\left[\omega_{\mathrm{LO}}t+\phi_N(t)\right]=\omega_{\mathrm{LO}}+\dot{\phi}_N(t), \tag{30}$$

or

$$\omega(t)=\omega_{\mathrm{LO}}+\delta\omega_{\mathrm{LO}}(t) \tag{31}$$

where the (angular) frequency noise takes the form

$$\delta\omega_{\mathrm{LO}}(t)=\dot{\phi}_N(t). \tag{32}$$

Following Eq. 8 we define the truncated Fourier transforms for phase and frequency noise fields

$$\Phi_T(\omega)\equiv\int_{-T/2}^{T/2}\phi_N(t)e^{-i\omega t}dt, \tag{33}$$

$$W_T(\omega)\equiv\int_{-T/2}^{T/2}\delta\omega_{\mathrm{LO}}(t)e^{-i\omega t}dt. \tag{34}$$

Then the phase and frequency noise PSDs are defined by

$$S^{(2)}_{\phi_N}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\Phi_T(\omega)\right|^2\right\rangle, \tag{35}$$

$$S^{(2)}_{\delta\omega_{\mathrm{LO}}}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|W_T(\omega)\right|^2\right\rangle. \tag{36}$$

Using the definition of $\Phi_T(\omega)$, the fact that $\delta\omega_{\mathrm{LO}}(t)=\dot{\phi}_N(t)$, and integrating by parts we find

$$W_T(\omega)=i\omega\Phi_T(\omega)+R, \tag{37}$$

$$R\equiv\phi_N(T/2)e^{-i\omega T/2}-\phi_N(-T/2)e^{i\omega T/2}, \tag{38}$$

where we have used the Fourier relation $d/dt\leftrightarrow i\omega$, and the residual term R comes from the integral limits after integrating by parts. Taking the modulus square we therefore obtain

$$\left|W_T(\omega)\right|^2=\omega^2\left|\Phi_T(\omega)\right|^2+R', \tag{39}$$

where the residual terms are absorbed into

$$R'=|R|^2+[i\omega\Phi_T(\omega)R^*]+[-i\omega\Phi_T(\omega)^*R]. \tag{40}$$

Consequently

$$S^{(2)}_{\delta\omega_{\mathrm{LO}}}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\omega^2\left|\Phi_T(\omega)\right|^2+R'\right\rangle \tag{41}$$

$$=\omega^2\left[\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\Phi_T(\omega)\right|^2\right\rangle\right]+\lim_{T\to\infty}\frac{1}{T}\left\langle R'\right\rangle. \tag{42}$$

The term in square brackets is the definition of $S^{(2)}_{\phi_N}(\omega)$ given in Eq. 35. The second term, on the other hand, vanishes in the limit $T\to\infty$ as the residual term R' does not grow sufficiently rapidly with T. We therefore set

$$\lim_{T\to\infty}\frac{1}{T}\left\langle R'\right\rangle=0 \tag{43}$$

from which it follows (see Eqs. 20-21, p. TN197 of Ref. [4])

$$S^{(2)}_{\delta\omega_{\mathrm{LO}}}(\omega)=\omega^2S^{(2)}_{\phi_N}(\omega). \tag{44}$$

Now define the noise field

$$\beta(t)\equiv\alpha\delta\omega_{\mathrm{LO}}(t)=\alpha\dot{\phi}_N(t) \tag{45}$$

by simply scaling the frequency noise by some factor α. Substituting this into the above derivation we obtain the slightly more general PSD relation

$$S^{(2)}_{\beta}(\omega)=\alpha^2S^{(2)}_{\delta\omega_{\mathrm{LO}}}=\alpha^2\omega^2S^{(2)}_{\phi_N}(\omega). \tag{46}$$

Clearly, substituting the unilateral counterparts to the PSDs in these relations simply multiplies both sides of the equation by a factor of 2. Consequently Eqs. 44 and 46 apply to both bilateral and unilateral PSDs.

B. Interpretation of $S_{\phi_N}(\omega)$: modulation theory

Here we provide an interpretation of the phase instability $S^{(2)}_{\phi_N}(\omega)$ defined in Eq. 35 as the ratio of the power in the sidebands relative to the power in the carrier. This relationship is set out in Ref. [4]. In this section we reproduce this result in a manner consistent with the notation introduced above. Consider the LO signal

$$V(t)=\Omega\cos[\omega_{\mathrm{LO}}t+\phi_N(t)], \tag{47}$$

where here the term Ω simply represents the amplitude of the sinusoidal signal and in a given realization over some window $[-\frac{T}{2},\frac{T}{2}]$, $\phi_N(t)$ has spectral components described by some Fourier spectrum. To obtain a simple interpretation of the resulting PSD $S_{\phi_N}(\omega)$, we restrict attention to the case where $\phi_N(t)$ consists of a single (phase-randomized) Fourier component

$$\phi_N(t)=\frac{\alpha_j}{2}\left(e^{i\omega_jt+\psi_j}+\text{c.c.}\right)=\alpha_j\cos(\omega_jt+\psi_j), \tag{48}$$

equivalent to imposing a narrow-band filter at Fourier frequency ω_j. The resulting waveform may be expressed in terms of an infinite comb of frequency components,

$$V(t)=\Omega\sum_{n=-\infty}^{\infty}J_n(\alpha_j)\cos[(\omega_{\mathrm{LO}}+n\omega_j)t+n\psi_j], \tag{49}$$

weighted by Bessel functions $J_n(x)$ and centred around the carrier frequency ω_{LO}. Assuming the phase fluctuations are small relative to the carrier frequency ($\alpha_j\ll\omega_{\mathrm{LO}}$) we may make the *small angle approximation* to obtain

$$V(t) \approx V_0(t) + V_+(t) + V_-(t), \tag{50}$$

where

$$V_0(t) \equiv \Omega \cos(\omega_{\rm LO} t) \qquad (\textit{LO/carrier}), \tag{51}$$
$$V_+(t) \equiv \frac{\Omega}{2}\alpha_j \cos[(\omega_+ t + \psi_j)] \qquad (\textit{blue sideband}), \tag{52}$$
$$V_-(t) \equiv \frac{\Omega}{2}\alpha_j \cos[(\omega_- t - \psi_j)] \qquad (\textit{red sideband}), \tag{53}$$

and $\omega_\pm \equiv \omega_{\rm LO} \pm \omega_j$ are the positive- and negative-detuned frequencies of the principle (first-order) sidebands on the carrier. Using Eqs. 8 and 9 we may now compute the PSDs for $\phi_N(t)$, $V_0(t)$ and $V_\pm(t)$.

Substituting Eq. 48 into Eq. 8 (or more specifically, Eq. 33), the Fourier transform of the time-gated phase fluctuation is given by

$$\Phi_T(\omega) = \frac{\alpha_j T}{2}\Big(e^{i\psi_j}\,\text{sinc}\,[T(\omega-\omega_j)/2] + e^{-i\psi_j}\,\text{sinc}\,[T(\omega+\omega_j)/2]\Big). \tag{54}$$

Substituting this into Eq. 9 (or more specifically, Eq. 35) the (bilateral) PSD is therefore given by

$$S^{(2)}_{\phi_N}(\omega) = \lim_{T\to\infty} \frac{\alpha_j^2}{2}\Big(\frac{T}{2}\text{sinc}^2\,[T(\omega-\omega_j)/2] + \frac{T}{2}\text{sinc}^2\,[T(\omega+\omega_j)/2]\Big), \tag{55}$$

where cross products in the modulus square between the *different* frequency components, $\text{sinc}[T(\omega \pm \omega_j)/2]$, are omitted as these vanish in the limit $T \to \infty$. Carrying out the limit, subject to the condition that the area under the $\text{sinc}^2(x)$ curve remains equal to π, we obtain

$$S^{(2)}_{\phi_N}(\omega) = \frac{\pi}{2}\alpha_j^2\left[\delta(\omega-\omega_j) + \delta(\omega+\omega_j)\right]. \tag{56}$$

Similarly,

$$S^{(2)}_{V_0}(\omega) = \frac{\pi}{2}\Omega^2\left[\delta(\omega-\omega_{\rm LO}) + \delta(\omega+\omega_{\rm LO})\right], \tag{57}$$
$$S^{(2)}_{V_\pm}(\omega) = \frac{\pi}{2}\left(\frac{\Omega\alpha_j}{2}\right)^2\left[\delta(\omega-\omega_\pm) + \delta(\omega+\omega_\pm)\right]. \tag{58}$$

Let $P_y(\omega, \Delta)$ denote the power in the signal $y(t)$ in (angular) frequency bandwidth Δ centred on (angular) frequency ω, namely over bandwidth $[\omega - \Delta/2, \omega + \Delta/2]$. Then

$$P_y(\omega,\Delta) = \frac{1}{2\pi}\int_{\omega-\Delta/2}^{\omega+\Delta/2} 2S^{(2)}_y(\omega')d\omega', \tag{59}$$

where the factor of 2 inside the integral accounts for the equal contribution to the total power in this bandwidth from the negative frequency domain of the bilateral PSD. In our case, for any $\Delta > 0$, we obtain

$$P_{\phi_N}(\omega_j,\Delta) = \frac{1}{2}\alpha_j^2, \tag{60}$$
$$P_{V_0}(\omega_{\rm LO},\Delta) = \frac{1}{2}\Omega^2, \tag{61}$$
$$P_{V_\pm}(\omega_\pm,\Delta) = \frac{1}{2}\left(\frac{\Omega\alpha_j}{2}\right)^2. \tag{62}$$

The ratio of the total power *in both sidebands* relative to the carrier is therefore given by

$$\frac{P_{V_+}(\omega_+,\Delta) + P_{V_-}(\omega_-,\Delta)}{P_{V_0}(\omega_{\rm LO},\Delta)} = \frac{1}{2}\alpha_j^2 = P_{\phi_N}(\omega_j,\Delta). \tag{63}$$

In particular, setting $\Delta = 1/2\pi$ rad $= 1$ Hz and dividing the power by the measurement bandwidth, we obtain

$$\frac{P_{\phi_N}(\omega_j,\Delta)}{\Delta} = \pi\alpha_j^2 \qquad [\text{rad}^2/\text{rad}]. \tag{64}$$

Using Eq. 56, this compares directly with the (unilateral) PSD for $\phi_N(t)$ (i.e. the phase instability)

$$S^{(1)}_{\phi_N}(\omega) = \pi\alpha_j^2\delta(\omega-\omega_j). \tag{65}$$

According to Eq. 63, the unilateral (one-sided) phase instability PSD $S^{(1)}_{\phi_N}(\omega)$ may therefore be interpreted as the ratio of the power in the both first-order sidebands relative to the power in the carrier measured over a 1 Hz bandwidth (compare with discussion surrounding Eq. 32, p. TN200 of Ref. [4]).

III. QUBIT ERRORS ARISING FROM PHASE NOISE

As detailed in section II, phase fluctuations in the LO map to frequency fluctuations resulting in the instantaneous LO frequency $\omega_{\rm LO}+\delta\omega_{\rm LO}(t)$, where $\delta\omega_{\rm LO}(t) = \dot{\phi}_N(t)$, and giving rise to the relationship between the PSDs for phase and frequency fluctuations expressed in Eq. 46. For a qubit driven by a (nominally) resonant LO, phase fluctuations therefore map to frequency detuning noise which give rise to dephasing errors [5, 6]. These errors are typically computed in terms of the associated "dephasing noise field" appearing in the qubit Hamiltonian in the *interaction picture rotating with the LO's instantaneous frequency*. The dephasing field appearing in this interaction picture, however, is different from $\delta\omega_{\rm LO}(t)$ by a pre-factor, which must be accounted for when computing errors in this way. In this section the qubit Hamiltonian in the interaction picture is derived to identify this pre-factor. Once identified, the correct relationship between the LO phase instability $S_{\phi_N}(\omega)$ and the dephasing noise PSD $S_z(\omega)$ is expressed.

A. Hamiltonian in interaction picture rotating with Instantaneous LO frequency

The Hamiltonian of a single qubit in the canonical "rotating frame" (i.e. rotating at the frequency ω_0 associated with the qubit's energy splitting) takes the generalized time-dependent form

$$H(t) = \big[\boldsymbol{h}(t) + \boldsymbol{\delta h}(t)\big]\boldsymbol{\sigma}. \tag{66}$$

Here $\boldsymbol{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)^T$ denotes a column vector of Pauli matrices, and the row vectors $\boldsymbol{h} = (h_x, h_y, h_z)$ and $\boldsymbol{\delta h} = (\delta h_x, \delta h_y, \delta h_z)$ denote the ideal (e.g. control) and fluctuating (noisy) Cartesian components of the respective Hamiltonian terms expressed in the basis of Pauli operators [7, 8]. The scalar noise fields $\delta h_{x,y,z}(t)$ model semi-classical processes in each of the three Cartesian directions. In particular, dephasing processes are accounted for by the dephasing noise field $\delta h_z(t)$, generating stochastic rotations about the $\hat{z}$-axis ($\propto \hat{\sigma}_z$) of the Bloch sphere.

In this paper, the canonical rotating frame is generalized to an interaction picture rotating with the LO's instantaneous frequency $\omega_{\mathrm{LO}} + \delta\omega_{\mathrm{LO}}(t)$. This interaction picture reduces to the canonical rotating frame in the case of an ideal ($\delta\omega_{\mathrm{LO}}(t) = \dot{\phi}_N(t) = 0$) resonant LO, since in this case the instantaneous frequency is given by the constant $\omega_{\mathrm{LO}} = \omega_0$. Restricting attention to the dephasing noise by setting $\delta h_{x,y}(t) = 0$, we show the total Hamiltonian in this interaction picture takes the form

$$H(t) = \boldsymbol{h}(t)\boldsymbol{\sigma} + \delta h_z(t)\hat{\sigma}_z, \tag{67}$$

$$\delta h_z(t) = \delta h_z^{(env)}(t) + \delta h_z^{(\mathrm{LO})}(t) \tag{68}$$

where the dephasing field separates into a component due strictly to *environmental dephasing*, $\delta h_z^{(env)}(t)$, and a component induced by phase fluctuations in the LO, $\delta h_z^{(\mathrm{LO})}(t)$, both on formally equal footing.

We begin with the physical system Hamiltonian ($\hbar = 1$) in the laboratory frame for a qubit possessing a nominal transition (angular) frequency ω_0, driven by a local oscillator with carrier frequency ω_{LO},

$$\begin{aligned} H_S =& \frac{1}{2}\omega_0\hat{\sigma}_z + \frac{1}{2}\delta\omega_0(t)\hat{\sigma}_z \\ &+ \Omega(t)\cos\Big(\omega_{\mathrm{LO}}t + [\phi_C(t) + \phi_N(t)]\Big)\hat{\sigma}_x. \end{aligned} \tag{69}$$

Appearing in this expression are the operators $\hat{\sigma}_i$ representing the Pauli matrices in the Cartesian coordinate basis. The first term corresponds to the Hamiltonian of the qubit under free evolution, while the second term captures environmental noise (e.g. magnetic field fluctuations), which effectively change the nominal qubit transition frequency by an amount $\delta\omega(t)$ as

$$\omega_0 \to \omega_0 + \delta\omega_0(t). \tag{70}$$

The third term corresponds to the qubit-field interaction with amplitude $\Omega(t)$ driven by a local oscillator with nominal carrier frequency ω_{LO}. The time-dependent phase of the LO is partitioned into a "control" component (desired phase evolution), $\phi_C(t)$, and a "noise" component (unwanted phase fluctuations), $\phi_N(t)$. The LO phase fluctuation produces a time-dependent frequency detuning through its time derivative, $\delta\omega_{\mathrm{LO}}(t) \equiv \dot{\phi}_N(t)$, effectively transforming the LO frequency as

$$\omega_{\mathrm{LO}} \to \omega_{\mathrm{LO}} + \delta\omega_{\mathrm{LO}}(t). \tag{71}$$

We now perform the transformation to the interaction picture rotating at the instantaneous LO frequency. This task is split into two steps. First, move to the frame rotating at the carrier frequency ω_{LO} (corresponding to the canonical rotating frame when the resonance condition $\omega_{\mathrm{LO}} = \omega_0$ holds). Second, perform the additional transformation associated with rotations due to the frequency detuning $\delta\omega_{\mathrm{LO}}(t)$.

The first of these transformations corresponds to the interaction-picture Hamiltonian

$$H_I^{(\omega_{\mathrm{LO}})} \equiv U_{\omega_{\mathrm{LO}}}^\dagger H_S U_{\omega_{\mathrm{LO}}} - H_{\omega_{\mathrm{LO}}} \tag{72}$$

where $U_{\omega_{\mathrm{LO}}}(t) \equiv \exp[-i\frac{\omega_{\mathrm{LO}}}{2}t\hat{\sigma}_z]$ is the evolution operator for the Hamiltonian term $H_{\omega_{\mathrm{LO}}} \equiv \frac{1}{2}\omega_{\mathrm{LO}}\hat{\sigma}_z$. Carrying out the transformation and making the rotating-wave approximation we obtain

$$\begin{aligned} H_I^{(\omega_{\mathrm{LO}})} =& \frac{1}{2}(\omega_0 - \omega_{\mathrm{LO}})\hat{\sigma}_z + \frac{1}{2}\delta\omega_0(t)\hat{\sigma}_z \\ &+ \frac{1}{4}\Omega(t)\Big\{e^{-i[\phi_C(t)+\phi_N(t)]}\hat{\sigma}_+ + e^{i[\phi_C(t)+\phi_N(t)]}\hat{\sigma}_-\Big\} \end{aligned} \tag{73}$$

where we define the qubit ladder operators $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$.

The second transformation corresponds to the interaction-picture Hamiltonian

$$H_I^{(\omega_{\mathrm{LO}},\dot{\phi}_N)} \equiv U_{\dot{\phi}_N}^\dagger H_I^{(\omega_{\mathrm{LO}})} U_{\dot{\phi}_N} - H_{\dot{\phi}_N} \tag{74}$$

where $U_{\dot{\phi}_N}(t) = \exp[-i\frac{\phi_N(t)}{2}\hat{\sigma}_z]$ is the evolution operator under the dephasing Hamiltonian $H_{\dot{\phi}_N} \equiv \frac{1}{2}\dot{\phi}_N(t)\hat{\sigma}_z$ induced by phase fluctuations in the LO. Setting the static LO detuning to zero ($\omega_0 - \omega_{\mathrm{LO}} = 0$) the transformed system Hamiltonian subject to LO phase fluctuations thereby takes the form

$$\begin{aligned} H_I^{(\omega_{\mathrm{LO}},\dot{\phi}_N)} =& \frac{1}{2}\delta\omega_0(t)\hat{\sigma}_z - \frac{1}{2}\dot{\phi}_N(t)\hat{\sigma}_z \\ &+ \frac{1}{2}\Omega(t)\Big\{\cos[\phi_C(t)]\hat{\sigma}_x + \sin[\phi_C(t)]\hat{\sigma}_y\Big\}. \end{aligned} \tag{75}$$

Comparing this with Eq. 67 we see the third term in Eq. 75 corresponds to the control component $\boldsymbol{h}(t)\boldsymbol{\sigma}$. The first two terms, on the other hand, represent the two dephasing terms identified in Eq. 68. Namely,

$$\delta h_z^{(env)}(t) = \frac{1}{2}\delta\omega_0(t) \tag{76}$$

$$\delta h_z^{(\mathrm{LO})}(t) = -\frac{1}{2}\delta\omega_{\mathrm{LO}}(t) = -\frac{1}{2}\dot{\phi}_N(t) \tag{77}$$

Arriving at this expression explicitly links LO phase fluctuations to a dephasing Hamiltonian that is *indistinguishable* from a generalized environmental bath Hamiltonian.

B. Relationship between $S_z(\omega)$ and $S_{\phi_N}(\omega)$

In the limit of a perfect qubit ($\delta\omega_0(t) = 0$), dephasing is induced by fluctuations in the LO phase exclusively. This is clear from Eqs. 68, 76 and 77 which, in this case, yield

$$\delta h_z(t) = \delta h_z^{(\mathrm{LO})}(t) = -\frac{1}{2}\delta\omega_{\mathrm{LO}}(t) = -\frac{1}{2}\dot{\phi}_N(t). \tag{78}$$

Comparing this with Eq. 46, we may identify $\delta h_z(t)$ with $\beta(t)$, setting $\alpha = -1/2$. Denoting the (bilateral) PSD of the dephasing field by $S_z^{(2)}(\omega)$ and using Eq. 46, we therefore obtain

$$S_z^{(2)}(\omega) = \frac{1}{4}S_{\delta\omega_{\mathrm{LO}}}^{(2)} = \frac{1}{4}\omega^2 S_{\phi_N}^{(2)}(\omega). \tag{79}$$

This relation also holds for the corresponding unilateral PSDs.

C. Relationship between $S_z(\omega)$ and tabulated single-sideband phase noise

Following Eq. 79, in the limit of a perfect qubit, the dephasing noise PSD describes fluctuations in the LO phase exclusively, as captured in the relation

$$S_z^{(1)}(\omega) = \frac{1}{4}\omega^2 S_{\phi_N}^{(1)}(\omega) \tag{80}$$

where $S_{\phi_N}^{(1)}(\omega)$ is the *phase instability*, defined as the *unilateral* PSD of the LO phase fluctuations. Although $S_{\phi_N}^{(1)}(\omega)$ is used in the metrological community, most LO manufacturers use a different metric to specify phase noise. Namely, the *single-sideband phase noise* defined by

$$\mathcal{L}(\omega) \equiv \frac{1}{2}S_{\phi_N}^{(1)}(\omega). \tag{81}$$

The phase noise $\mathcal{L}(\omega)$ is typically represented in logarithmic units of dBc/Hz, which we capture explicitly by writing (see Eq. 26, p. TN199 of Ref. [4])

$$\tilde{\mathcal{L}}(\omega) = 10\log_{10}[\mathcal{L}(\omega)]. \tag{82}$$

The Fourier frequency ω is then interpreted as an *offset* from the carrier, with $\tilde{\mathcal{L}}(\omega)$ taking units dBc/Hz and indicating the level of the sideband power relative to the carrier measured over a 1 Hz band. We convert from $\tilde{\mathcal{L}}(\omega)$ into linear units via the relationship $S_{\phi_N}^{(1)}(\omega) = 2\cdot 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}}$. Combining with Eq. 80 thus yields the final relationship

$$S_z^{(1)}(\omega) = \frac{1}{2}\omega^2 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}} \tag{83}$$

taking (logarithmic) single-sideband phase-noise specifications to unilateral dephasing power spectral densities.

D. Expressions for fidelity under phase noise

The quantities derived above may now be directly incorporated into expressions for the fidelity of arbitrary single-qubit operations in the presence of (LO-induced) pure-dephasing noise using Ref. [8]

$$\mathcal{F}_{av}(\tau) \approx \frac{1}{2}\{1 + \exp[-\chi(\tau)]\}, \tag{84}$$

$$\chi(\tau) = \left(\frac{1}{\pi}\right)\int_0^\infty \frac{d\omega}{\omega^2}S_z^{(1)}(\omega)\sum_{l\in\, x,y,z} G_{z,l}(\omega) \tag{85}$$

$$= \left(\frac{1}{4\pi}\right)\int_0^\infty d\omega S_{\phi_N}^{(1)}(\omega)\sum_{l\in\, x,y,z} G_{z,l}(\omega) \tag{86}$$

$$= \left(\frac{1}{2\pi}\right)\int_0^\infty d\omega 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}}\sum_{l\in\, x,y,z} G_{z,l}(\omega). \tag{87}$$

The transfer functions for the control, $G_{z,l}(\omega)$, may be calculated analytically as presented in Sec. IV. The latter two expression for $\chi(\tau)$ simply re-express the overlap integral in terms of various quantities encountered in describing phase noise.

IV. ANALYTIC FORMS OF FILTER FUNCTIONS

Here we give the explicit form for the filter-transfer functions for the qubit operations detailed in the main text. Namely, Ramsey (free evolution), spin echo, primitive π-pulse, and DCG/WAMF π-pulse. In the dephasing noise quadrature, the filter function in all cases takes the generic form

$$F_z(\omega) = \sum_{l\in\, x,y,z} G_{z,l}(\omega) \tag{88}$$

where the transfer functions for the control, $G_{z,l}(\omega)$, capture contributions along all Cartesian directions (indexed by l). In general all three components are non-zero as dephasing noise induces both dephasing and amplitude-damping rotations [8, 9] during non-commuting control operation (e.g. $\propto \hat{\sigma}_x$).

In the following sections we give the functional form only for the nonzero transfer functions $G_{z,l}(\omega)$ for each of the qubit operations listed above.

A. Ramsey sequence

For free evolution (*i.e.* zero driving field) over a duration τ the only nonzero transfer function component takes the form [10–13]

$$G_{zz}(\omega) = 4\sin^2\left(\frac{\omega\tau}{2}\right). \tag{89}$$

B. Spin echo

Spin echo is the DES version of the identity operation. This takes the form of free evolution over duration τ, but with an instantaneous π_x rotation at time $\tau/2$. The nonzero transfer function component takes the form

$$G_{zz}(\omega) = \left|1 + e^{i\omega\tau} - 2e^{i\omega\tau/2}\right|^2 = 16\sin^4\left(\frac{\omega\tau}{4}\right). \quad (90)$$

C. Primitive π-pulse ($\hat{X}$ gate)

We consider a finite-duration primitive π-pulse (or $\hat{X}$ gate), defined as a driven rotation ($\propto \hat{\sigma}_x$) through angle π about the $\hat{x}$-axis of the Bloch sphere. For a duration τ, the primitive $\hat{X}$ takes a constant Rabi rate $\Omega = \pi/\tau$. There are two nonzero transfer functions:

$$G_{zz}(\omega) = \left|\frac{\omega^2}{\omega^2 - \Omega^2}\left(e^{i\omega\tau} + 1\right)\right|^2, \quad (91)$$

$$G_{zy}(\omega) = \left|\frac{i\omega\Omega}{\omega^2 - \Omega^2}\left(e^{i\omega\tau} + 1\right)\right|^2. \quad (92)$$

The second line arises because of the non-commuting nature of the control ($\propto \hat{\sigma}_x$) and the dephasing noise ($\propto \hat{\sigma}_z$), as described in [8, 9].

D. DCG/WAMF π-pulse

This is a finite-strength sequence performing a net π rotation via three distinct steps, and has been identified both as a *dynamically corrected gate* (DCG) and a *Walsh amplitude modulated filter* (WAMF) [6, 14]. Let $\hat{R}_x(\theta, \tau)$ denote a primitive rotation of the Bloch vector about the $\hat{x}$-axis ($\propto \hat{\sigma}_x$) through an angle θ over a duration τ with Rabi rate $\Omega = \theta/\tau$ (e.g. $\hat{R}_x(\pi, \tau) \equiv \hat{X}$). Then the DCG/WAMF sequence of total duration τ is defined by the operator sequence

$$\hat{R}_x\left(\pi, \frac{\tau}{4}\right)\hat{R}_x\left(\pi, \frac{\tau}{2}\right)\hat{R}_x\left(\pi, \frac{\tau}{4}\right), \quad (93)$$

with transfer function components

$$G_{zy}(\omega) = \left|\frac{i4\pi\omega\tau e^{i\frac{\omega\tau}{2}}\left[\left(\tau^2\omega^2 + 8\pi^2\right)\cos\left(\frac{\omega\tau}{4}\right) + 2\left(\tau^2\omega^2 - 4\pi^2\right)\cos\left(\frac{\omega\tau}{2}\right)\right]}{\tau^4\omega^4 - 20\pi^2\tau^2\omega^2 + 64\pi^4}\right|^2, \quad (94)$$

$$G_{zz}(\omega) = \left|\frac{2\tau^2\omega^2 e^{i\frac{\omega\tau}{2}}\left[12\pi^2\cos\left(\frac{\omega\tau}{4}\right) + \left(\tau^2\omega^2 - 4\pi^2\right)\cos\left(\frac{\omega\tau}{2}\right)\right]}{\tau^4\omega^4 - 20\pi^2\tau^2\omega^2 + 64\pi^4}\right|^2. \quad (95)$$

V. THERMAL NOISE FLOOR

Here we consider the contribution to infidelity from the fundamental limit associated with a thermal noise floor. For instance, the theoretical minimum noise power for an oscillator coupled into 50 Ω RF cable is given by $P_{min} = kTB$ for bandwidth B, temperature T, and where k is the Boltzmann constant. Normalized to a 1Hz bandwidth this is expressed as absolute powers, either in W/Hz or dBm/Hz.

We assume an LO at 0 dBm with the minimal possible phase noise. Namely, with phase noise given by the power in the (white) thermal noise floor relative to the power in the carrier. We further impose a high-frequency cutoff ω_c on to the otherwise white thermal noise to account for filtering due to e.g. coaxial cable cutoff frequencies or narrowband filtering. Hence, the thermal noise floor may be characterized by

$$\tilde{\mathcal{L}}(\omega) = \begin{cases} \tilde{\mathcal{L}}_{min} & 0 \leq \omega \leq \omega_c \\ 0 & \omega > \omega_c \end{cases}. \quad (96)$$

where the value of $\tilde{\mathcal{L}}_{min}$ in dBc/Hz is assumed to be given by the value of P_{min} in dBm/Hz. These correspondences are shown in Table I for $T = 4$ K and $T = 290$ K.

T [K]	P_{min} [W/Hz]	P_{min} [dBm/Hz]	$\tilde{\mathcal{L}}_{min}$ [dBc/Hz]
4	5.5×10^{-23}	-192.6	-192.6
290	4.0×10^{-21}	-174.0	-174.0

TABLE I. Lower bounds on phase noise set by thermal noise floor.

The infidelity $\mathcal{I} = 1 - \mathcal{F}$ associated with this noise floor is given by

$$\mathcal{I} = 1 - \frac{1}{2}\Big[1 + e^{-\chi(\tau)}\Big], \tag{97}$$

where

$$\chi(\tau) = \frac{1}{2\pi}\int_0^\infty d\omega 10^{\frac{\tilde{\mathcal{L}}(\omega)}{10}} F_z(\omega) \tag{98}$$

$$= \frac{1}{2\pi} 10^{\frac{\tilde{\mathcal{L}}_{min}}{10}} \int_0^{\omega_c} d\omega F_z(\omega) \tag{99}$$

and $F_z(\omega) = \sum_{l\in\ x,y,z} G_{z,l}(\omega)$ is the filter function. Thus, the infidelity is given by the integral of the filter function over the band $[0, \omega_c]$ scaled by a constant factor associated with the strength of the thermal noise floor. For the qubit operations described in the main text we can obtain analytic expressions for this integral. We assume the cutoff frequency ω_c is fast compared to the duration τ of the qubit operation:

$$\omega_c \gg \frac{1}{\tau} \iff \omega_c\tau \gg 1. \tag{100}$$

In this case the bulk of the thermal noise floor lies in the bandwidth beyond the stopband of the filter function and the integral is very well approximated by its asymptotic value, linear in the cutoff frequency. Specifically

$$\chi(\tau) = \frac{\kappa\omega_c}{2\pi} 10^{\frac{\tilde{\mathcal{L}}_{min}}{10}}, \tag{101}$$

where

$$\kappa = 2 \qquad (\textit{Ramsey}), \tag{102}$$

$$\kappa = 6 \qquad (\textit{spin echo}), \tag{103}$$

$$\kappa = 2 \qquad (\textit{primitive } \pi\textit{-pulse}), \tag{104}$$

$$\kappa = 2 \qquad (\textit{DCG/WAMF } \pi\textit{-pulse}). \tag{105}$$

We derive these expressions below.

1. Ramsey sequence

In the case of free evolution we have

$$\int_0^{\omega_c} d\omega F_z(\omega) = \int_0^{\omega_c} 4\sin\left(\frac{\omega\tau}{2}\right)^2 d\omega \tag{106}$$

$$= 2\omega_c\left(1 - \frac{\sin(\omega_c\tau)}{\omega_c\tau}\right) \tag{107}$$

$$\approx 2\omega_c, \qquad \omega_c\tau \gg 1. \tag{108}$$

Consequently we arrive at a final approximate expression for the coherence integral

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{\tilde{\mathcal{L}}_{min}}{10}} (2\omega_c) \tag{109}$$

is proportional to the cutoff, with proportionality factor associated with strength of the thermal noise floor. Using the appropriate filter function, similar results hold for other qubit operations.

2. Spin echo

For the spin echo sequence we derive

$$\int_0^{\omega_c} d\omega F_z(\omega) = 6\omega_c. \tag{110}$$

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{\tilde{\mathcal{L}}_{min}}{10}} (6\omega_c). \tag{111}$$

3. Primitive π-pulse ($\hat{X}$ gate)

The form of the filter function becomes more complicated for nonzero-duration driven operations. For the primitive π pulse we have

$$F_z(\omega) = \frac{2\pi^2 z^2}{(\pi^2 - z^2)^2} + \frac{2z^4}{(\pi^2 - z^2)^2} + \frac{2\pi^2 z^2\cos(z)}{(\pi^2 - z^2)^2} + \frac{2z^4\cos(z)}{(\pi^2 - z^2)^2}, \tag{112}$$

where we introduce the dimensionless variable $z = \omega\tau$. The main contribution to the integral over the band $[0, \omega_c]$ for $\omega_c \gg 1/\tau$ comes from those terms in Eq. 112 which survive the limit $z \to \infty$. We find

$$\lim_{z\to\infty} \frac{2\pi^2 z^2}{(\pi^2 - z^2)^2} = 0,$$
$$\lim_{z\to\infty} \frac{2z^4}{(\pi^2 - z^2)^2} = 2,$$
$$\lim_{z\to\infty} \frac{2\pi^2 z^2\cos(z)}{(\pi^2 - z^2)^2} \to \left[\lim_{z\to\infty} \frac{2\pi^2 z^2}{(\pi^2 - z^2)^2}\right]\cos(z) = 0,$$
$$\lim_{z\to\infty} \frac{2z^4\cos(z)}{(\pi^2 - z^2)^2} \to \left[\lim_{z\to\infty} \frac{2z^4}{(\pi^2 - z^2)^2}\right]\cos(z) = 2\cos(z),$$

where in the last two lines we formally separate the oscillating (and bounded) cosine factors in order to extract the *functional* form of the surviving terms. Consequently,

$$F_z(\omega) \approx 2\left(1 + \cos(\omega\tau)\right), \qquad \omega\tau \gg 1, \tag{113}$$

and the integral is well approximated by

$$\int_0^{\omega_c} F_z(\omega)d\omega \approx \int_0^{\omega_c} 2\left(1 + \cos(\omega\tau)\right)d\omega \tag{114}$$

$$= \omega_c\left(2 + \frac{\sin(\omega_c\tau)}{\omega_c\tau}\right) \tag{115}$$

$$\approx 2\omega_c, \qquad \omega_c\tau \gg 1. \tag{116}$$

Hence

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{\tilde{\mathcal{L}}_{min}}{10}} (2\omega_c). \tag{117}$$

4. DCG/WAMF π-pulse

Similarly, expanding the filter function for DCG/WAMF π-pulse and taking the limit, the only surviving term takes the form

$$\lim_{z\to\infty} \frac{4z^8 \cos(\frac{z}{2})^2}{(64\pi^4 - 20\pi^2 z^2 + z^4)^2} \to 4\cos^2\left(\frac{z}{2}\right). \tag{118}$$

Consequently,

$$F_z(\omega) \approx 4\cos^2\left(\frac{\omega\tau}{2}\right), \qquad \omega\tau \gg 1, \tag{119}$$

and the integral is well approximated by

$$\int_0^{\omega_c} F_z(\omega) d\omega \approx \int_0^{\omega_c} 4\cos^2\left(\frac{\omega\tau}{2}\right) d\omega \tag{120}$$

$$= 2\omega_c\left(1 + \frac{\sin(\omega_c\tau)}{\omega_c\tau}\right) \tag{121}$$

$$\approx 2\omega_c, \qquad \omega_c\tau \gg 1. \tag{122}$$

Hence

$$\chi(\tau) = \frac{1}{2\pi} 10^{\frac{\tilde{\mathcal{L}}_{min}}{10}} (2\omega_c). \tag{123}$$

VI. PHASE NOISE IN LO UPCONVERSION

An experimental LO, operated near resonance with the qubit Larmor frequency will generally be upconverted from a lower-frequency phase reference. This process results in additional phase noise; assuming noiseless upconversion by a factor of N to reach the desired carrier, the carrier phase noise in log units $\mathcal{L}_{\text{carrier}}(\omega) \to \mathcal{L}_{\text{Ref}}(\omega) + 20\log N$ dB [4]. The second term on the right indicates that the frequency upconversion leads to an effective increase in the fractional phase instability of the carrier. The synthesis chain will also generally add some phase noise on top of this quantity.

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