

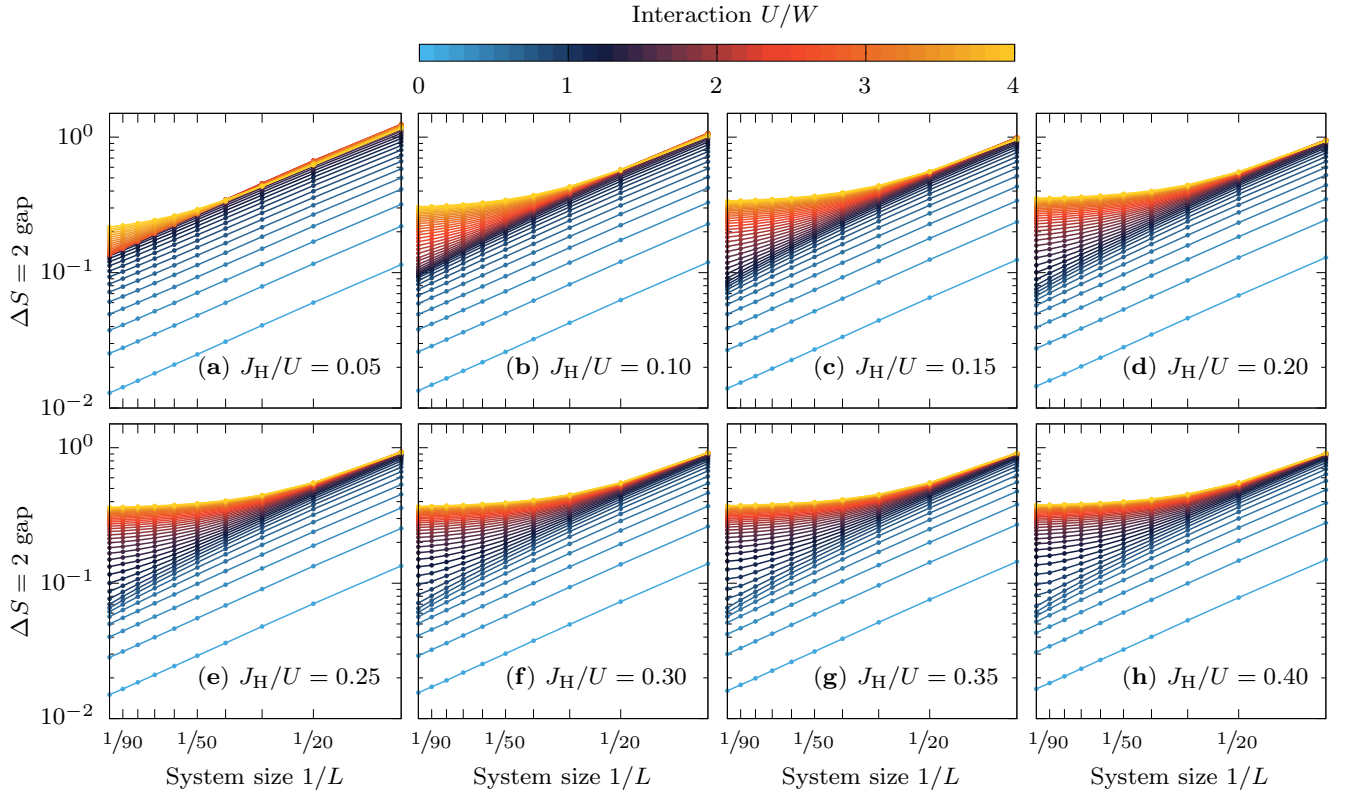
## SUPPLEMENTARY INFORMATION

### Transition to the Haldane phase driven by electron-electron correlations

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#### SUPPLEMENTARY NOTE 1: GAP ANALYSIS.

In Supplementary Fig. 1, we present the finite-size  $1/L$  and interaction  $U$  dependence of the  $\Delta S = 2$  gap (the magnon gap  $\Delta_S/J$ ) for various values of the Hund exchange  $J_H/U = 0.05, 0.10, \dots, 0.40$ . The main text displays the results of  $1/L$  extrapolations of this data in Fig. 2B.

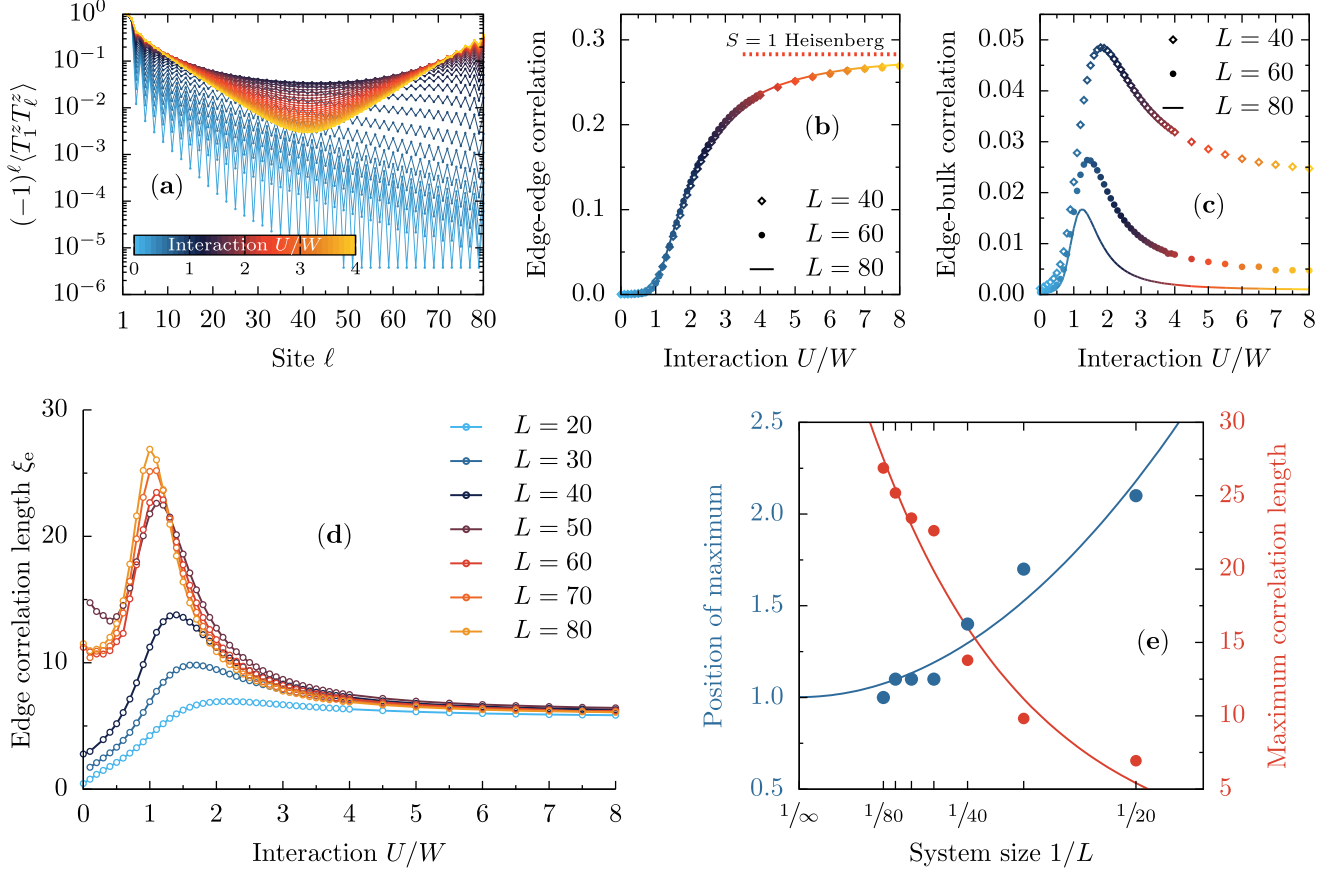


Supplementary Figure 1. **Spin gaps.** Finite-size scaling of  $\Delta S = 2$  excitations (magnon gaps) for  $J_H/U = 0.05, 0.10, \dots, 0.40$  and  $L \in \{10, 20, \dots, 100\}$ . Line color-code represents the value of interaction  $U$ . All data in units of spin exchange  $J = 2t^2/(U + J_H)$ . The saturation to the finite value (to the Haldane gap  $\Delta_S/J \simeq 0.41$ ) is clearly visible in all panels.

#### SUPPLEMENTARY NOTE 2: STATIC AND DYNAMIC SPIN CORRELATIONS.

In the main text, we have described how the dynamical spin-spin correlation, i.e., the non-local Green's function  $\langle\langle T_1^z T_\ell^z \rangle\rangle_{\omega=0}$ , behave vs. distance  $\ell$  and strength of the interaction  $U$ . Here, we present additional results for the static spin correlations,  $\langle T_1^z T_\ell^z \rangle = \langle \text{gs} | T_1^z T_\ell^z | \text{gs} \rangle$ .

In Supplementary Fig. 2, we present the analysis of the static  $\langle T_1^z T_\ell^z \rangle$ , similar to the one presented in Fig. 3 of the main text. As evident from panels (a-d), the overall behaviour of  $\langle T_1^z T_\ell^z \rangle$  is almost identical to the zero-frequency  $\omega = 0$  data. The main difference between the static and the dynamic correlation function can be observed in the extracted edge correlation length  $\xi_e$  close to the transition  $U \sim U_c$  [compare Fig. 3C of the main text and Supplementary Fig. 2(d)]. I.e., the edge correlation length extracted from  $\langle\langle T_1^z T_\ell^z \rangle\rangle_{\omega=0}$  data is much sharper than the



Supplementary Figure 2. **Static spin correlations.** (a) Distance  $\ell$  dependence of static spin-spin correlations  $(-1)^\ell \langle T_1^z T_\ell^z \rangle$  for various values of interaction  $U$  (denoted by color code). The results are normalized by the  $\ell = 1$  value of the correlation function. (b) Edge-edge  $|\langle T_1^z T_L^z \rangle|$  and (c) edge-bulk  $|\langle T_1^z T_{L/2}^z \rangle|$  spin correlations vs. interaction strength  $U/W$ . At  $U_c$ , one observes the appearance of finite edge-edge correlations, saturating at  $U \gg W$  to the value given by the  $S = 1$  Heisenberg model (red dashed line). (d) Edge correlation length  $\xi_e$  vs. interaction  $U$  strength, extracted from  $(-1)^\ell \langle T_1^z T_\ell^z \rangle = a \exp(-\ell/\xi_e)$  for  $\ell < L/2$ . (e) System-size dependence of the position of (blue points, left y-axis) and the value (red points, right y-axis) of the maximum edge correlation length. All data calculated for  $J_H/U = 0.25$ .

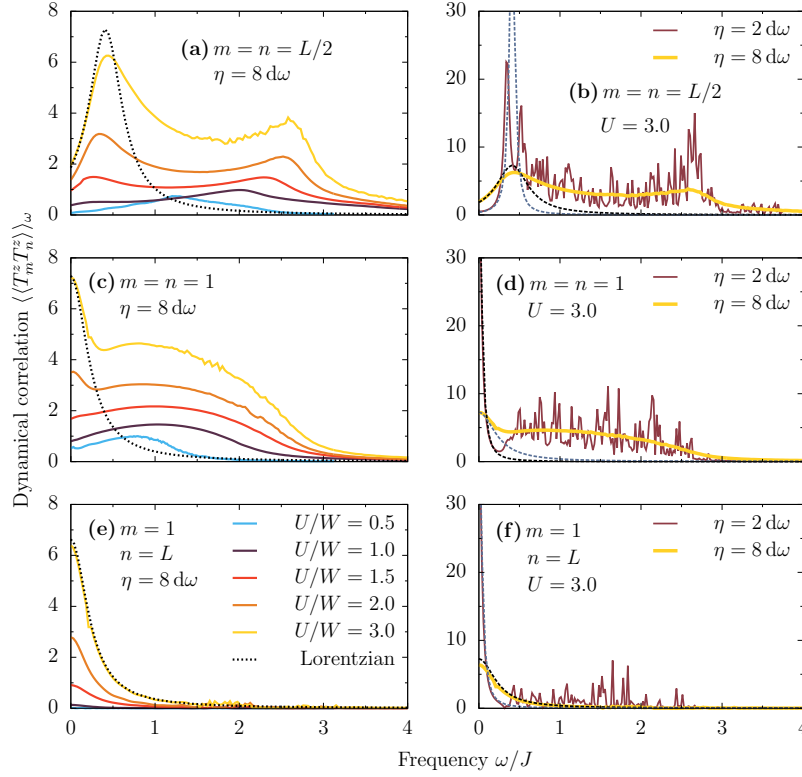
one extracted from the static  $\langle T_1^z T_\ell^z \rangle$ . Nevertheless, the scaling with the system size [see Supplementary Fig. 2(e)] of the position of the maximum of  $\xi_e$  as well as the value itself indicate a transition at  $U_c$ .

To understand the difference between static and dynamic results, consider the sum rule relating (at zero temperature) both of these quantities, i.e.,

$$\langle T_m^z T_n^z \rangle = \int_0^\infty d\omega \langle \langle T_m^z T_n^z \rangle \rangle_\omega. \quad (1)$$

It is evident that both approaches would yield the same behaviour if  $\langle \langle T_m^z T_n^z \rangle \rangle_\omega = \delta(\omega)$ . However, our analysis presented in Supplementary Fig. 3 indicates that the behaviour of the non-local Green's function  $\langle \langle T_m^z T_n^z \rangle \rangle_\omega$  strongly depends on the pair of sites  $(m, n)$  considered. Note that, in the presented analysis, we have used finite broadening  $\eta$  [see Eq. (5)]. Consequently, all sharp features of the spectrum are broadened by a Lorentzian.

Within the fermionic Haldane phase,  $U > U_c$ , the spectrum of the bulk of the system,  $m \sim n \sim L/2$ , has only the incoherent part  $\omega > 0$ , and it is gapped; see Supplementary Fig. 3(a,b). Such behaviour is expected because  $(m, n)$  elements contribute (via the Fourier transform) to the overall dispersion relation presented in Fig. 1 of the main text. On the other hand, the correlation between the edge  $m = 1$  and the rest of the system needs more attention: (i) consistent with the phenomenology of zero-energy edge modes, the edge-edge  $(m, n) = (1, L)$  dynamical correlations contain only the  $\omega = 0$   $\delta$ -function (broaden by the Lorentzian in our numerical investigation), see



Supplementary Figure 3. **Dynamical spin correlations**  $\langle T_m^z T_n^z \rangle_\omega$ . Shown are the frequency  $\omega$  dependence of the local Green's function in (a,b) the bulk of the system  $m = n = L/2$  and in (b,c) at the edge  $m = n = 1$ . (d,e) The non-local Green function between edges of the system ( $m = 1, n = L$ ). In all plots, color solid curves represent data for various values of interaction  $U/W = 0.5, 1.0, \dots, 3.0$ ,  $J_H/U = 0.25$ , and  $\eta = 8d\omega$  (left column) and  $\eta = 2d\omega$  (right column). The dashed line represents the Lorentzian.

Supplementary Fig. 3(c,d). Consequently  $\langle T_1^z T_\ell^z \rangle \simeq \langle T_1^z T_\ell^z \rangle_{\omega=0}$ . (ii) Edge-bulk correlation ( $1, \sim L/2$ ) vanishes, again, consistent with the presence of the edge states (not shown). (iii) However, in the proximity of the edges (for  $m \sim n \sim 1$ ), the dynamical correlations contain both coherent  $\omega \sim 0$  (edge mode) and incoherent  $\omega > 0$  parts [see Supplementary Fig. 3(e,f)], and  $\langle T_1^z T_{m \sim 1}^z \rangle \neq \langle T_1^z T_{m \sim 1}^z \rangle_{\omega=0}$ .

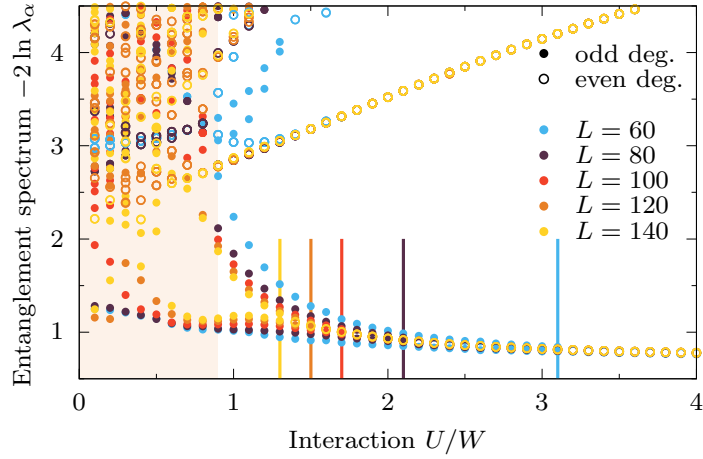
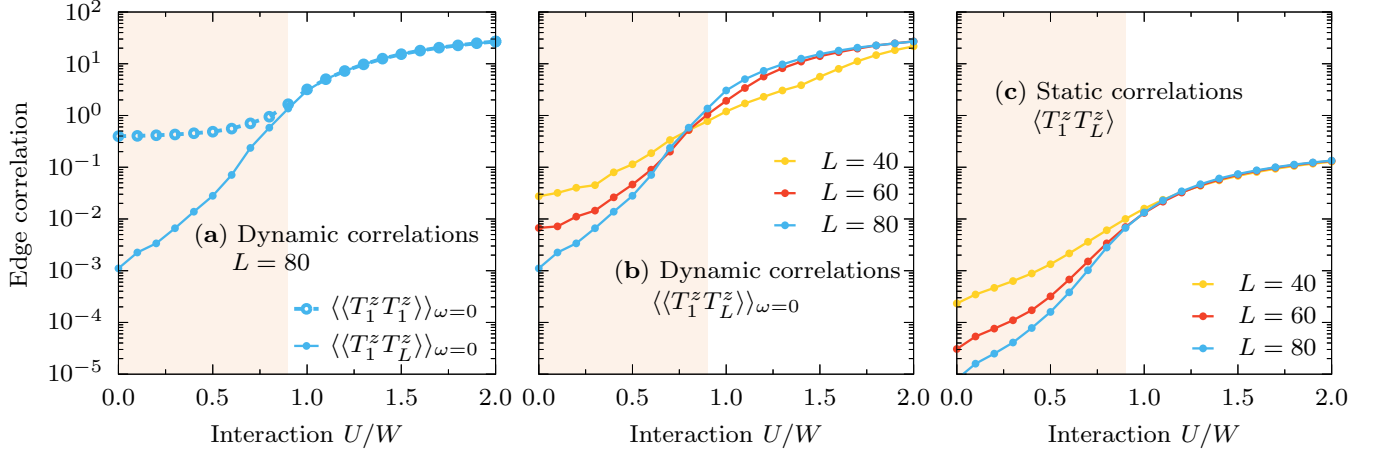
Finally, in Supplementary Fig. 4, we present additional evidence for the topological phase transition at  $U = U_c \simeq 0.9W$  (for  $J_H/U = 0.25$ ) from spin correlation data.

(1) In panel (a), we compare the local Green's function  $n = m = 1$  and non-local edge-edge correlations  $n = 1, m = L$  at zero-frequency  $\omega = 0$ . The change in the local value is related to the development of the magnetic moment  $\mathcal{S}$ . On the other hand, the non-local (edge-edge) probes only the appearance of edge zero-modes (as discussed in the main text). It is evident from the presented results that both quantities merge at  $U = U_c$ , consistent with the presence of the zero frequency edge modes for  $U > U_c$ .

(2) In panels (b) and (c), we present the same data as in Fig. 3B of the main text and Supplementary Fig. 2(b), respectively, in a log- $y$  scale. Evidently, for  $U > U_c$ , the non-zero correlation functions do not originate from finite-size effects. On the other hand, for  $U < U_c$ , the values of edge-edge correlations (both static and dynamic) decay exponentially with system size.

### SUPPLEMENTARY NOTE 3: ENTANGLEMENT SPECTRUM.

Supplementary Fig. 5 depicts additional results for the entanglement spectrum  $-2 \ln \lambda_\alpha$ . Consistent with the discussion presented in the main text, the value of the interaction  $U$  for which the spectrum is evenly degenerate moves to  $U_c$  with increasing system size  $L$  (see also Fig. 4 of the main text). Furthermore, it is evident from the presented results that the  $\lambda_\alpha$  spectrum does not contain any apparent structure for  $U < U_c$ , and the condition [i.e.,  $\ln \lambda_{i+1} - \ln \lambda_i \leq 0.01$ ] of evenly degenerate entanglement spectrum is not fulfilled.



Supplementary Figure 5. **Entanglement entropy spectrum.** Interaction  $U/W$  dependence of the entanglement spectrum  $-2 \ln \lambda_\alpha$ , obtained for  $J_H/U = 0.25$  and  $L = 60, 80, \dots, 140$  site systems divided in half. The vertical lines depict the value of interaction at which the lowest 50 eigenvalues  $\lambda_\alpha$  are evenly degenerate. The shaded region depicts the topologically trivial phase.