

Supplementary information for software [NLSM_monte_carlo](#)

Software requirements

- The software was written in [Python](#) and has been tested on [Windows 10](#).
- Before using the [NLSM_monte_carlo](#) software, users should have [Python](#) version 3.11.7 or higher installed, together with the following packages: [numpy](#), [numba](#), [pandas](#).
- [Jupyter Notebook](#) is required for the demo.

Demo

- Save [NLSM_monte_carlo](#) and [run_NLSM](#) in a directory of your choice. Open [run_NLSM](#) using [Jupyter Notebook](#). Set the variable [begpath](#) in the first cell to the path of the directory in which you have saved [NLSM_monte_carlo](#) and [run_NLSM](#).
- Create a directory to hold the results of the simulation. Set the variable [dat_path](#) in cell 3 of the notebook to the path of this directory. Run cells 1-3. For the parameters included in [run_NLSM](#) the run time is approximately 9 minutes.
- A folder containing the results is created in the folder that you have created in the previous item. To view the results set the variable [res](#) in cell 4 to the path of the directory created by the simulation, and run the cell.

Calling [NLSM_monte_carlo](#)

Input parameters involved in setting up the Monte Carlo simulation:

- [dat_path](#) - (str) Holds the path of the directory, where the results are to be saved.
- [L](#) - (int) The in-plane size of the system. Each plane contains $L \times L$ sites.
- [Lz](#) - (int) The number of planes. This should be an even number since the model consists of bilayers.
- [N_sweep](#) - (int) Total number of Monte Carlo sweeps.
- [N_skip](#) - (int) Number of sweeps for thermalization.
- [corr.t](#) - (int) Number of sweeps between consecutive measurements.
- [seed](#) - (int) Seed for random number generation.

Input parameters defining the NLSM model (Eqs. 2 and 3 of the main text). The code assumes $\rho_s = 1$ and $\lambda = 1$:

- [temperature](#) - (float) Temperature (T).
- [mag_field](#) - (float) The magnetic field (not used for the purpose of the present study).
- [g](#) - (float) The effective CDW mass (g).
- [dg](#) - (float) The CDW mass anisotropy due to the chains (Δg).
- [dg_strain](#) - (float) The CDW mass anisotropy due to the strain (Δg_s).
- [U_int](#) - (float) The intra-bilayer CDW interaction coupling ($\tilde{U}$).
- [U_int.bi](#) - (float) The inter-bilayer CDW interaction coupling (U).

- **J_int** - (float) The intra-bilayer Josephson coupling ($\tilde{J}$).
- **J_int.bi** - (float) The inter-bilayer Josephson coupling (J).
- **w** - (float) Set to zero. Not used in the present model.
- **V** - (float) The amplitude of the disorder within a disc (V).
- **V_local** - (float) The code allows to include also disorder on the planes in addition to the disorder on the chain layers (see. Ref. 39 in the main text). The in-plane disorder is modeled as random Gaussian fields whose standard deviation is given by **V_local**.
- **gamma** - (float) The ratio between the coupling to the disorder on the inner and outer planes (γ).
- **d_length** - (int) The radius of the disorder discs (r_d).
- **N_defect** - (int) The number of disorder discs per bilayer.
- **onsite** - (bool) False if the disorder discs are to be centered on centers of plaquettes. True if the discs are to be centered on sites.

Output of NLSM_monte_carlo

The output is a dictionary that contains the input parameter, and in addition:

- **disorder_config** - A 4-dimensional array of size $L \times L \times L_z \times 4$ containing the total disorder potential to which the CDW fields Φ^α , $\alpha = 1, 2, 3, 4$ couple on the site (x, y, z) . The fields Φ^1, Φ^2 correspond to the real and imaginary parts of the complex field Φ^b of the main text, while Φ^3, Φ^4 correspond to the real and imaginary parts of Φ^a .
- **G_ab** - A 6-dimensional array of size $6 \times L \times L \times L_z/2 \times 2 \times 2$ containing the correlations

$$G_{\mu\mu'}^\alpha(\mathbf{q}, q_z) = \sum_{\mathbf{r}\mathbf{r}'} \sum_{jj'} e^{-i[\mathbf{q} \cdot (\mathbf{r}-\mathbf{r}') + q_z(j-j')]} \langle \phi_{j\mu}^\alpha(\mathbf{r}) \phi_{j'\mu'}^\alpha(\mathbf{r}') \rangle,$$

where $\phi^1 = \text{Re}(\psi)$, $\phi^2 = \text{Im}(\psi)$, and $\phi^\alpha = \Phi^{\alpha-2}$ for $\alpha = 3, 4, 5, 6$. The array indices stand for $(\alpha, q_x, q_y, q_z, \mu, \mu')$. The momenta take the values $q_{x,y} = \frac{2\pi}{L} n_{x,y}$, where $n_{x,y} = 0, \dots, L-1$, and $q_z = \frac{4\pi}{L_z} n_z$ with $n_z = 0, \dots, \frac{L_z}{2} - 1$.

- **n_a** - A 4-dimensional array of size $L \times L \times L_z \times 6$ containing $\langle [\phi^\alpha(x, y, z)]^2 \rangle$.
- **helicity** - A 1-dimensional array of size $(\text{N_sweep} - \text{N_skip})/\text{corr_t}$ containing the measurements of the superconducting stiffness calculated in the course of the simulations.
- **order** - A 4-dimensional array of size $L \times L \times L_z \times 6$ containing the final Monte Carlo measurement of the fields $\phi^\alpha(x, y, z)$.