Peer Review File

Observation of monopole topological mode

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REVIEWERS' COMMENTS

Reviewer #1 (Remarks to the Author):

This is a 3rd round report for the manuscript "Monopole topological resonators," by Cheng et al.

Overall, I appreciate the authors' detailed responses and extensive revisions to my previous report. In particular, I very much appreciate the re-written section on the system's construction, I think the authors have done an excellent job making this complicated topic more accessible. And, I am happy to recommend this manuscript for publication.

I would urge the authors to address one small point prior to publication -- On page 3, the authors discuss how their system is in symmetry class BDI, and how this class has an integer invariant. However, it only has an integer invariant in 1D, in 2D and 3D this class is always trivial. I assume that the authors are using some form of dimensional reduction such that their topological charge reveals itself as the invariant of an effective 1D BDI Hamiltonian. If a simple explanation of this process exists, could it be added here to the main text? If not, could a quick explanation be provided in the supplement (with an associated pointed in the main text)?

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We thank the reviewer for recommending our paper for publication.

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We thank the reviewer for catching this point, which was not written precisely in the previous version. The confusion comes from the subtle difference in “topological classification” and the less-known “topological-defect classification”.

The reviewer mentioned that, in symmetry class BDI, there is only an integer topological invariant in 1D, but in 2D and 3D it’s always trivial. This is true for the topological classification of bulk-boundary correspondence from D-dimension bulk to D-1-dimension boundary. But boundary is only a special case of the topological-defect.

The full classification from D-dimension bulk to D-2 and D-3-dimension topological defects, for bulk-topological-defect correspondence, was done in the following reference.

[10] “Topological defects and gapless modes in insulators and superconductors”, Jeffrey C. Y. Teo and C. L. Kane, Phys. Rev. B 82, 115120
Below we show the screenshots of the related figure and table in this reference.

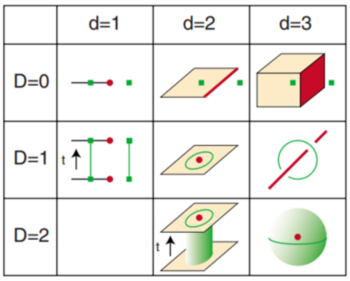

FIG. 1. (Color online) Topological defects characterized by a D parameter family of d-dimensional Bloch-BdG Hamiltonians. Line defects correspond to $d-D=2$ while point defects correspond to $d-D=1$. Temporal cycles for point defects correspond to $d-D=0$.

As listed in Fig. 1., all point defects including kinks in 1D, vortex in 2D, and monopoles in 3D correspond to $\delta = d - D = 1$.

TABLE I. Periodic table for the classification of topological defects in insulators and superconductors. The rows correspond to the different Altland Zirnbauer (AZ) symmetry classes while the columns distinguish different dimensionalities, which depend only on $\delta=d-D$.

		Symmetry			$\delta=d-D$							
s	AZ	Θ^2	Ξ^2	Π^2	0	1	2	3	4	5	6	7
0	A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
1	AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	AI	1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
2	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
3	DIII	−1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
4	AII	−1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
5	CII	−1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
6	C	0	−1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
7	CI	1	−1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

From TABLE I. we know that in symmetry class BDI, a Z topological invariant characterizes all systems with $\delta = d - D = 1$, including the kink, vortex, and monopole.

To clarify this point, we improved the statement in the main text as follows:

"In topological-defect classifications [10], the kink, vortex, and monopole zero modes in 1D, 2D, and 3D all belong to the same Altland-Zirnbauer symmetry class BDI with integer invariant Z under time-reversal and chiral symmetries"