nature portfolio

Peer Review File

Stability of Optical Knots in Atmospheric Turbulence

Corresponding Author: Professor Natalia Litchinitser

This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
This paper presents an interesting investigation into the stability of optical knots in thin, phase-turbulent-like, phase errors. Using a heater and folder path, optical turbulence is emulated over a range of scintillation indices, and it is shown that the number of crossing points in an SLM-generated optical knot is maintained. As far as I know, this is the first observation of this effect, and the paper could be of considerable interest to many researchers focusing on sensing applications. The paper is well written and presented, with a clear narrative for the projected impact of the research. I feel with suitable adjustments the paper could be suitable for publication in Nature Communications.

Areas of improvement:

1. Weak Turbulence
The authors clearly articulate that their results are valid only for the weak turbulence regime. However, it is not entirely clear just how weak the turbulence generated is as there are no images of measured beam profiles to indicate the level of turbulence. There is one image in Fig. S4, which seems to be very lightly aberrated by the turbulent propagation. If this very selective image is representative of the worst turbulence, any optical field under similar conditions, including beams that carry OAM, would similarly not observe any substantial change in vortex position, i.e. a line of darkness, and therefore the fact that the beams are not adversely affected by turbulence would be expected based on the substantial literature for weak turbulence, such as 22 in the paper or more wide cited work by Patterson, https://doi.org/10.1103/PhysRevLett.94.153901. To improve this area of the paper I would suggest:

(i) The authors clearly show experimental data that indicates the lowest, highest, and median turbulence achieved over the emulator. This will allow readers to clearly see the level of turbulence.

(ii) If turbulence is truly equivalent to a single phase screen in their experimental channel, I feel the authors should more clearly justify why one would expect optical knots to be more effected by turbulence, compared to other similarly complex optical fields. This will assist readers in clearly understanding why the result is not simply direct analogous to other structured optical fields.

2. Calibration of Turbulence
The current results are verified by considering the effective scintillation index measured over the channel, however an estimation of the fried parameter, r0, or Cn2 for the experimental channel are not presented. As the authors are using a small beam, to emulate the propagation of a 6mm beam over a 270m channel, the main aberration could be mostly be tip/tilt as the r0 is substantially bigger then the beams size. If so, the aberrations would not be expected to have any impact of knotted beams. To improve this area of the paper I would suggest:

(i) The authors determine from the experimentally recovered phase patterns for a gaussian beam propagated over the channel, determine the fried parameter and Cn2 for the experimental link. Alternatively, a measurement of the optical centre of mass could also provide insight into the specific aberration being induced by the turbulent emulator.

(ii) Provide more details about the calculation for scintillation index, specifically mentioning the number of pixels and time used for this calculation.

3. Turbulence-induced knots

The propagation through any complex medium will induce knots and loops of darkness, as shown in previous work by O'Holleran (2006), https://doi.org/10.1364/OE.14.003039. In any real-world channel, it would be expected that vortex pair formation will occur over suitable propagation lengths, where these could generate knots that can intersect with the generated knots. The paper would benefit from a discussion around the potential impact of environmental-generated knots.

4. Application to communications
The paper suggests these knots could be used for communication; however, it's not clear how the authors envisage this being implemented. I would suggest a reference or small discussion on how knots could be used within a communication system for information encoding and why it would be an effective communication strategy.

5. Projection in strong turbulence
The abstract correctly indicates that the number of crossing points within the knots might not be preserved is very likely to be true. I would suggest the authors delve a little deeper into this, and maybe draw reference to vortex splitting that occurs in high-order OAM fields during propagation, https://doi.org/10.1103/PhysRevA.56.4064, and in turbulence, https://doi.org/10.1088/1367-2630/aaae9e, as similar behaviour would surely disrupt the knot structure.

Reviewer #2

(Remarks to the Author)
In this manuscript, the authors studied the stability of optical knots in atmospheric turbulence theoretically and experimentally. They showed that the topological structure of optical knots is relatively stable in the weak-turbulence regime, while, in a strong-turbulence regime, some correction algorithms may need to be added to maintain the shape of the knot. Moreover, they showed that the Trefoil knot can transform into a Hopf link or an unknot because the effect of the turbulent medium. It is an interesting work, so it can be considered for publication in Nature Communications. However, the following issues should be addressed in the revision:
1. To recover the knot transmitting through the turbulence in experiments, the authors extracted the complex field from the recorded interferogram, and calculated the 3D field using light field propagation codes and then recovered the knot structure. I wondered why don't they experimentally record the light field at every transverse plane directly? Will it take more time?
2. Why did the authors use five LG vortex beams to construct a 3D light field with singularity knots? Will it be better if we use more LG beams?
3. I suggest the authors can revise the abstract, because the readers can not get more important information from it. In the abstract, the authors claimed that they demonstrate that the number of crossings (the topological invariant) can be preserved in the weak turbulence regime but may not be conserved in the stronger turbulence conditions, which is somewhat ordinary because it is very easy to imagine it.
4. It is shown that the knots will be opened up and reconnected in turbulence since the inner singularity experiences less wandering than the outer one, and the LG mode spectrum is also discussed to show the spectral dispersion. More details should be given to show how the spectral broadening associated with the reconnections of knots.
5. At the beginning of line 148, it looks some words are missing.

Reviewer #3

(Remarks to the Author)
This article discusses the stability of optical knots in atmospheric turbulent environments, and claims that even though the mathematical topological stability of knots, i.e., the number of singularity line crossings, are unchanged, the knotted fields present differently under different levels of turbulence strength. Also, the paper presents some optimization method to improve the stability of optical knots by adjusting the amplitudes and beam wandering parameters. Overall, it is a great work that gives me a unique point of view to understand the stability and chaotic, the optical beams, and the fields, and thereby, I've got some inspiration. The paper is well-written. The figures are illustrated clearly. Before I recommend it for publication, please find my comments below:

1. The title of this article is 'Knots of Darkness in Atmospheric Turbulence'. However, the keywords of this article should be optical knots, which shows more times. So, could the authors clarify why the title is not 'The Stability of Optical Knots in Atmospheric Turbulence'?
2. Could the authors give some reference or specific illustration for 'the singularity lines of the knots', 'beam waist'? That would be more friendly for the readers in other research areas.
3. In Fig. 3, do (b) and (e) show the phase distortion? And do (c) and (f) show the amplitude distortion? Moreover, how the k1 plane and k2 plane is selected? Are these planes representative?
4. For me, the relationship between the optical knots and the wavefront is not intuitive. Please justify.
5. In both the art files and the supplementary materials, it seems to me that there is no equation for the optical/trefoil knot coefficient. I think Eq. S3 and Eq. S4 are the expressions for $c_{p,l}$. So please give the exact expressions for the optical knots with a brief introduction, and I also recommend them to appear in the main text of your article, with minor parts or at least with a reference.
6. I think the background and motivation of this research takes more than half of the abstract, which might be too much. It might be better to enlarge the proportion of your research work and results. You decide.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors have addressed all my point and I'm happy to support publication of the work in nature communications.

Reviewer #2

(Remarks to the Author)
This manuscript has been improved greatly, so it can be accepted for publication now.

Reviewer #3

(Remarks to the Author)
The authors have well addressed all my comments.

Response to the Reviewers

Reviewer #1

Reviewer's comment: *"This paper presents an interesting investigation into the stability of optical knots in thin, phase-turbulent-like, phase errors. Using a heater and folder path, optical turbulence is emulated over a range of scintillation indices, and it is shown that the number of crossing points in an SLM-generated optical knot is maintained. As far as I know, this is the first observation of this effect, and the paper could be of considerable interest to many researchers focusing on sensing applications. The paper is well written and presented, with a clear narrative for the projected impact of the research. I feel with suitable adjustments the paper could be suitable for publication in Nature Communications."*

Authors' response: The authors would like to thank the reviewer for his/her opinion about their work.

Reviewer's comment: *"The authors clearly articulate that their results are valid only for the weak turbulence regime. However, it is not entirely clear just how weak the turbulence generated is as there are no images of measured beam profiles to indicate the level of turbulence. There is one image in Fig. S4, which seems to be very lightly aberrated by the turbulent propagation. If this very selective image is representative of the worst turbulence, any optical field under similar conditions, including beams that carry OAM, would similarly not observe any substantial change in vortex position, i.e. a line of darkness, and therefore the fact that the beams are not adversely affected by turbulence would be expected based on the substantial literature for weak turbulence, such as 22 in the paper or more wide cited work by Patterson, https://doi.org/10.1103/PhysRevLett.94.153901. "*

Authors' response: Regarding the validity of the results, we would like to clarify that they are valid over a wide range of turbulence strengths. We conclude that optical vortex knots are stable within the weak turbulence regime but transform into other knotted states for stronger turbulence levels. The turbulence regimes considered here are shown in Section 1 of the Supplementary Materials, chosen in terms of the scintillation index. However, we agree that an additional figure containing a few examples of turbulence distortion on the knotted beam might be beneficial. The following text and figure are now included in Section 7 of the Supplementary Materials, line 598, where it shows additional examples of measured optical knots.

Text modifications:

Additionally, amplitude and phase distributions of the measured knotted field using the turbulence chamber for multiple temperature settings are shown in Fig. S9. Here, three different realizations are plotted for temperature (scintillation) values of (a) = 30 ℃ ($\sigma_I^2 = 0.0068$), (b) $T = 45$ ℃ ($\sigma_I^2 = 0.0418$), and (c) $T = 65$ ℃ ($\sigma_I^2 = 0.0931$). At higher turbulence strengths, more complex and smaller-scale beam alterations are applied.

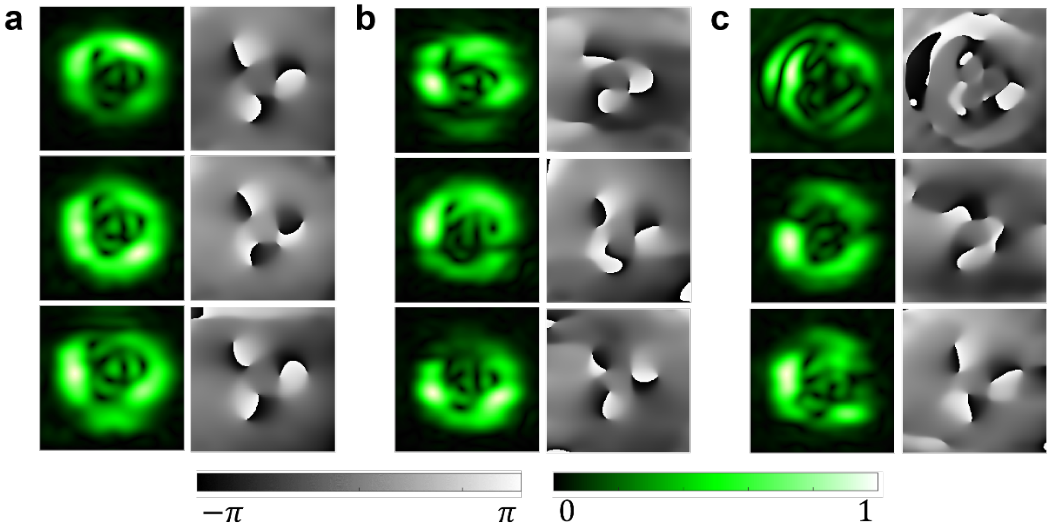

Fig. S9. Examples of amplitude and phase distributions measured at different temperature (scintillation) values in the turbulence chamber. Each row represents different realizations at (a) = 30 ℃ ($\sigma_I^2 = 0.0068$), (b) $T = 45$ ℃ ($\sigma_I^2 = 0.0418$), and (c) $T = 65$ ℃ ($\sigma_I^2 = 0.0931$).

Reviewer's comment: *"To improve this area of the paper I would suggest: (i) The authors clearly show experimental data that indicates the lowest, highest, and median turbulence achieved over the emulator. This will allow readers to clearly see the level of turbulence."*

Authors' response: This comment is addressed in Fig. S1(a), which shows different turbulence levels achieved experimentally. In particular, we show the scintillation level as a function of the temperature of the chamber used to calibrate the turbulence levels generated by the turbulence emulator.

Reviewer's comment: *"(ii) If turbulence is truly equivalent to a single phase screen in their experimental channel, I feel the authors should more clearly justify why one would expect optical knots to be more effected by turbulence, compared to other similarly complex optical fields. This will assist readers in clearly understanding why the result is not simply direct analogous to other structured optical fields."*

Authors' response: We thank the reviewer for raising this important question and address two important points to answer it.

First, we find that, in fact, a single-phase screen cannot accurately simulate thick, long turbulence. Single-phase screens do not take the diffraction of the optical beams into account. Indeed, due to the diffraction, the beam sizes of the constituent components of the knot (different LG beams) change along the propagation direction and, therefore, are affected differently by the turbulence at different planes along the propagation direction. Although the knot is formed within the Rayleigh length of the beam, the turbulence links are typically much longer than the Rayleigh range. This is the reason why the turbulence chamber was used to obtain the results presented in the main text of the manuscript, while on the numerical simulations, we used multiple phase screens along the propagation distance to include the effects of diffraction properly. In the previous version of the manuscript, we had additional experimental and numerical results using single-phase screens included in the Supplementary Materials for the case of the weak turbulence, but we decided to remove them as it has caused some confusion with the main results using thick, long turbulence in a hot air chamber.

Second, the comparison between optical knots and other structured light beams, such as OAM beams, in the presence of turbulence is not straightforward due to the important differences in their topological properties. Indeed, the typical measurements of OAM-carrying beams lead to a spectrum of states with associated crosstalk between them, meaning the perturbed beam has a superposition of OAM states and might (for the weak/medium turbulence levels) or might not (for the strong turbulence level) be centered around the initial OAM state. On the contrary, a different characterization technique had to be used in the optical knots case since both relative amplitudes and phases of the constituent LG components determine whether the knot topology is preserved after the turbulence. Due to these inherent differences between 2D and 3D topologies, we focused on studying the dynamics of knotted optical topologies in the presence of turbulence and how their topologies transform into other states. To highlight these differences to readers, we added the following words in line 222:

Text modifications:

> Additionally, it is important to highlight a major difference in the topological properties between 2D and 3D structured fields while interacting with turbulence. Typically, OAM-carrying beams lead to a spectrum of states upon perturbation by a turbid media, possessing an associated crosstalk between them [20]. Here, the spectrum might or might not be centered around the initial topological mode, depending on the turbulence strength. In the optical knots case, the topology is determined by the knot type (trefoil, Hopf link, or an unknot) recovered at the receiver, that depends on both relative amplitudes and phases of the constituent LG components.

Reviewer's comment: *The current results are verified by considering the effective scintillation index measured over the channel, however an estimation of the fried parameter, r0, or Cn2 for the experimental channel are not presented. As the authors are using a small beam, to emulate the propagation of a 6mm beam over a 270m channel, the main aberration could be mostly be tip/tilt as the r0 is substantially bigger then the beams size. If so, the aberrations would not be expected to have any impact of knotted beams.*

Authors' response: As requested by the reviewer, the values of r_0 and C_n^2 were measured and included in line 473.

Text modifications:

> To estimate the refractive index structure parameter C_n^2 of the experimental system, we relied on statistical measurements from the beam wandering of a Gaussian beam. Such a relationship is given by $C_n^2 = 0.328\langle r_c^2\rangle D^{1/3}L^{-3}$, where D is the beam diameter, L is the turbulence link length, and $\langle r_c^2\rangle = \langle x^2\rangle + \langle y^2\rangle$ is the variance of the beam wander

displacement on the detector [S1]. The curves showing the experimentally estimated C_n^2 using 2000 different realizations, alongside the numerical values, are shown in Fig. S2 (a). The Fried parameter r_0 can be determined by the relation $r_0 = 1.68(C_n^2 L k^2)^{-3/5}$, with the wavenumber $k = 2\pi/\lambda$ already introduced in the main text. Figure S2 (b) depicts the curve with the experimental estimations for r_0, superposed with the numerical counterparts. Notice that our laboratory scale experimental setup was carefully designed to mimic a realistic free-space optical link based on a $w_0 = 6$ mm beam propagating over a $L = 270$ m distance [S2], making use of the Fresnel scaling to ensure the same optical properties in both scenarios. In particular, to simulate similar values of scintillations in simulations and experiments, stronger turbulence had to be realized while using the hot air chamber to compensate for the shorter propagation distance and smaller beam size. For instance, simply using the mathematical expression for the Rytov variance $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$ to estimate the order of magnitude of C_n^2 for such shorter propagation distance, here $L_{\mathrm{lab}} = 1.5$ m, gives the same order of magnitude we observe in Fig. S2 (a).

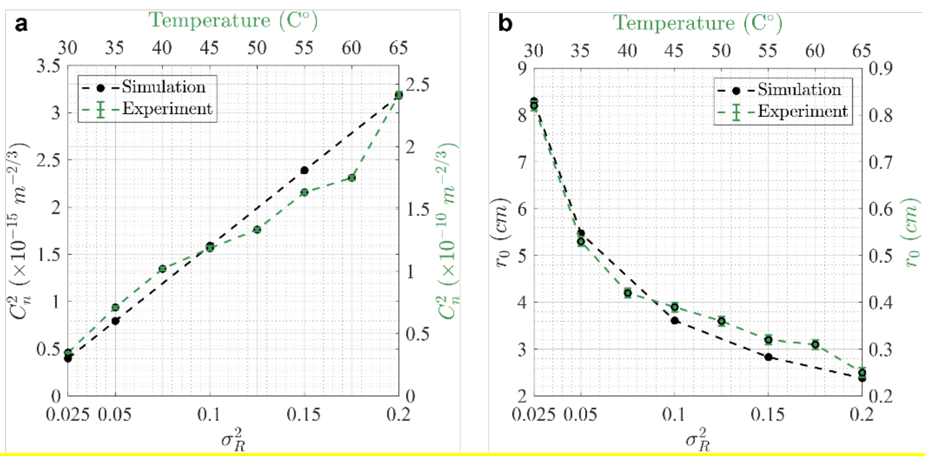

Fig. S2. (a) Experimental estimation of the refractive index structure parameter C_n^2 with respect to temperature values of the turbulence chamber (green), together with the parameter used in the simulations (black) with respect to the Rytov variance σ_R^2. Here, 2000 experimental realizations were performed. **(b)** The respective Fried parameter r_0 values for both simulations (black) and experiments (green) are similar to panel **(a)**.

These measurements are now also mentioned in the main manuscript, line 130, as follows:

Text modifications:

The calibration curve is included in section S1 of Supplementary Materials, alongside the estimated values of the refractive index structure C_n^2 and the Fried parameter r_0.

Regarding the type of aberrations, as shown in the new Fig. S9, the aberrations imposed by the turbulence considered in this paper cannot be reduced to tip/tilt-like perturbations. These aberrations do not substantially affect the knot as they primarily lead to the spatial shifts of the optical beam, maintaining its knotted structure. The text of the manuscript was modified as follows, located in line 603:

Text modifications:

Figure S10 shows how tilt-like aberrations affect the knotted beams, where the amplitude and phase distributions and corresponding 3D singularity lines are shown. Here, we numerically apply tilts along the x-direction in the form of $E_{tilt} = E_0 \exp(ik_x x)$ as (a) $k_x = k_0 \times 10^{-5}$, (b) $k_x = 2.5k_0 \times 10^{-5}$, and (c) $k_x = 5k_0 \times 10^{-5}$. In all cases, the trefoil topology is maintained. Such types of aberrations correspond to the 1st order aberrations in terms of Zernike polynomials, suggesting the higher-order terms are required to substantially change the knots' topology.

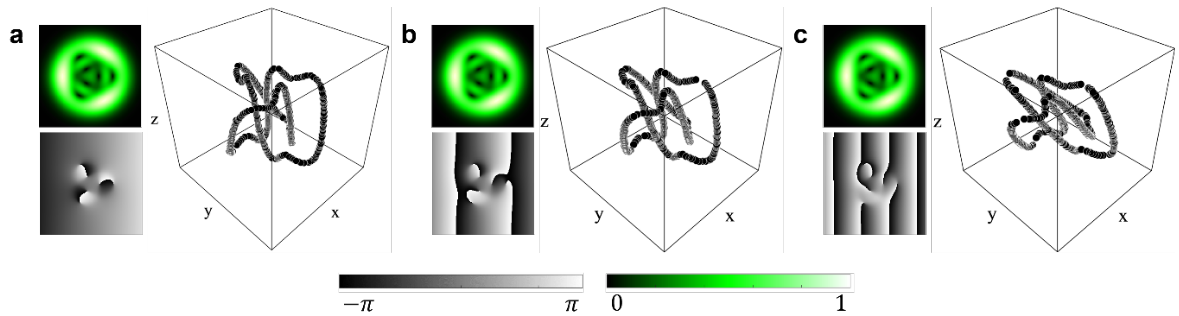

Fig. S10. Examples of tilt-like aberrations applied to the optical trefoil knot. Each tilt along the $\boldsymbol{x}$-direction has a magnitude of (a) $\boldsymbol{k_x = k_0} \times \mathbf{10^{-5}}$, (b) $\boldsymbol{k_x = 2.5k_0} \times \mathbf{10^{-5}}$, and (c) $\boldsymbol{k_x = 5k_0} \times \mathbf{10^{-5}}$. The corresponding amplitude, phase distributions, and resulting 3D singularity lines are shown.

Reviewer's comment: *To improve this area of the paper I would suggest:*

(i) The authors determine from the experimentally recovered phase patterns for a gaussian beam propagated over the channel, determine the fried parameter and Cn2 for the experimental link. Alternatively, a measurement of the optical centre of mass could also provide insight into the specific aberration being induced by the turbulent emulator."

Authors' response: As mentioned in the previous response, C_n^2 and r_0 values were measured and plotted in Fig. S2 along with the exact numerical parameters considered in this study, starting at line 472.

Reviewer's comment: *"(ii) Provide more details about the calculation for scintillation index, specifically mentioning the number of pixels and time used for this calculation."*

Author's response: The following text has been added to Section 1 of the Supplementary Materials, line 455, regarding the calculation of the scintillation index:

> Figure S1 (a) shows the calibration curve, which agrees with the numerical simulations predictions. The scintillation index $\sigma_I^2 = \langle I^2(\mathbf{0})\rangle / \langle I(\mathbf{0})\rangle^2 - 1$ is calculated by measuring the on-axis intensity values of a Gaussian beam propagating through the turbulence link. For each realization, we average the intensity values within the nine central pixels for both numerical and experimental evaluations. Here, 2000 different realizations were performed.

Reviewer's comment: *"The propagation through any complex medium will induce knots and loops of darkness, as shown in previous work by O'Holleran (2006), https://doi.org/10.1364/OE.14.003039. In any real-world channel, it would be expected that vortex* *pair formation will occur over suitable propagation lengths, where these could generate knots that can intersect with the generated knots. The paper would benefit from a discussion around the potential impact of environmental-generated knots."*

Authors' response: We thank the reviewer for this very important comment. Indeed, closed vortex lines can be observed upon propagation of particular field superpositions in both free-space and complex media due to the generation of new components of the modal spectrum. To address the connection between environment-generated knots and the original knots, we added the following text to Section 6 of the Supplementary Materials, line 581:

Text modifications:

> It is noteworthy that closed vortex lines may form during light propagation in any complex media [S5]. They result from the modal spectrum broadening upon interaction with the environment, e.g., turbulence [S6]. In particular, such a phenomenon is observed in our work and reveals itself in the reconnection events, where the addition or redistribution of optical modes resulting from the interactions with the environment leads to a change in the vortex line trajectories. The knotted vortex trajectories presented in this work differ from environmental-generated knots as they are mathematically designed to possess specific topological properties and shapes, contrasting with spontaneously generated closed vortex trajectories.

Reviewer's comment: *"The paper suggests these knots could be used for communication; however, it's not clear how the authors envisage this being implemented. I would suggest a reference or small discussion on how knots could be used within a communication system for information encoding and why it would be an effective communication strategy."*

Authors' response: Although this work was mainly focused on developing a fundamental understanding of the effect of turbulence on the optical knots' properties, we included the following references that discuss the potential applications of optical knots for classical and quantum communications:

[4]. Larocque, H., et al., Optical framed knots as information carriers. Nature Communications, 2020. 11(1).

[5]. Ollikainen, T., et al., Decay of a Quantum Knot. Physical Review Letters, 2019. 123(16).

[6]. Hall, D.S., et al., Tying quantum knots. Nature Physics, 2016. 12(5): p. 478.

[7]. Kong, L.-J., et al., High capacity topological coding based on nested vortex knots and links. Nature Communications, 2022. 13(1): p. 2705.

Reviewer's comment: *"The abstract correctly indicates that the number of crossing points within the knots might not be preserved is very likely to be true. I would suggest the authors delve a little deeper into this, and maybe draw reference to vortex splitting that occurs in high-order OAM fields during propagation, https://doi.org/10.1103/PhysRevA.56.4064, and in turbulence, https://doi.org/10.1088/1367-2630/aaae9e, as similar behaviour would surely disrupt the knot structure."*

Authors' response: Indeed, there is a similar behavior between the change of topologies in OAM-carrying beams and optical knots. In both cases, the initial higher-order topological state transitions into a lower-order one. In the case of the OAM beams, a single higher-order OAM beam tends to split into multiple topological charges ± 1 beams, while higher-order optical knots tend to transform into lower-order knots through reconnection events. However, since the optical knots are formed by multiple singularities with ± 1 charges, we do not expect to see the same higher-order topological charge breaking as in the OAM-carrying beams case. The following statement in line 217 of the main text:

Text modifications:

> The knot transformation from a higher-order topology to a lower-order one is similar to the higher-order charge splitting in OAM-carrying beams. In the case of the OAM beams, any perturbations in complex media lead to higher-order topological charges splitting into multiple ± 1 topological charges. In the case of optical knots, we observe a similar behavior, but instead of reducing the topological charge, higher-order optical knots tend to transform into lower-order knots through reconnection events.

Reviewer #2

Reviewer's comment: *"In this manuscript, the authors studied the stability of optical knots in atmospheric turbulence theoretically and experimentally. They showed that the topological structure of optical knots is relatively stable in the weak-turbulence regime, while, in a strong-turbulence regime, some correction algorithms may need to be added to maintain the shape of the knot. Moreover, they showed that the Trefoil knot can transform into a Hopf link or an unknot because the effect of the turbulent medium. It is an interesting work, so it can be considered for publication in Nature Communications. "*

Authors' response: We appreciate the reviewer's consideration of our work for publication in Nature Communications. Please find below the point-by-point responses for the addressed issues.

Reviewer's comment: *"However, the following issues should be addressed in the revision: To recover the knot transmitting through the turbulence in experiments, the authors extracted the complex field from the recorded interferogram, and calculated the 3D field using light field propagation codes and then recovered the knot structure. I wondered why don't they experimentally record the light field at every transverse plane directly? Will it take more time?"*

Authors' response: Time is crucial in the measurements performed. We must ensure that the measurements for each realization are done such that the temporal, dynamic changes in the turbulence itself are insignificant, so each optical knot is recovered under nearly static turbulence conditions. Measuring multiple planes will lead to the same results as those shown in the manuscript if the scanning suffices these temporal requirements, most probably using a superfast delay stage to cover the entire longitudinal distance of the knot fast enough.

Reviewer's comment: *"Why did the authors use five LG vortex beams to construct a 3D light field with singularity knots? Will it be better if we use more LG beams?"*

Authors' response: The mathematical derivation of a structured beam carrying a trefoil knot embedded in its 3D vortex lines naturally leads to 5 different LG modes with specific azimuthal and radial orders. As treated in Section 2 of the Supplementary Materials, this is done by projecting the corresponding Milnor polynomial into the LG basis. Naturally, the number of modes would change according to the orthonormal basis chosen. However, one could add more radial modes and numerically recalibrate their relative amplitude weightings, even though this is not standard practice as it does not follow the rigorous mathematical derivation described in the Supplementary Materials and previous works cited accordingly in the manuscript. An example of an optical trefoil knot carrying additional LG modes is shown in Ref. [30] of the main text, "Zhong, J.Z., et al., *Reconstructing the topology of optical vortex lines with single-shot measurement.* Applied Physics Letters, 2021. **119**(16)."

Reviewer's comment: *"I suggest the authors can revise the abstract, because the readers can not get more important information from it. In the abstract, the authors claimed that they demonstrate that the number of crossings (the topological invariant) can be preserved in the weak turbulence regime but may not be conserved in the stronger turbulence conditions, which is somewhat ordinary because it is very easy to imagine it."*

Authors' response: Following the reviewer's suggestion, we modified the abstract as follows (beginning in line 12):

Text modifications:

> Here, we experimentally and theoretically investigate the effects of atmospheric turbulence on optical knot stability and demonstrate that the number of crossings (the topological invariant) can be preserved in the weak turbulence regime but may not be conserved in the stronger turbulence conditions despite the topological nature of the knotted structure. Such topology transitions occur through reconnection events, where the additional optical modes resulting from the interaction of the knot with the turbulent medium change the vortex lines in space. Additionally, we propose an optimization algorithm to maximize the distance between the phase singularities at each longitudinal plane, aiming to facilitate measurements of optical knots and improve their performance in the presence of turbulence.

Reviewer's comment: *"It is shown that the knots will be opened up and reconnected in turbulence since the inner singularity experiences less wandering than the outer one, and the LG mode spectrum is also discussed to show the spectral dispersion. More details should be given to show how the spectral broadening associated with the reconnections of knots."*

Authors' response: The manuscript discusses how the reconnection events are linked to the modal spectrum distribution after interaction with turbulence. However, what specific modes lead to such events remains unknown, and our group is currently exploring this.

Reviewer's comment: *"At the beginning of line 148, it looks some words are missing."*

Authors' response: Thank you for noticing the typo. The word "directly" was inserted at the beginning of that sentence, line 151.

Reviewer #3

Reviewer's comment: *"This article discusses the stability of optical knots in atmospheric turbulent environments, and claims that even though the mathematical topological stability of knots, i.e., the* *number of singularity line crossings, are unchanged, the knotted fields present differently under different levels of turbulence strength. Also, the paper presents some optimization method to improve the stability of optical knots by adjusting the amplitudes and beam wandering parameters. Overall, it is a great work that gives me a unique point of view to understand the stability and chaotic, the optical beams, and the fields, and thereby, I've got some inspiration. The paper is well-written. The figures are illustrated clearly. Before I recommend it for publication, please find my comments below:"*

Authors' response: We thank the reviewer for the kind words. Please find below the detailed responses to each comment.

Reviewer's comment: *"The title of this article is 'Knots of Darkness in Atmospheric Turbulence'. However, the keywords of this article should be optical knots, which shows more times. So, could the authors clarify why the title is not 'The Stability of Optical Knots in Atmospheric Turbulence'?"*

Authors' response: The term 'Knots of Darkness' was chosen to refer to the fact that optical vortex knots are formed within the closed trajectories of their phase singularities, which inherently appear in regions without light or dark regions. However, we agree that the suggested title fits better the scope of the manuscript. We now changed the title to 'Stability of Optical Knots in Atmospheric Turbulence'.

Reviewer's comment: *"Could the authors give some reference or specific illustration for 'the singularity lines of the knots', 'beam waist'? That would be more friendly for the readers in other research areas."*

Authors' response: To clarify what 'singularity lines' stand for in this context, we added the following clarification in the caption of Fig. 2 :

> Each black dot refers to a phase singularity, which, after connecting all dots in 3D, forms the closed singularity line of a trefoil knot.

The beam waist radius is the smallest beam size along the propagation direction, typically defined as $z = 0$. To address this, we added the following comment in line 107:

> Here, w_0 stands for the beam waist radius (or the beam's radius at $z = 0$, where its size is the smallest throughout the propagation), λ is the wavelength, and L is the turbulence link length.

Reviewer's comment: *"In Fig. 3, do (b) and (e) show the phase distortion? And do (c) and (f) show the amplitude distortion? Moreover, how the k1 plane and k2 plane is selected? Are these planes representative?"*

Authors' response: In Fig. 3, we plot the amplitude (insets) and phase distributions (gray plots), along with the corresponding 3D singularity lines, for a trefoil before and after optimization. The k_1 and k_2 planes, shown in Fig. 3, are simply examples of two distinct planes, as the optimization algorithm minimizes the designed cost function in all longitudinal planes across the optical knot.

Reviewer's comment: *"For me, the relationship between the optical knots and the wavefront is not intuitive. Please justify."*

Authors' response: In general terms, a wavefront can be defined as a complex transverse surface of a given optical field with constant phase accumulation at a specific longitudinal position. For instance, a plane wave has a constant transverse wavefront at different longitudinal positions along the free-space propagation direction. On the other hand, paraxial beams such as Gaussian, Hermite-Gaussian (HG), and Laguerre-Gaussian (LG) modes accumulate extra phase shifts due to the Gouy phase and wavefront curvature terms (For more details, please see Refs. [21] and [29] of the main manuscript, and rewritten for simplicity in equation S3 of the manuscript for the LG case. In our case, we shape the wavefront following equation 1, with more details in Section 2 of the Supplementary Materials. The optical knot structure we observe by tracing the closed trajectories of the phase singularities in 3D space arises from this wavefront shaping. On a simple, intuitive level, the wavefront of an optical knot can perhaps be visualized as a result of 3D interference of all the constituent LG components with the appropriate coefficients.

Reviewer's comment: *"In both the art files and the supplementary materials, it seems to me that there is no equation for the optical/trefoil knot coefficient. I think Eq. S3 and Eq. S4 are the expressions for $c_{p,l}$. So please give the exact expressions for the optical knots with a brief introduction, and I also recommend them to appear in the main text of your article, with minor parts or at least with a reference."*

Authors' response: The exact coefficients for both unoptimized and optimized trefoil knots are shown in the caption of Fig. 3 in the main manuscript. Here, the values for each $c_{p,l}$ corresponding to the expression for the trefoil knot (equation 1) are explicitly displayed, where the unoptimized case can be directly obtained using equation S4. Naturally, the optimized coefficients are obtained upon minimizing the cost function $\left[min\left(\left|\vec{r}_i^k - \vec{r}_j^k\right|\right)\right]^{-1}$. We chose to include the derivation of these coefficients in the Supplementary Materials since it is a well-known procedure from Ref. [1], vastly cited in this work and many previous publications.

Reviewer's comment: *"I think the background and motivation of this research takes more than half of the abstract, which might be too much. It might be better to enlarge the proportion of your research work and results. You decide."*

Authors' response: We appreciate the reviewer's suggestion to revise the abstract. A portion of the background and motivation was removed, and the abstract now reads as follows:

> Topology, which originated as a mathematical concept, nowadays advances the understanding of many branches of science and technology, from elementary particle physics and cosmology to condensed matter physics. While the topological stability of mathematical knots implies robustness to perturbations and suggests their potential as information carriers, the behavior of optical knots in perturbative environments such as atmospheric turbulence is largely unexplored. Here, we experimentally and theoretically investigate the effects of atmospheric turbulence on optical knot stability and demonstrate that the number of crossings (the topological invariant) can be preserved in the weak turbulence regime but may not be conserved in the stronger turbulence conditions despite the topological nature of the knotted structure. Such topology transitions occur by means of reconnection events, where the additional optical modes resulting from the interaction of the knot with the turbulent medium change the vortex lines in space. Additionally, we propose an optimization algorithm to maximize the distance between the phase singularities at each longitudinal plane, aiming to facilitate measurements of optical knots and improve their performance in the presence of

turbulence. The turbulent medium is created in the laboratory using a turbulence chamber consisting of a hot plate allied with a convective transverse airflow. The optical knots are reconstructed by single-shot measurements of the complex fields, and the resilience of the knot topology is analyzed for various realistic turbulence strengths. These studies may give rise to entirely new approaches to the three-dimensional spatially resolved probing of turbulence.