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Peer Review File

Nonlinear wave propagation governed by a fractional derivative

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This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The manuscript "Nonlinear wave propagation governed by a fractional derivative" reports an experimental study on the generation and characterization of temporal solitons based on fractional nonlinear wave equation, which is interesting. The manuscript is also well-written with fruitful discussion. However, the method is conventional, lacking novelty. Therefore, I believe this work may be more suitable for other journals. Here, I would like to provide some comments for authors' reference.
1. The layout of figures is not normal enough, more details should be added, such as adding units in Fig. 1.
2. In line 271, “In experiments such as in Fig. 3(a), their energy thus varies by a factor 27 as the pulse length varies by a factor 3 in contrast with the approximate constant energy of Hilbert-NLS solitons.” which is not clear in the figure.
3. In line 255, please add more explanations about peaks of the blue curve on the boundary.
4. In caption of Fig.5, the dispersion coefficients for (a) and (b) are different, please explain the purpose of this setting, why cannot use the same value.
5. In line 72 of the supplementary, there is no dashed green line in Fig. 2(a) in the main text.
6. In line 87 of the supplementary, it should be Fig. 3(a)?
7. In line 89 of the supplementary, the np=1,3,5, but the caption in Fig. 4 np=1,2,5.

Reviewer #2

(Remarks to the Author)
This is a nice contribution on the theoretical and experimantal front as it relates to pulses in media with fractional diffraction (the authors use of the word soliton, even when this is not an integrable system should be noted) and specific when the fractional parameter is 1 leading to a Hilbert transorm leading dispersion operator. There has been a boost on the research in fractional NLSE, but very few (in fact on I can think of) with experimental verification.

The theoretical aspects to corroborate outputs (ex. universaily on pulse energy) is consistent and validates results.

Only missing, possible applications perhaps because there may not be one.

I reommned submission after addressing the following points:

1. After equation 4 the authors write: T"his means that, depending on the initial condition, nominally stationary solutions either disperse or diverge (“blow up”)". It is not clear to me if indeed some instabilityies would lead to blow up. They need to either verify numerically or provide a reference specific to \alpha = 1. The VK test is useful does not provide the outcome of the instability.

2. While the authors correctly predics equation 4 based on equations 3,4, the pulses created in the laser cavity may follow more a mean field (Lugiato-Lefever) equation which can produce pulses with such universality. I would like to see some clarification.

Beyond addressing these requests, I think the paper is suitable for publication

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors replied my comments, but still there are some unclear things to be addressed. The work reports nonlinear wave propagation governed by a fractional derivative, which is not a fresh topic at all. I agree with the authors that previous papers mainly reported theoretical results, and experimental results are rare. The main challenging is that how to find a physical system that can be described by the fractional Laplacian.
1. The authors claim that “The experiments are carried out in a fibre laser in which the nonlocal temporal effects can be programmed through the dispersion: the dependence of the propagation constant of the light β on the angular frequency ω [37]”, so can one believe that they found a system that indeed its dispersion can be described by fractional Laplacian?
2. It seems that the system should be elaborated as the authors spoke “We consider a dispersion relation in a co-moving frame of the form $\beta(\omega)=-|\beta 1||(\omega-\omega 0)|$”. Is such a dispersion relation common in real world?
3. The governing equation is Hilbert-NLS equation which is closely connected with the NLS equation as shown by the authors in the method section. But why the authors introduce the Hilbert transform but not use the traditional NLS directly? What is the feasible scope about the Hilbert-NLS equation?
4. Why the parabolic dispersion can be lifted in authors’ setup?

All in all, even though the authors made revisions, the results are still not well addressed. Especially, why the authors choose the dispersion relation in a co-moving frame and the Hilbert-NLS equation is not clear to me. I encourage the authors pay more efforts in the presentation and explain their results in a clearer way, even though the authors themselves know the details well.

Reviewer #2

(Remarks to the Author)
This is a novel result both oh theory and experiments producing optical pulses in a medium having fractional dispersion.

The work is of high quality and the revised version merits publication

Version 2:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
I appreciate the authors' patience in explaining the comments. Indeed, the revised work is a good job and a promising development on realization of the fractional Laplacian. In my opinion, this work is ready to be published in NC without change.

Response to Reviewer 1

We thank the referee for their review of our manuscript, and we were pleased to read their comments.

Comment 1: The manuscript "Nonlinear wave propagation governed by a fractional derivative" reports an experimental study on the generation and characterization of temporal solitons based on fractional nonlinear wave equation, which is interesting. The manuscript is also well-written with fruitful discussion. However, the method is conventional, lacking novelty. Therefore, I believe this work may be more suitable for other journals.

Response 1: The novelty of our work lies in the theoretical and experimental investigation of nonlinear pulses governed by a fractional derivative. While physical systems ruled by fractional derivatives have been widely studied theoretically and numerically across various fields, including optics, realizing the associated experiments has remained a major challenge. For the experiments reported here, we leveraged the flexibility of our experimental setup to generate these novel nonlinear pulses. Therefore our manuscript focuses on the thorough characterization of their unique properties, such as their spectral-temporal profiles and energy scaling, rather than on our setup. We modified a sentence in the revised manuscript to clarify the novel aspect of our work.

"Here we provide such a demonstration: we report the generation and full characterization of temporal solitons governed by a fractional Laplacian. The experiments are carried out in a fibre laser in which the nonlocal temporal effects can be programmed through the dispersion: the dependence of the propagation constant of the light β on the angular frequency ω [37]."

Comment 2: The layout of figures is not normal enough, more details should be added, such as adding units in Fig. 1.

Response 2: Figure 1 illustrates some of the fundamental concepts of the novel solitons reported in this work. Following the referee's comment, we modified Fig. 1 to improve readability. We added labels on the horizontal and vertical axes. In Fig. 1(a) and (e), we included two vertical axes to represent the linear dispersion (β) and the inverse group velocity ($1/\mathrm{v_g}$). Since Fig. 1 is conceptual, the units are arbitrary. We modified the caption to clarify the description of the figure.

"**Concept of the Hilbert-NLS solitons.** (a)-(d) Conventional solitons can form in the presence of quadratic dispersion. (a) The dispersion (red) and (inverse) group velocity (dashed green) vary smoothly with frequency ω; (b) At low intensities I, dispersion stretches pulses in time. Solitons can form at high intensities with smooth (c) spectral and (d) temporal profiles that decay exponentially. Hilbert-NLS solitons (e) arise in the presence of dispersion relation Eq. (1) which has a discontinuous (disc.) derivative (red) and associated discontinuous (inverse) group velocity (dashed green) as a function of frequency; (f) At low intensities the dispersion causes input pulses to split in two. At high intensities solitons form which have (g) a spectrum with discontinuous derivative, and (h) non-exponential decay in time. The units are arbitrary and the intensity profiles in (c), (d), (g) and (h) are on logarithmic scales."

We also modified the description of the figure in the main text for clarification.

"The experiments are carried out in a fibre laser in which the nonlocal temporal effects can be programmed through the dispersion: the dependence of the propagation constant of the light β on the angular frequency ω [37]. We consider a *dispersion relation* in a co-moving frame of the form"

"This dispersion relation and associated inverse group velocity are conceptually illustrated in Fig. 1(e), and should be compared to the parabolic dispersion relation for conventional NLS solitons in Fig. 1(a)."

Comment 3: In line 271, "In experiments such as in Fig. 3(a), their energy thus varies by a factor 27 as the pulse length varies by a factor 3 in contrast with the approximate constant energy of Hilbert-NLS solitons." which is not clear in the figure.

Response 3: This sentence is referring to the energy-width scaling of pure-quartic solitons, as reported in Ref. [37], but the corresponding curve is not shown in Fig. 3(a). We removed the sentence to avoid confusion.

Comment 4: In line 255, please add more explanations about peaks of the blue curve on the boundary.

Response 4: The sharp low and high-frequency spectral peaks seen in Fig. 2(a) and Fig. 5 correspond to Kelly sidebands. These are well-known dispersive waves that arise from perturbations when solitons propagates through the cavity (see Refs. [37, 46]). We added a sentence in the revised manuscript to clarify this point.

"The sharp spectral peaks at low and high frequencies in Fig. 2(a) correspond to Kelly sidebands, resonant dispersive waves arising from perturbations when the soliton propagates around the cavity [37, 46]. They can be ignored in our analysis."

Comment 5: In caption of Fig.5, the dispersion coefficients for (a) and (b) are different, please explain the purpose of this setting, why cannot use the same value.

Response 5: The units of the coefficient β_α depends on α, and therefore cannot be directly compared. We chose the particular values for β_α for $\alpha = 5/4$ and $\pi/2$ so they lead to the generation of pulses with similar spectral bandwidth as in Fig. 1 for $\alpha = 1$. We added a sentence to clarify this point in the revised manuscript.

"The numerical values of $|\beta_\alpha|/\alpha!$ are chosen such that the generated solitons have bandwidths that are similar to that for $\alpha = 1$."

We also corrected an error in the definition of the dispersion relation $\beta(\omega)$ in the original submission. We modified the caption of Fig. (5) and added a sentence in the main text of the revised manuscript.

"Using the same experimental approach as discussed above, we apply a phase profile $\phi(\omega) = -\frac{|\beta_\alpha|}{\alpha!}|(\omega-\omega_0)|^\alpha L$ with $\alpha_1 = 5/4$ and $\alpha_2 = \pi/2$. Here the $|\beta_\alpha|$ are constants with units of $\mathrm{m}\cdot\mathrm{s}^{-\alpha}$, and $\alpha!$ is evaluated using the Gamma function, i.e., $\alpha! = \Gamma(\alpha+1)$."

"**Fractional NLS solitons for different values of Lévy indices α.** Dispersion relations $\beta(\omega) = -\frac{|\beta_\alpha|}{\alpha!}|(\omega-\omega_0)|^\alpha$ are indicated by solid orange curves, whereas dashed orange curves correspond to the quartic dispersion included to provide stability. Solid blue curves are measured spectra whereas numerically calculated spectra are red dashed. (a) $\alpha_1 = 5/4$ and $\beta_{\alpha_1}/\alpha_1! = -30\ \mathrm{ps}^{\alpha_1}\ \mathrm{km}^{-1}$; (b) $\alpha_2 = \pi/2$ and $\beta_{\alpha_2}/\alpha_2! = -15\ \mathrm{ps}^{\alpha_2}\ \mathrm{km}^{-1}$; and (c) $\alpha_3 = 2$ and $\beta_2/2! = -10.7\ \mathrm{ps}^2\ \mathrm{km}^{-1}$, the dispersion of single-mode fibre (see Methods)."

Comment 6: In line 72 of the supplementary, there is no dashed green line in Fig. 2(a) in the main text.

Response 6: Thank you. This sentence was referring to Fig. 2(a) in the Supplementary material.

Supplementary figures are now labeled with the prefix S to avoid ambiguity.

Comment 7: In line 87 of the supplementary, it should be Fig. 3(a)?

Response 7: We corrected this typo in the revised Supplementary.

Comment 8: In line 89 of the supplementary, the np=1,3,5, but the caption in Fig. 4 np=1,2,5.

Response 8: We corrected this typo in the revised Supplementary.

Response to Reviewer 2

We thank the referee for their review of our manuscript, and we are pleased to read his/her comments on the quality of our work.

Comment 1: Only missing, possible applications perhaps because there may not be one.

Response 1: We added a sentence in the conclusion to discuss the properties of these solitons that could be used in photonics applications.

“These solitons have striking properties, summarised below, for which applications may be developed in future, for example in telecommunications or in laser physics.”

Comment 2: After equation 4 the authors write: "This means that, depending on the initial condition, nominally stationary solutions either disperse or diverge (“blow up”)". It is not clear to me if indeed some instabilityies would lead to blow up. They need to either verify numerically or provide a reference specific to $\alpha = 1$. The VK test is useful does not provide the outcome of the instability.

Response 2: We now realise that our discussion of solution stability was confusing. We have now moved most of this discussion to the Method section, The only reference to stability in the main text reads:

“As discussed in more detail in the Method section, the stationary solutions of Eq. (2) are marginally stable. To address this we add a small amount of negative quartic dispersion at frequencies far from the central frequency (see Methods) where the spectral soliton amplitude is low (orange curve in Fig. 2(a)).”

In the Method section we now added the following passages:

“In our non-conservative system this instability manifests as a spectral asymmetry. In our numerical simulations with initial phase noise, the soliton spectrum and energy fluctuate after each round trip. Such behavior can be arrested by applying a small additional perturbation [53, 54], as discussed in more detail below.”

“Even though the addition of a small amount of quartic dispersion leads to slight changes in the generated solitons, the overall effects—while most clear in our pulse energy measurements (see Fig. 3)—are nonetheless quite modest.”

Therefore, any references to the outcome of the instability has been removed.

Comment 3: While the authors correctly predict equation 4 based on equations 3,4, the pulses created in the laser cavity may follow more a mean field (Lugiato-Lefever) equation which can produce pulses with such universality. I would like to see some clarification.

Response 3: Our experiments utilize a passively mode-locked fibre laser, which is inherently a non-conservative system. However, both in this manuscript and in previous studies [37, 42], we find that our theoretical model—based on a conservative generalized nonlinear Schrödinger equation—not only accurately describes our experimental results (see Fig. 3), but is also consistent with full numerical simulations using an iterative cavity map (see inset of Fig. 2(a) in the main manuscript). We are therefore confident that the modeling in the manuscript is appropriate. We added a short discussion in the conclusion.

"Our theoretical model is sufficient for all the cases considered here even though it is based on a conservative Hilbert nonlinear Schrödinger equation. It is in very good agreement with both our experimental results and with numerical simulations based on an iterative map model, consistent with earlier studies of stationary solutions in similar systems with non-fractional dispersion relations [37]."

Summary of changes to the manuscript

- Bottom of p. 2: We modified the discussion on previous works in the introduction.
  "Indeed, the *linear* fractional propagation of light beams has already been studied theoretically [32-34], and demonstrated experimentally [35, 36]."

  "In fact, any comprehensive experimental investigations of a fractional Laplacian with a potential, either linear or nonlinear, have yet to be reported."

- Bottom of p. 2: We clarified the novel aspect of our work to address the first comment of Reviewer #1.
  "Here we provide such a demonstration: we report the generation and full characterization of temporal solitons governed by a fractional Laplacian. The experiments are carried out in a fibre laser in which the nonlocal temporal effects can be programmed through the dispersion: the dependence of the propagation constant of the light β on the angular frequency ω [37]. We consider a *dispersion relation* in a co-moving frame of the form"

- Figure 1: We modified Fig. 1 to address the second comment of Reviewer #1. We added one more y-axis to Fig. 1(a) and 1(e). The lines are thicker. The green lines present the inverse group velocity in Fig. 1(a) and 1(e) are changed to the dashed lines. Labels of Fig.1 (a-h) are modified.

- Figure 1 caption, p. 3: We revised the caption of Fig. 1 to address the second comment of Reviewer #1.
  "**Concept of the Hilbert-NLS solitons.** (a)-(d) Conventional solitons can form in the presence of quadratic dispersion. (a) The dispersion (red) and (inverse) group velocity (dashed green) vary smoothly with frequency ω; (b) At low intensities I, dispersion stretches pulses in time. Solitons can form at high intensities with smooth (c) spectral and (d) temporal profiles that decay exponentially. Hilbert-NLS solitons (e) arise in the presence of dispersion relation Eq. (1) which has a discontinuous (disc.) derivative (red) and associated discontinuous (inverse) group velocity (dashed green) as a function of frequency; (f) At low intensities the dispersion causes input pulses to split in two. At high intensities solitons form which have (g) a spectrum with discontinuous derivative, and (h) non-exponential decay in time. The units are arbitrary and the intensity profiles in (c), (d), (g) and (h) are on logarithmic scales."

- Middle of p. 3: We modified the information on the dispersion and inverse group velocity in the main text to be consistent with the caption of Fig. 1.
  "This dispersion relation and associated inverse group velocity are conceptually illustrated in Fig. 1(e), and should be compared to the parabolic dispersion relation for conventional NLS solitons in Fig. 1(a)."

- Top of p. 4: We slightly modified a sentence on our works for other cases of fractional derivative.
  "Although we focus on the case with $\alpha = 1$, the flexibility of our experimental apparatus allows for the generation of solitons at any value of the Lévy index α, and we additionally demonstrate the cases $\alpha = 5/4$ and $\alpha = \pi/2$ (Section 3.4).

- Top of p. 5: We slightly modified the sentence describing a property of Hilbert transform.
"This relies on the property of Hilbert transforms that if $g(t) = \mathcal{H}(f(t))$, then $g(at) = \mathcal{H}(f(at))$ for any real, positive a [41]."

- Middle of p. 5: We now indicated that the discussion about the soliton stability has been moved to the Method section, to address the second comment of Reviewer #2.
"As discussed in more detail in the Method section, the stationary solutions of Eq. (2) are marginally stable. To address this we add a small amount of negative quartic dispersion at frequencies far from the central frequency (see Methods) where the spectral soliton amplitude is low (orange curve in Fig. 2(a))."

- Bottom of p. 5: We added a short discussion on the Kelly sidebands appearing in the measured spectra (Figs. 2 and 5), to address the fourth comment of Reviewer #1.
"The sharp spectral peaks at low and high frequencies in Fig. 2(a) correspond to Kelly sidebands, resonant dispersive waves arising from perturbations when the soliton propagates around the cavity [37, 46]. They can be ignored in our analysis."

- Middle of p. 6: We modified the sentence comparing the experimental data and numerical solutions of the Hilbert-NLS equation.
"It is in good agreement with the experiments and confirms that the nonlinear propagation is governed by the Hilbert-NLS equation Eq. 2."

- Bottom of p. 6: We modified the font used to make it consistent with the rest of the manuscript.
"The discontinuous derivative in the spectrum causes $u(t)$ to be proportional to $1/t^2$ (i.e., $I(t) \propto 1/t^4$) as $|t| \to \infty$ (see Fig. 1(h)), and is a manifestation of the nonlocal nature of the fractional Laplacian; the solutions thus decay algebraically unlike the exponential decay of conventional solitons [40]."

- Top of p. 7: We modified sentences to improve the English expression.
"The time-bandwidth product $\Delta f\, \Delta t$—the product of the full-width at half maximum (FWHM) of the spectral and temporal intensities—is characteristic for a particular pulse shape."

  "The small discrepancy of the measured time-bandwidth product with the theoretical value is likely due to residual chirp in our dispersion-managed laser configuration [37]."

- Middle and bottom of p. 7: We modified the sentence to improve the English expression.
"On the other hand, the expected energy of conventional solitons for the parameters of our laser cavity, is shown by the dashed green curve and follows a $1/\Delta t$-dependence (see Methods) [37, 40]."

  "The observation that the soliton energy is approximately constant as the pulse width varies by a factor 3 indicates once more the clear difference between the solitons we generate and conventional NLS solitons."

  "The energy of each pulse is then calculated by integrating the recorded temporal trace and dividing by n_p."

- Top of p. 8: We modified the sentence to improve the English expression.
"In contrast, in our work, it is entirely due to the phase properties of the cavity."

- Figure 3, p. 8: We changed the solid green curve to a dashed green curve to improve the figure. We modified the caption accordingly.

"The dashed green line shows the energy scaling of the conventional NLS soliton for our cavity fibre parameters (see Methods)."

- Bottom of p. 8: We modified the sentence to improve the English expression.
  "We repeated the measurements from Section 3.2 but for different values of the dispersion coefficient β_1."

- Middle of p. 9: We modified the definition of the dispersion coefficients used in our experiments, to address the fifth comment of Reviewer #1, and to improve the English expression.
  "Using the same experimental approach as discussed above, we apply a phase profile $\phi(\omega) = -\frac{|\beta_\alpha|}{\alpha!}|(\omega-\omega_0)|^\alpha L$ with $\alpha_1 = 5/4$ and $\alpha_2 = \pi/2$. Here the $|\beta_\alpha|$ are constants with units of $\mathrm{m}\cdot\mathrm{s}^{-\alpha}$, and $\alpha!$ is evaluated using the Gamma function, i.e., $\alpha! = \Gamma(\alpha+1)$. The numerical values of $|\beta_\alpha|/\alpha!$ are chosen such that the generated solitons have bandwidths that are similar to that for $\alpha = 1$. We again apply weak negative quartic dispersion to improve the soliton stability."

  "The measured spectra (solid blue curve) corresponding to $\alpha_1 = 5/4$ and $\alpha_2 = \pi/2$ are shown in Figs. 5(a) and (b) respectively, and are again in very good agreement with the numerically predicted profiles (red dashed) and previous numerical studies [17, 48]. When no phase is applied by the pulse shaper, the laser operates in the conventional NLS soliton regime ($\alpha_3 = 2$). The intrinsic anomalous negative second-order dispersion of the fibre then balances the Kerr nonlinearity (see Methods) and the output pulses exhibit the well-known hyperbolic secant spectrum, as seen for comparison in Fig. 5(c)."

- Figure 5, p. 10: We modified the caption of the figure to correct the definition of the dispersion coefficients used in our experiments, to address the fifth comment of Reviewer #1.
  "**Fractional NLS solitons for different values of Lévy indices α.** Dispersion relations $\beta(\omega) = -\frac{|\beta_\alpha|}{\alpha!}|(\omega-\omega_0)|^\alpha$ are indicated by solid orange curves, whereas dashed orange curves correspond to the quartic dispersion included to provide stability. Solid blue curves are measured spectra whereas numerically calculated spectra are red dashed. (a) $\alpha_1 = 5/4$ and $\beta_{\alpha_1}/\alpha_1! = -30\ \mathrm{ps}^{\alpha_1}\ \mathrm{km}^{-1}$; (b) $\alpha_2 = \pi/2$ and $\beta_{\alpha_2}/\alpha_2! = -15\ \mathrm{ps}^{\alpha_2}\ \mathrm{km}^{-1}$; and (c) $\alpha_3 = 2$ and $\beta_2/2! = -10.7\ \mathrm{ps}^2\ \mathrm{km}^{-1}$, the dispersion of single-mode fibre (see Methods)."

- Bottom of p. 10: We modified the sentence discussing the flexibility of our experimental setup.
  "However, we show in Section 3.4 that our experimental approach is flexible enough to dial in any value $1 \leq \alpha \leq 2$, allowing in-depth studies of other fractional regimes [49]."

- Top of p. 11: We added a sentence in the conclusion to discuss potential applications based on these novel solitons, to address the first comment of Reviewer #2.
  "These solitons have striking properties, summarised below, for which applications may be developed in future, for example in telecommunications or in laser physics."

- Middle of p. 11: We added a sentence in the conclusion to discuss the validity of our theoretical model applied to our expeirmental results, to address the third comment of Reviewer #2.
  "Our theoretical model is sufficient for all the cases considered here even though it is based on a conservative Hilbert nonlinear Schrödinger equation. It is in very good agreement with both our experimental results and with numerical simulations based on an iterative map model, consistent with earlier studies of stationary solutions in similar systems with non-fractional dispersion relations [37]."

- Middle of p. 11: We modified a sentence to improve the English expression.
  "As such, our results can act as a starting point for a new wave of experimental investigations of linear and nonlinear wave phenomena in media with fractional dispersion relations."

- Bottom of p. 11 (Methods): We modified the definition of the dispersion relation and associated operator to make it consistent with equations in the main text.
"For the conventional NLS equation the dispersion relation takes the form $\beta = -1/2\,|\beta_2|\omega^2$ [40], where β is the wavenumber, ω is the frequency and β_2 is a coefficient that expresses the strength of the effect. In this case $\mathcal{O}_D = (|\beta_2|/2)(\partial^2/\partial t^2)$."

- Top of p. 12 (Methods): We slightly modified Eq. (7) in the Methods section, to make it consistent with the main text.
"The Hilbert transform has the property that

$$\mathcal{H}(u(t)) = i\mathcal{F}^{-1}(\mathrm{sgn}(\omega)\mathcal{F}(u(t))), \tag{1}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively."

- Middle of p. 12 (Methods): We added a sentence to refer to previous works on the stability of conservative solitons, to address the second comment of Reviewer #2.
"The condition for stability of solitons in conservative systems has been discussed by many authors [17, 52]."

- Bottom of p. 12 (Methods): We modified the discussion on the soliton stability in our experiments and added a sentence, to address the second comment of Reviewer #2.
"The finding that $s_c = 0$ follows from the scaling relation derived above and is equivalent to the observation that the energy E does not depend on μ, and it is thus an immediate consequence of the fact that the energy does not depend on the pulse width. In our non-conservative system this instability manifests as a spectral asymmetry. In our numerical simulations with initial phase noise, the soliton spectrum and energy fluctuate after each round trip. Such behavior can be arrested by applying a small additional perturbation [53, 54], as discussed in more detail below."

- Bottom of p. 13 (Methods): We added a short discussion on the effects of quartic dispersion used to stabilized the solitons in our experiments, and to address the second comment of Reviewer #2.
"Even though the addition of a small amount of quartic dispersion leads to slight changes in the generated solitons, the overall effects—while most clear in our pulse energy measurements (see Fig. 3)—are nonetheless quite modest."

- Top of p. 14 (Acknowledgments): We added a sentence in the Acknowledgment section of the revised manuscript.
"The authors thank Professor Panayotis Kevrekidis for useful discussions in the early phases of this work."

- Reference #53: Reference 53 of the submitted manuscript has been removed.

- Reference #35: Reference 35 has been added to the revised manuscript.

- Reference #49: Reference 49 has been added to the revised manuscript.

- Supplementary: Supplementary figures and equations are labeled with the prefix S to avoid ambiguity.

- Supplementary: The first paragraph-p4, line 87: We corrected the typo.
"results similar to Fig. 3(a) in the main text."

- Supplementary: The second paragraph-p4, line 89: We corrected the typo.
"These results are also confirmed by the average output spectra for $n_p = 1$, 2 and 5 solitons are shown in Fig. S4."

Response to Reviewer 1

We thank the referee for their review of our revised manuscript. We believe that the referee's comments are all related and stem from a misunderstanding of our approach.

Comment 1: The authors claim that "The experiments are carried out in a fibre laser in which the nonlocal temporal effects can be programmed through the dispersion: the dependence of the propagation constant of the light β on the angular frequency ω [37]", so can one believe that they found a system that indeed its dispersion can be described by fractional Laplacian?

Response 1: We regret that our previous description was evidently not sufficiently clear. We have completely rewritten the associated passage to explain how our dispersion is equivalent to a fractional Laplacian. We added a passage in the introduction in which we remind the reader of the relationship between time and frequency domain representations:

"To understand how a fractional Laplacian is represented in our optical experiment, we first consider the conventional case of the nonlinear Schrödinger equation, in which a second derivative operator $(|\beta_2|/2)(\partial^2/\partial t^2)$, where β_2 is the dispersion coefficient, in the time domain, corresponds to a parabolic dispersion relation in frequency. This can be seen by application the operator to $e^{-i\omega t}$, where ω is the frequency, which leads to $-|\beta_2|\omega^2/2$ (see Fig 1(a) and (b)). This relationship is valid in a co-moving frame in which any constant speed, represented by a linear relationship, vanishes. Similarly, a fractional Laplacian is represented in the frequency domain by an associated (non-parabolic) dispersion relation. As discussed in more detail in Section 3, we conduct our experiments using a fibre laser in which arbitrary dispersion relations can be applied by the use of a programmable phase mask [37].

Here we consider the *dispersion relation*

$$\beta(\omega) = -|\beta_1||(\omega - \omega_0)|, \tag{1}$$

and show in Section 2 that it corresponds to a fractional Laplacian with $\alpha = 1$."

Having emphasised the relationship between the time and frequency domains, we added a passage in Section 2 that establishes that the dispersion relation mentioned at the end of the previous passage in the time domain corresponds to a fractional Laplacian.

"The second term in this equation corresponds to the fractional Laplacian. The operator $\mathcal{H}$ denotes the Hilbert transform [41], an integral, (i.e. nonlocal) operator. The Hilbert transform, in essence, reverses the sign of the amplitudes of the negative frequency components of its argument, thereby expressing the absolute value in Eq. (1), followed by a multiplication by i (see also Eq (8))). More precisely, in contrast to Section 1 we find $\mathcal{H}(\partial/\partial t)e^{-i\omega t} = -i\omega\mathcal{H}(e^{-i\omega t}) = -i\omega\, i\,\mathrm{sgn}(\omega)e^{-i\omega t} = |\omega|e^{-i\omega t}$ [41] as illustrated in Fig. 1(f). To see that this term corresponds to the fractional Laplacian $(-\nabla^2)^{1/2}$ we consider the square of the operator, i.e.

$$\begin{aligned}\left(-\nabla^2\right)^{1/2}\left(-\nabla^2\right)^{1/2}\varphi &= \mathcal{H}\left(\frac{\partial}{\partial t}\right)\mathcal{H}\left(\frac{\partial}{\partial t}\right)\varphi \\ &= \mathcal{H}\mathcal{H}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}\right)\varphi = -\frac{\partial^2\varphi}{\partial t^2} = -\nabla^2\varphi,\end{aligned} \tag{2}$$

where we used that the Hilbert transform and taking a derivative commute and that $\mathcal{H}\mathcal{H} = -\mathcal{I}$, with $\mathcal{I}$ the identity operator [41]. This finding is consistent with Kwasnicki who showed that the operator in the second term in Eq. (2) corresponds to the fractional Laplacian with $\alpha = 1$ according to Riesz [21]."

Additional information is given in the Methods section. Finally, we further modified both Fig. 1 to emphasise the close relationship between the time and frequency domains and the associated caption.

Comment 2: It seems that the system should be elaborated as the authors spoke "We consider a dispersion relation in a co-moving frame of the form $\beta(\omega) = -|\beta_1|(\omega - \omega_0)|$". Is such a dispersion relation common in real world?

Response 2: Such a dispersion relation is not common and in fact, we are not aware of any material or waveguides with such dispersion. This is why fractional systems have been almost exclusively studied theoretically and numerically. Our unique approach offers a way to experimentally achieve such dispersion (and therefore fractional Laplacian). We added a sentence in the introduction to refer more explicitly to the section in our manuscript describing our experimental approach.

"As discussed in more detail in Section 3, we conduct our experiments using a fibre laser in which arbitrary dispersion relations can be applied by the use of a programmable phase mask [37]."

Comment 3: The governing equation is Hilbert-NLS equation which is closely connected with the NLS equation as shown by the authors in the method section. But why the authors introduce the Hilbert transform but not use the traditional NLS directly? What is the feasible scope about the Hilbert-NLS equation?

Response 3: In the traditional nonlinear Schrödinger equation, the dispersion relation is a parabola. This equation is well-known but is not associated with fractional derivatives and we therefore only consider it here as a comparison in Fig. 1. By using a programmable phase mask in our laser cavity we can change the dispersion to the one mentioned in Comment 2. As discussed in our response to Comment 1, this corresponds to a fractional Laplacian in the time domain.

Comment 4: Why the parabolic dispersion can be lifted in authors' setup?

Response 4: In our experimental setup the intracavity spectral pulse-shaper serves two aims: (i) to compensate the second- and third-order dispersion introduced by the optical fibres, so that the net-cavity dispersion of these orders is negligible; and (ii) to generate a dominant dispersion profile which follows Eq. (1). Therefore, the second- and third- order dispersion experienced by the solitons over one cavity roundtrip is equal to zero in this case. This approach has been successfully used in previous experimental studies. See for example Refs. [37, 42, 44, 49]). More details are also provided in the first Section of the Supplementary Information.

Response to Reviewer 2

We thank the referee for their time spent reviewing our work. Their comments helped us improving the quality of our manuscript.

Comment 1: This is a novel result both oh theory and experiments producing optical pulses in a medium having fractional dispersion.

The work is of high quality and the revised version merits publication.

Summary of changes to the manuscript

- Top of p. 2: We corrected the name of the mathematician de l'Hôpital.
  "The idea of a fractional derivative was raised as early as 1695 in a correspondence between Leibniz and de l'Hôpital [2]."

- Bottom of p. 2 and top p. 3: We added a passage in the introduction to explain why the linear dispersion used in our work corresponds to a fractional Laplacian operator in the time domain, to address the first comment of Reviewer #1.
  "To understand how a fractional Laplacian is represented in our optical experiment, we first consider the conventional case of the nonlinear Schrödinger equation, in which a second derivative operator $(|\beta_2|/2)(\partial^2/\partial t^2)$, where β_2 is the dispersion coefficient, in the time domain, corresponds to a parabolic dispersion relation in frequency. This can be seen by application the operator to $e^{-i\omega t}$, where ω is the frequency, which leads to $-|\beta_2|\omega^2/2$ (see Fig 1(a) and (b)). This relationship is valid in a co-moving frame in which any constant speed, represented by a linear relationship, vanishes. Similarly, a fractional Laplacian is represented in the frequency domain by an associated (non-parabolic) dispersion relation. As discussed in more detail in Section 3, we conduct our experiments using a fibre laser in which arbitrary dispersion relations can be applied by the use of a programmable phase mask [37].
  Here we consider the *dispersion relation*
  $$\beta(\omega) = -|\beta_1||(\omega - \omega_0)|, \tag{3}$$
  and show in Section 2 that it corresponds to a fractional Laplacian with $\alpha = 1$."

- Figure 1: We modified Fig. 1 to address the first comment of Reviewer #1. We added one more column. The new Figs. 1(a) and (b) now emphasis the equivalence between the time and frequency dispersion operators. We modified the figure numbering in the text accordingly.

- Bottom of p. 4 and top p. 5: We added a passage in Section 2 to explain how the linear dispersion used in our work leads to the Hilbert nonlinear Schrödinger equation, to address the first comment of Reviewer #1.
  "The second term in this equation corresponds to the fractional Laplacian. The operator $\mathcal{H}$ denotes the Hilbert transform [41], an integral, (i.e. nonlocal) operator. The Hilbert transform, in essence, reverses the sign of the amplitudes of the negative frequency components of its argument, thereby expressing the absolute value in Eq. (1), followed by a multiplication by i (see also Eq (8)). More precisely, in contrast to Section 1 we find $\mathcal{H}(\partial/\partial t)e^{-i\omega t} = -i\omega\mathcal{H}(e^{-i\omega t}) = -i\omega\, i\,\mathrm{sgn}(\omega)e^{-i\omega t} = |\omega|e^{-i\omega t}$ [41] as illustrated in Fig. 1(f). To see that this term corresponds to the fractional Laplacian $(-\nabla^2)^{1/2}$ we consider the square of the operator, i.e.
  $$\begin{aligned}\left(-\nabla^2\right)^{1/2}\left(-\nabla^2\right)^{1/2}\varphi &= \mathcal{H}\left(\frac{\partial}{\partial t}\right)\mathcal{H}\left(\frac{\partial}{\partial t}\right)\varphi = \\ &= \mathcal{H}\mathcal{H}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}\right)\varphi = -\frac{\partial^2\varphi}{\partial t^2} = -\nabla^2\varphi,\end{aligned} \tag{4}$$
  where we used that the Hilbert transform and taking a derivative commute and that $\mathcal{H}\mathcal{H} = -\mathcal{I}$, with $\mathcal{I}$ the identity operator [41]. This finding is consistent with Kwasnicki who showed that the operator in the second term in Eq. (2) corresponds to the fractional Laplacian with $\alpha = 1$ according to Riesz [21]. The Hilbert transform is further discussed in the Methods section."

- Top and bottom of p. 11: We slightly modified three sentences.
  "We have provided the first comprehensive experimental realization of fractional solitons by harnessing the interaction between fractional dispersion, associated with a fractional Laplacian, and the Kerr nonlinearity in a fibre laser cavity."

  "In a sense this case, for which $\alpha = 1$ and the conventional NLS, for which $\alpha = 2$ are extremes in the usual interval $0 \leq \alpha \leq 2$ since for $\alpha < 1$ the solitons are unstable (see Methods)."

  "While our experiments were carried out in an optics context, our results are universal and agnostic to the particular physical embodiment of the governing equation. As such, our results can act as a starting point for a new wave of experimental investigations of linear and nonlinear wave phenomena in media with fractional dispersion relations."

- Top of p. 12: We modified a sentence describing the dispersion operator in the Methods section, to make it consistent with other changes in the introduction, to address the first comment of Reviewer #1.
  "As discusssed in Section 1, for the conventional NLS equation the dispersion relation takes the form $\beta = -1/2\,|\beta_2|\omega^2$ [40], where β is the wavenumber, in which case $\mathcal{O}_D = (|\beta_2|/2)(\partial^2/\partial t^2)$."