

Universal quantum computation using Ising anyons from a non-semisimple Topological Quantum Field Theory

Corresponding Author: Dr Aaron Lauda

This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Attachments originally included by the reviewers as part of their assessment can be found at the end of this file.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)

This paper proposes a framework for topological quantum computation using non-semisimple analogs of topological quantum field theories in 2+1 dimensions. The conventional theory of Ising anyons, which is believed to describe excitations in the $\nu = 5/2$ fractional quantum Hall state, is not universal for quantum computation via braiding of quasiparticles. However, the authors show that the non-semisimple theory introduces new anyon types that extend the Ising framework. By adding just one new anyon type, universal quantum computation can be achieved through braiding alone, opening new avenues for realizing fault-tolerant quantum computing in topologically ordered systems.

It appears that the authors propose a new direction for topological quantum computation, specifically through the exploration of novel topological phases of matter that are described by non-semi-simple topological quantum field theory (TQFT). I am inclined to think this story is novel enough to recommend for publication, but I want to ask some questions to fix the evaluation.

My only concern is the physical realization of such non-semi-simple topological orders. I apologize if this has already been addressed elsewhere, but I would appreciate it if the authors could include a more detailed discussion of how the Hamiltonian for this non semi-simple theory can be constructed, to make the paper more self-contained. I suspect that a realization of this theory may require a non-Hermitian Hamiltonian or some non-unitary setup. It would be helpful if the authors could explore whether such a setup could be interpreted as a form of decoherence affecting certain Hermitian phases.

Reviewer #2

(Remarks to the Author)

See attachment.

Reviewer #3

(Remarks to the Author)

The authors consider a non-semisimple representation of the quantum group of the Ising model and argue that the inclusion of the additional anyon label (α) will render this representation universal for topological quantum computing, i.e. computing by braids only. They continue with defining fusion rules, encoding qubits and specific recipes for single and two-qubit gates.

A drawback of this representation is that the anyon Hilbert space is no longer positive definite, and the corresponding braids don't always lead to unitary operations. The authors get around the general non-unitarity by encoding their qubits such that the braid operations in the computational space are always unitary and only leakage into non-computational space involves non-unitary operations. This encoding is defined by combining an α label with pairs of σ labels, in a tree structure, such that n qubits are encoded by one α and $2n$ σ s.

They then define basic braid matrices, which is straightforward for the first qubit but gets more and more expensive for the qubits further away from the α particle. They argue that combinations of these braid matrices form a dense cover for $SU(2)$, leading to arbitrary single-qubit gates.

For entangling two-qubit gates they build on ideas from Refs. [30-33,52] and reduce the problem into finding a braid that simultaneously approximates two diagonal matrices, one in $SU(2)$ and one in $SU(1,1)$, such that the second element of the former and the first element of the latter form the non-trivial gate acting on the target qubit. The diagonal form of these matrices will ensure no leakage. I did not follow all the details for this construction but it looks correct.

The work is original and interesting. As far as I know, this is the first detailed study of a non-semisimple model for quantum computing. Overall, the paper is well-written and the arguments are sound. I am, however, concerned about two points.

1. I am not sure if I understand their pentagon equation, since the order of fusion seems to make a difference. For example, it seems to me that:

$((\alpha \times \sigma) \times \sigma) \times \sigma \times \sigma$ leads to 6 α states, whereas

$((\sigma \times \sigma) \times \sigma) \times \sigma \times \alpha = ((1 + \psi + 1 + \psi) \times \alpha)$ leads to 4.

So, unless I am missing something, the consistency of fusion seems to be problematic.

2. To what extent these models are topologically protected? Given that non-unitary CFTs are usually representatives of gapless systems, it is not clear to me how to define (and less how to quantify) topological robustness in these models.

A few typos:

Page 2, last paragraph of Sec I: "disordered" systems

Page 4, first column, the equation in "see equation" is missing.

Page 5, first column, the sentence that starts with "Fix the basis ..." doesn't sound correct grammatically.

Page 5, first column, there is an extra verb in "... so that the resulting gate is approximates ..."

Page 5, second column, just above equation II.10, there is an undefined D in the definition of $D_{-}(1,1)$, then again, a few lines below D is defined in terms of $D_{-}(1,1)$.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)

The revised manuscript proposes a framework for universal topological quantum computation (TQC) using Ising anyons derived from a non-semisimple Topological Quantum Field Theory (TQFT), thereby overcoming the known limitation that standard Ising anyons (as realized in the $\nu = 5/2$ fractional quantum Hall state) cannot achieve universality via braiding alone. The authors introduce a new anyon type, labeled α , arising from non-semisimple extensions of $SU(2)_k$ TQFTs. This anyon enables universal braiding-based computation when held fixed and combined with standard Ising anyons.

The authors develop a detailed encoding scheme for qubits, demonstrate that the computational Hilbert space remains positive-definite (despite the overall theory being pseudo-Hermitian), and construct explicit braiding matrices. They rigorously prove universality for single-qubit gates and design entangling gates with exponentially suppressed leakage, adapting and generalizing Reichardt's compilation algorithm to the indefinite unitary setting. These techniques are demonstrated concretely for the case $\alpha = 2 + 2/5$. The manuscript also discusses possible physical realizations via fractional quantum Hall systems and connections to logarithmic conformal field theory.

The authors have fully addressed the criticisms raised in the original review. The revised manuscript presents a well-motivated and technically solid contribution that opens a new direction in topological quantum computation by leveraging non-semisimple TQFTs. The encoding is explicit, the gate constructions are rigorous, and the physical relevance is reasonably discussed. I am satisfied with the revisions and recommend the manuscript for acceptance.

Reviewer #2

(Remarks to the Author)

The authors have addressed all of my comments. I recommend it for publication.

Reviewer #3

(Remarks to the Author)

Open Access This Peer Review File is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

In cases where reviewers are anonymous, credit should be given to 'Anonymous Referee' and the source.

The images or other third party material in this Peer Review File are included in the article's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.

To view a copy of this license, visit <https://creativecommons.org/licenses/by/4.0/>

UNIVERSAL QUANTUM COMPUTATION USING ISING ANYONS FROM A NON-SEMISIMPLE TOPOLOGICAL QUANTUM FIELD THEORY

We would like to thank the referees for their careful reading of this submission. Below we have detailed our responses.

Motivation for changing the value of α . In response to inquiries from readers regarding the computation of the braid w (renamed W in the current submission) we decided to modify the value of α from $1 + 1/5$ to $2 + 2/5$ with the goal of simplifying the computation of W . At this value of α , the braid $(b_2^{\alpha\psi\sigma\sigma})^2$ has lower non-unitary leakage than at $\alpha = 1 + 1/5$, making it easier to find a braid with leakage less than 1. In particular, the difficulty in calculating the old form of $W = (b_2^{\alpha\psi\sigma\sigma})(b_1^{\alpha\sigma\psi\sigma})^2(b_2^{\alpha\sigma\psi\sigma})^2(b_2^{\alpha\psi\sigma\sigma})$ lies in the calculation of matrices such as $(b_1^{\alpha\psi\sigma\sigma}) : \text{Hom}(\alpha \pm 1, \alpha\psi\sigma\sigma) \rightarrow \text{Hom}(\alpha \pm 1, \alpha\sigma\psi\sigma)$. These operations are between different vector spaces and computing them involves subtleties in the in the basis ordering. In contrast, the new definition $W = (b_2^{\alpha\psi\sigma\sigma})^2(b_1^{\alpha\psi\sigma\sigma})^2(b_2^{\alpha\psi\sigma\sigma})^2(b_1^{\alpha\psi\sigma\sigma})^2(b_2^{\alpha\psi\sigma\sigma})^{-2}$ only requires computing $(b_1^{\alpha\psi\sigma\sigma})^2$ and $(b_2^{\alpha\psi\sigma\sigma})^2$, both of which have the same source and target space.

This different value of α affects the rest of the paper only in minor ways: the eigenvalues for the single-qubit braid at $\alpha = 2 + 2/5$ is different, but the technique for showing that the matrix has infinite order is unchanged. For $\alpha \in (2, 3)$, the signature of the two-qubit Hilbert space goes from $(+, +, +, +, +, -)$ to $(+, +, +, +, -, +)$ (note that the computational space is still positive-definite).

Finally, the diagonal matrix used in Reichardt's algorithm slightly changes form from $\begin{pmatrix} e^{\frac{i\pi}{5}} & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{5}} \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i3\pi}{5}} \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i3\pi}{5}} \end{pmatrix}$, without affecting the result of Reichardt's algorithm in any way.

Reviewer 1.

- My only concern is the physical realization of such non-semi-simple topological orders. I apologize if this has already been addressed elsewhere, but I would appreciate it if the authors could include a more detailed discussion of how the Hamiltonian for this non semi-simple theory can be constructed, to make the paper more self-contained. I suspect that a realization of this theory may require a non-Hermitian Hamiltonian or some non-unitary setup. It would be helpful if the authors could explore whether such a setup could be interpreted as a form of decoherence affecting certain Hermitian phases.

Response. *We have added several paragraphs near the end of the introduction that we hope better explain the potential physical relevance. The decoherence interpretation would be very interesting to pursue. We don't have specific insights, but to us, this seems like a very promising direction.*

Reviewer 2.

- (pg. 4) "A standard approach from [48] for showing density in $SU(2)$ is to specify two non-commuting braids b_1, b_2 that generate an infinite subgroup of $SU(2)$. In particular, it suffices to show that either b_1 or b_2 has infinite order." This is incorrect. A counterexample is as follows.

Let $b_1 = \text{diag}(e^{i\pi\alpha}, e^{-i\pi\alpha})$ for any irrational value α and b_2 be the Pauli X gate (multiplied by $-i$). Then b_1 and b_2 do not commute, b_1 has infinite order, b_2 has finite order, but b_1, b_2 do NOT generate a dense subgroup of $SU(2)$. The correct statement should require both b_1 and b_2 to be infinite order and non-commuting. See Lemma 2 of (<https://arxiv.org/pdf/quant-ph/9906129>), though the original result is due to Kitaev. This, however, doesn't affect the universality result in the paper. This is because, for the gates in the paper, although b_1 has infinite order and b_2 has finite order, one can let $b'_2 = b_2 b_1 b_2^{-1}$, it turns out b'_2 does not commute with b_1 and of course it has infinite order. For the counterexample above, b'_2 would commute with b_1 .

Response. Following the Reviewer's suggestion, we corrected the statement of the density theorem and redefined b_2 as $b_2 b_1 b_2^{-1}$. We also verified that the commutator of b_1 and b_2 is indeed nonzero for all non-integer values of α .

- (pg. 5), "For unitary matrices, the off-diagonal entries are less than or equal to 1, so iterating this procedure suppresses the off-diagonal entries to arbitrary accuracy". This is true as long as the off-diagonal entry to start with does not have norm equal to 1, such as the Pauli X gate. So, at the top left of page 6, in addition to check $|\langle \psi_2 | \omega | \psi_3 \rangle| < 1$, you also need to check the second condition $|\langle \psi_0 | \omega | \psi_1 \rangle| \neq 1$, which should in general holds, but still needs to verify. In particular, for the actual example following that, one needs to verify the second condition.

Response. We extended the requirement for convergence of Reichardt's algorithm to include both the unitary and non-unitary leakage terms: "if $|\langle \psi_0 | W | \psi_1 \rangle| < 1$ and $|\langle \psi_2 | W | \psi_3 \rangle| < 1$ ". We also reported the magnitude of both leakage terms (highlighting that they are both less than one) in the first example.

- (pg. 5) A noteworthy remark is that the method used to construct (approximately) leakage-free entangling gates via recursive sequences bears resemblance to the approach described in Section 4.3 of arXiv:1904.01731, which addresses the construction of leakagefree entangling gates in the context of Fibonacci anyons. Both methods appear to be inspired by earlier work of Reichardt.

Response. We agree that this reference is valuable. We added a reference and will likely revisit this work in a future paper where we expand on this model.

- 4. (pg. 6) In the paragraph "Performing a brute force search over...", there is a switch of notations on the anyon types from Greek letters, e.g. σ, ψ to numerical labels, e.g. 1,2.

Response. Done.

- The rest are mostly typos. For example, in the left column of page 4, "i+1st", "see equation" (equation number is missing), " $\sigma^{\alpha\sigma\sigma}$ " (should the first σ be b ?); On the right column of page 5, " $D(1,1) = e^{-i\pi 5D}$ " (should D be $D(2)$?). In Eqn (II.11), $Vk+1$ should be $\omega k+1$.

Response. We corrected these typos.

Reviewer 3. I am, however, concerned about two points.

- (1) I am not sure if I understand their pentagon equation, since the order of fusion seems to makes a difference. For example, it seems to me that:

$$(((\alpha \times \sigma) \times \sigma) \times \sigma) \times \sigma \mapsto \text{six } \alpha \text{ states}$$

states, whereas

$$(((\sigma \times \sigma) \times \sigma) \times \sigma) \times \alpha = ((1 + \psi + 1 + \psi) \times \alpha) \mapsto \text{four } \alpha \text{ states}$$

So, unless I am missing something, the consistency of fusion seems to be problematic.

Response. We added a comment in Appendix D to remind the reader about the nonsemisimple modules that must be included when verifying the pentagon in the nonsemisimple framework.

In greater detail than we provide in the article, here, we compare the relevant fusion rules side-by-side:

fusion	semisimple framework	non-semisimple framework
$\sigma \times \sigma$	$\mathbb{1} + \psi$	$\mathbb{1} + \psi$
$\psi \times \sigma$	σ	$\sigma + \mathbf{0}$
$\psi \times \psi$	$\mathbb{1}$	$\mathbb{1} + P_2$
$\mathbf{0} \times \sigma$	DNE	P_2

where $\mathbf{0}$ corresponds to the α -particle type indexed by $(\mathbb{R} \setminus \mathbb{Z}) \cup 4\mathbb{Z}$ whose parameter is zero.

Consequently, in the non-semisimple setting, we calculate that

$$\begin{aligned}
 ((\sigma \times \sigma) \times \sigma) \times \sigma &= ((\mathbb{1} + \psi) \times \sigma) \times \sigma \\
 &= ((\mathbb{1} \times \sigma) + (\psi \times \sigma)) \times \sigma \\
 &= (\sigma + \sigma + \mathbf{0}) \times \sigma \\
 &= 2(\sigma \times \sigma) + (\mathbf{0} \times \sigma) \\
 &= \mathbb{1} + \psi + \mathbb{1} + \psi + P_2
 \end{aligned}$$

Therefore, in the semisimple Ising framework, it is true that $((\sigma \times \sigma) \times \sigma) \times \sigma = \mathbb{1} + \psi + \mathbb{1} + \psi$. However, in the non-semisimple setting, new particle types (highlighted in blue above) with vanishing quantum dimensions augment the fusion rules appearing in the semisimple framework. As the referee deduced, we have four α fusion channels coming from $(\mathbb{1} + \psi + \mathbb{1} + \psi) \times \alpha$. In our paper, we choose an irrational α for our qubit encoding. In which case, the remaining two α channels come from fusing $P_2 \times \alpha = (\alpha - 4) + (\alpha - 2) + (\alpha - 2) + \alpha + \alpha + (\alpha + 2) + (\alpha + 2) + (\alpha + 4)$.

We thought other readers may be confused by this, so we expanded the discussion in Section II.A in addition to some additional clarification in Appendix D. We also opted to switch from using 0 for the module $V_0 = S_{3/2}$ as we thought this would cause confusion and we thought the 3/2 would better connect the reader to the usual theory of spin.

- (2) To what extent these models are topologically protected? Given that non-unitary CFTs are usually representatives of gapless systems, it is not clear to me how to define (and less how to quantify) topological robustness in these models.

Response. *We added more discussion about this in the introduction as to why we think these theories could be connected with gapped systems.*

A few typos:

- Page 2, last paragraph of Sec I: “disordered” systems
- 4, first column, the equation in (“equation number” is missing).
- Page 5, first column, the sentence that starts with “Fix the basis ...” doesn’t sound correct grammatically.
- Page 5, first column, there is an extra verb in “... so that the resulting gate is approximates ...”
- Page 5, second column, just above equation II.10, there is an undefined D in the definition of $D(1, 1)$, then again, a few lines below D is defined in terms of $D(1, 1)$.

Response. *We corrected these typos.*

Report for

Universal quantum computation using Ising anyons from a non-semisimple Topological Quantum Field Theory

The manuscript proposes a novel framework for topological quantum computing based on *non-semisimple* topological quantum field theories (TQFTs) in 2+1 dimensions. In particular, it investigates a non-semisimple version of the well-known Ising-type TQFT.

The Ising TQFT is among the most extensively studied models in topological quantum computing. It features three anyon types: the vacuum $\mathbf{1}$, a fermion ψ , and the non-Abelian Ising anyon σ . Experimental evidence suggests that Ising anyons may emerge in systems such as the fractional quantum Hall state at filling factor $5/2$. However, from a computational standpoint, the braiding of Ising anyons generates only Clifford gates, which can be efficiently simulated on a classical computer. Even when supplemented by fusion operations, the Ising anyons are not believed to support universal quantum computation.

In contrast, non-semisimple (and hence non-unitary) TQFTs have attracted increasing attention in recent years due to their rich mathematical structure and physical relevance. These theories often yield more refined topological invariants than their semisimple counterparts, and have connections to non-Hermitian physics, such as logarithmic conformal field theories. Furthermore, they provide a foundation for constructing Levin-Wen-type lattice models with pseudo-Hermitian gapped Hamiltonians. Thus, exploring their applications in quantum computing is both timely and compelling.

While the conventional (unitary) Ising TQFT includes only three anyon types, the non-semisimple version considered in this work introduces an infinite family of additional anyon types labeled by a continuous real parameter α . For a fixed choice of α , the authors construct a multi-anyon fusion space involving one α -anyon and several Ising anyons. This fusion space decomposes into a unitary sector and a non-unitary sector, with the unitary component identified as the computational (multi-qubit) subspace. The two-qubit space in particular resides in a six-dimensional fusion space, requiring careful treatment of leakage outside the computational subspace.

Employing the F - and R -symbols of the non-semisimple theory, the authors compute the unitary gates corresponding to generators of the affine braid group. Universality for single-qubit gates is achieved via two non-commuting braids with infinite order. For two- and multi-qubit gates, the authors construct approximately leakage-free entangling gates, thereby establishing universality of their proposed computational model.

In my view, this represents a significant theoretical advancement in the study of topological quantum computation. The work highlights the potential of non-semisimple TQFTs to overcome limitations inherent in semisimple models such as the Ising TQFT. Although an experimental realization of non-semisimple Ising anyons may be a long-term goal, the theoretical implications of this study are profound.

The manuscript is generally well written. I provide several specific comments below. Subject to appropriate revisions addressing those points, I recommend the manuscript for publication.

1. (pg. 4) “A standard approach from [48] for showing density in $SU(2)$ is to specify two non-commuting braids b_1, b_2 that generate an infinite subgroup of $SU(2)$. In particular, it

suffices to show that either b_1 or b_2 has infinite order.” This is incorrect. A counterexample is as follows. Let $b_1 = \text{diag}(e^{i\pi\alpha}, e^{-i\pi\alpha})$ for any irrational value α and b_2 be the Pauli X gate (multiplied by $-i$). Then b_1 and b_2 do not commute, b_1 has infinite order, b_2 has finite order, but b_1, b_2 do NOT generate a dense subgroup of $\text{SU}(2)$. The correct statement should be requiring both b_1 and b_2 to be infinite order and non-commuting. See Lemma 2 of (<https://arxiv.org/pdf/quant-ph/9906129>), though the original result is due to Kitaev. This, however, doesn’t affect the universality result in the paper. This is because, for the gates in the paper, although b_1 has infinite order and b_2 has finite order, one can let $b'_2 = b_2 b_1 b_2^{-1}$, it turns out b'_2 does not commute with b_1 and of course it has infinite order. For the counterexample above, b'_2 would commute with b_1 .

2. (pg. 5), “For unitary matrices, the off-diagonal entries are less than or equal to 1, so iterating this procedure suppresses the off-diagonal entries to arbitrary accuracy”. This is true as long as the off-diagonal entry to start with does not have norm equal to 1, such as the Pauli X gate. So, at the top left of page 6, in addition to check $|\langle\psi_2|\omega|\psi_3\rangle| < 1$, you also need to check the second condition $|\langle\psi_0|\omega|\psi_1\rangle| \neq 1$, which should in general holds, but still needs to verify. In particular, for the actual example following that, one needs to verify the second condition.
3. (pg. 5) A noteworthy remark is that the method used to construct (approximately) leakage-free entangling gates via recursive sequences bears resemblance to the approach described in Section 4.3 of arXiv:1904.01731, which addresses the construction of leakage-free entangling gates in the context of Fibonacci anyons. Both methods appear to be inspired by earlier work of Reichardt.
4. (pg. 6) In the paragraph “Performing a brute force search over....”, there is a switch of notations on the anyon types from Greek letters, e.g. σ, ψ , to numerical labels, e.g. 1,2.
5. The rest are mostly typos. For example, in the left column of page 4, “i+1st”, “see equation ” (equation number is missing), “ $\sigma^{\alpha\sigma\sigma}$ ” (should the first σ be b ?); On the right column of page 5, “ $D_{(1,1)} = e^{-\frac{i\pi}{5}} D$ ” (should D be $D_{(2)}$?). In Eqn (II.11), V_{k+1} should be ω_{k+1} .